Extracting quadratic propagators in one-loop amplitudes

第五届全国场论与弦论学术研讨会

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Outline

Backgrounds: Feynman diagrams, CHY and BCJ

Helpful tools expansion formulas at on-/off-shell levels

Quadratic propagators in YMS and YM

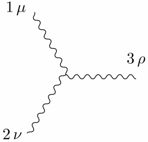
Quadratic propagators in double-YMS, EYM and GR

I. Scattering amplitudes: from Feynman diagrams to BCJ and CHY

Main features of Yang-Mills and gravity Feynman diagrams

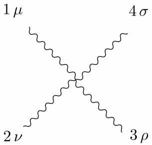
Yang-Mills	Gravity
3-, 4-point vertices	Infinite number of vertices
Color-dressed	Color singlet
External lines ϵ^{μ}	External lines $\epsilon^{\mu\nu}$

Yang-Mills: Feynman diagrams and color decomposition



$f_{a_1 a_2 a_3}\left[\eta^{\mu\nu}(k_1 - k_2)^\rho + \text{cyc(1,2,3)}\right]$

(a)



$-i\left[f_{a_1 a_2 e} f_{a_3 a_4 e}(\eta^{\mu\rho}\eta^{\sigma\rho} - \eta^{\mu\sigma}\eta^{\nu\rho})\right] + \text{cyc(2,3,4)}$

(b)

SU(N) gauge field

$$f^{abc} = -\frac{i}{\sqrt{2}}\left[\mathrm{tr}(t^a t^b t^c) - \mathrm{tr}(t^b t^a t^c)\right]$$

$$(t^a)_{i_1}^{\ j_1}\,(t^a)_{i_2}^{\ j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{N}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}$$

Color decompositions

Berends, Giele, 87; Mangano,Parke,Xu, 88

$$M = \sum_{\sigma\in S_n/Z_n} \mathrm{tr}\left[t^{a_{\sigma(1)}}\ldots t^{a_{\sigma(n)}}\right]$$

$$\times A(\sigma(1)\ldots\sigma(n)) \leftarrow$$ Color-ordered amplitude

Del Duca, Dixon, Maltoni, 99

$$M = \sum_{\sigma\in S_{n-2}} if^{a_1 a_{\sigma(2)} e_1}\ldots if^{e_{n-2} a_{\sigma(n-1)} a_n}$$

$$\times A(1,\sigma(2)\ldots\sigma(n-1),n)$$

Gravity: Feynman diagrams

3-graviton vertex

$$-2\eta_{\alpha\beta}\eta_{\mu\nu}\eta_{\sigma\rho}p\cdot p-2\eta_{\alpha\beta}\eta_{\mu\nu}\eta_{\sigma\rho}p\cdot q+5/2\eta_{\alpha\beta}\eta_{\mu\nu}\eta_{\sigma\rho}p\cdot l+\eta_{\alpha\beta}\eta_{\mu\nu}\eta_{\sigma\rho}q\cdot l+1/2\eta_{\alpha\beta}\eta_{\mu\nu}\eta_{\sigma\rho}l\cdot l+$$
$$2\eta_{\alpha\beta}\eta_{\mu\nu}p_\sigma p_\rho+3/2\eta_{\alpha\beta}\eta_{\mu\nu}p_\sigma q_\rho-\eta_{\alpha\beta}\eta_{\mu\nu}p_\sigma l_\rho+3/2\eta_{\alpha\beta}\eta_{\mu\nu}p_\rho q_\sigma-\eta_{\alpha\beta}\eta_{\mu\nu}p_\rho l_\sigma+2\eta_{\alpha\beta}\eta_{\mu\nu}q_\sigma q_\rho-$$
$$3/2\eta_{\alpha\beta}\eta_{\mu\nu}q_\sigma l_\rho-3/2\eta_{\alpha\beta}\eta_{\mu\nu}q_\rho l_\sigma+\eta_{\alpha\beta}\eta_{\mu\nu}l_\sigma l_\rho+\eta_{\alpha\beta}\eta_{\mu\sigma}\eta_{\nu\rho}p\cdot p+1/2\eta_{\alpha\beta}\eta_{\mu\sigma}\eta_{\nu\rho}p\cdot q-$$
$$3/2\eta_{\alpha\beta}\eta_{\mu\sigma}\eta_{\nu\rho}p\cdot l-\eta_{\alpha\beta}\eta_{\mu\sigma}\eta_{\nu\rho}q\cdot q+1/2\eta_{\alpha\beta}\eta_{\mu\sigma}\eta_{\nu\rho}q\cdot l-\eta_{\alpha\beta}\eta_{\mu\sigma}\eta_{\nu\rho}l\cdot l-2\eta_{\alpha\beta}\eta_{\mu\sigma}p_\nu p_\rho-$$
$$3\eta_{\alpha\beta}\eta_{\mu\sigma}p_\nu q_\rho+2\eta_{\alpha\beta}\eta_{\mu\sigma}p_\nu l_\rho-2\eta_{\alpha\beta}\eta_{\mu\sigma}p_\rho q_\nu+\eta_{\alpha\beta}\eta_{\mu\sigma}p_\rho l_\nu+4\eta_{\alpha\beta}\eta_{\mu\sigma}q_\nu q_\rho-\eta_{\alpha\beta}\eta_{\mu\sigma}q_\nu l_\rho-$$
$$\eta_{\alpha\beta}\eta_{\mu\sigma}q_\rho l_\nu+2\eta_{\alpha\beta}\eta_{\mu\sigma}l_\nu l_\rho+\eta_{\alpha\beta}\eta_{\mu\rho}\eta_{\nu\sigma}p\cdot p+1/2\eta_{\alpha\beta}\eta_{\mu\rho}\eta_{\nu\sigma}p\cdot q-3/2\eta_{\alpha\beta}\eta_{\mu\rho}\eta_{\nu\sigma}p\cdot l-\eta_{\alpha\beta}\eta_{\mu\rho}\eta_{\nu\sigma}q\cdot$$
$$q+1/2\eta_{\alpha\beta}\eta_{\mu\rho}\eta_{\nu\sigma}q\cdot l-\eta_{\alpha\beta}\eta_{\mu\rho}\eta_{\nu\sigma}l\cdot l-2\eta_{\alpha\beta}\eta_{\mu\rho}p_\nu p_\sigma-3\eta_{\alpha\beta}\eta_{\mu\rho}p_\nu q_\sigma+2\eta_{\alpha\beta}\eta_{\mu\rho}p_\nu l_\sigma-$$
$$2\eta_{\alpha\beta}\eta_{\mu\rho}p_\sigma q_\nu+\eta_{\alpha\beta}\eta_{\mu\rho}p_\sigma l_\nu+4\eta_{\alpha\beta}\eta_{\mu\rho}q_\nu q_\sigma-\eta_{\alpha\beta}\eta_{\mu\rho}q_\nu l_\sigma-\eta_{\alpha\beta}\eta_{\mu\rho}q_\sigma l_\nu+2\eta_{\alpha\beta}\eta_{\mu\rho}l_\nu l_\sigma+$$
$$2\eta_{\alpha\beta}\eta_{\sigma\rho}p_\mu p_\nu+2\eta_{\alpha\beta}\eta_{\sigma\rho}p_\mu q_\nu-\eta_{\alpha\beta}\eta_{\sigma\rho}p_\mu l_\nu+3\eta_{\alpha\beta}\eta_{\sigma\rho}p_\nu q_\mu-2\eta_{\alpha\beta}\eta_{\sigma\rho}p_\nu l_\mu-4\eta_{\alpha\beta}\eta_{\sigma\rho}q_\mu q_\nu+$$
$$2\eta_{\alpha\beta}\eta_{\sigma\rho}q_\mu l_\nu+2\eta_{\alpha\beta}\eta_{\sigma\rho}q_\nu l_\mu-2\eta_{\alpha\beta}\eta_{\sigma\rho}l_\mu l_\nu+4\eta_{\alpha\mu}\eta_{\beta\nu}\eta_{\sigma\rho}p\cdot p+6\eta_{\alpha\mu}\eta_{\beta\nu}\eta_{\sigma\rho}p\cdot q-6\eta_{\alpha\mu}\eta_{\beta\nu}\eta_{\sigma\rho}p\cdot l-$$
$$4\eta_{\alpha\mu}\eta_{\beta\nu}\eta_{\sigma\rho}q\cdot q+6\eta_{\alpha\mu}\eta_{\beta\nu}\eta_{\sigma\rho}q\cdot l-3\eta_{\alpha\mu}\eta_{\beta\nu}\eta_{\sigma\rho}l\cdot l-4\eta_{\alpha\mu}\eta_{\beta\nu}p_\sigma p_\rho-3\eta_{\alpha\mu}\eta_{\beta\nu}p_\sigma q_\rho+\eta_{\alpha\mu}\eta_{\beta\nu}p_\sigma l_\rho-$$
$$3\eta_{\alpha\mu}\eta_{\beta\nu}p_\rho q_\sigma+\eta_{\alpha\mu}\eta_{\beta\nu}p_\rho l_\sigma+4\eta_{\alpha\mu}\eta_{\beta\nu}q_\sigma q_\rho-\eta_{\alpha\mu}\eta_{\beta\nu}q_\sigma l_\rho-\eta_{\alpha\mu}\eta_{\beta\nu}q_\rho l_\sigma+2\eta_{\alpha\mu}\eta_{\beta\nu}l_\sigma l_\rho-$$
$$4\eta_{\alpha\mu}\eta_{\beta\sigma}\eta_{\nu\rho}p\cdot p-6\eta_{\alpha\mu}\eta_{\beta\sigma}\eta_{\nu\rho}p\cdot q+6\eta_{\alpha\mu}\eta_{\beta\sigma}\eta_{\nu\rho}p\cdot l+4\eta_{\alpha\mu}\eta_{\beta\sigma}\eta_{\nu\rho}q\cdot q-6\eta_{\alpha\mu}\eta_{\beta\sigma}\eta_{\nu\rho}q\cdot l+$$
$$4\eta_{\alpha\mu}\eta_{\beta\sigma}\eta_{\nu\rho}l\cdot l+4\eta_{\alpha\mu}\eta_{\beta\sigma}p_\nu p_\rho+4\eta_{\alpha\mu}\eta_{\beta\sigma}p_\nu q_\rho-6\eta_{\alpha\mu}\eta_{\beta\sigma}p_\nu l_\rho+2\eta_{\alpha\mu}\eta_{\beta\sigma}p_\rho q_\nu-2\eta_{\alpha\mu}\eta_{\beta\sigma}p_\rho l_\nu-$$
$$4\eta_{\alpha\mu}\eta_{\beta\sigma}q_\nu q_\rho+2\eta_{\alpha\mu}\eta_{\beta\sigma}q_\nu l_\rho+2\eta_{\alpha\mu}\eta_{\beta\sigma}q_\rho l_\nu-4\eta_{\alpha\mu}\eta_{\beta\sigma}l_\nu l_\rho-4\eta_{\alpha\mu}\eta_{\beta\rho}\eta_{\nu\sigma}p\cdot p-6\eta_{\alpha\mu}\eta_{\beta\rho}\eta_{\nu\sigma}p\cdot$$
$$q+6\eta_{\alpha\mu}\eta_{\beta\rho}\eta_{\nu\sigma}p\cdot l+4\eta_{\alpha\mu}\eta_{\beta\rho}\eta_{\nu\sigma}q\cdot q-6\eta_{\alpha\mu}\eta_{\beta\rho}\eta_{\nu\sigma}q\cdot l+4\eta_{\alpha\mu}\eta_{\beta\rho}\eta_{\nu\sigma}l\cdot l+4\eta_{\alpha\mu}\eta_{\beta\rho}p_\nu p_\sigma+$$
$$4\eta_{\alpha\mu}\eta_{\beta\rho}p_\nu q_\sigma-6\eta_{\alpha\mu}\eta_{\beta\rho}p_\nu l_\sigma+2\eta_{\alpha\mu}\eta_{\beta\rho}p_\sigma q_\nu-2\eta_{\alpha\mu}\eta_{\beta\rho}p_\sigma l_\nu-4\eta_{\alpha\mu}\eta_{\beta\rho}q_\nu q_\sigma+2\eta_{\alpha\mu}\eta_{\beta\rho}q_\nu l_\sigma+$$
$$2\eta_{\alpha\mu}\eta_{\beta\rho}q_\sigma l_\nu-4\eta_{\alpha\mu}\eta_{\beta\rho}l_\nu l_\sigma+4\eta_{\alpha\mu}\eta_{\nu\sigma}p_\beta p_\rho+2\eta_{\alpha\mu}\eta_{\nu\sigma}p_\beta q_\rho-4\eta_{\alpha\mu}\eta_{\nu\sigma}p_\beta l_\rho+4\eta_{\alpha\mu}\eta_{\nu\sigma}p_\rho q_\beta-$$
$$2\eta_{\alpha\mu}\eta_{\nu\sigma}p_\rho l_\beta-4\eta_{\alpha\mu}\eta_{\nu\sigma}q_\beta q_\rho+2\eta_{\alpha\mu}\eta_{\nu\sigma}q_\beta l_\rho+2\eta_{\alpha\mu}\eta_{\nu\sigma}q_\rho l_\beta-4\eta_{\alpha\mu}\eta_{\nu\sigma}l_\beta l_\rho+4\eta_{\alpha\mu}\eta_{\nu\rho}p_\beta p_\sigma+$$
$$2\eta_{\alpha\mu}\eta_{\nu\rho}p_\beta q_\sigma-4\eta_{\alpha\mu}\eta_{\nu\rho}p_\beta l_\sigma+4\eta_{\alpha\mu}\eta_{\nu\rho}p_\sigma q_\beta-2\eta_{\alpha\mu}\eta_{\nu\rho}p_\sigma l_\beta-4\eta_{\alpha\mu}\eta_{\nu\rho}q_\beta q_\sigma+2\eta_{\alpha\mu}\eta_{\nu\rho}q_\beta l_\sigma+$$
$$2\eta_{\alpha\mu}\eta_{\nu\rho}q_\sigma l_\beta-4\eta_{\alpha\mu}\eta_{\nu\rho}l_\beta l_\sigma-4\eta_{\alpha\mu}\eta_{\sigma\rho}p_\beta p_\nu-2\eta_{\alpha\mu}\eta_{\sigma\rho}p_\beta q_\nu+2\eta_{\alpha\mu}\eta_{\sigma\rho}p_\beta l_\nu-4\eta_{\alpha\mu}\eta_{\sigma\rho}p_\nu q_\beta+$$
$$2\eta_{\alpha\mu}\eta_{\sigma\rho}p_\nu l_\beta+4\eta_{\alpha\mu}\eta_{\sigma\rho}q_\beta q_\nu-2\eta_{\alpha\mu}\eta_{\sigma\rho}q_\beta l_\nu-2\eta_{\alpha\mu}\eta_{\sigma\rho}q_\nu l_\beta+\eta_{\alpha\sigma}\eta_{\beta\rho}\eta_{\mu\nu}p\cdot p+\eta_{\alpha\sigma}\eta_{\beta\rho}\eta_{\mu\nu}p\cdot q-$$
$$\eta_{\alpha\sigma}\eta_{\beta\rho}\eta_{\mu\nu}p\cdot l+\eta_{\alpha\sigma}\eta_{\beta\rho}\eta_{\mu\nu}q\cdot q-3/2\eta_{\alpha\sigma}\eta_{\beta\rho}\eta_{\mu\nu}q\cdot l+\eta_{\alpha\sigma}\eta_{\beta\rho}\eta_{\mu\nu}l\cdot l-\eta_{\alpha\sigma}\eta_{\beta\rho}p_\mu p_\nu-2\eta_{\alpha\sigma}\eta_{\beta\rho}p_\mu q_\nu+$$
$$2\eta_{\alpha\sigma}\eta_{\beta\rho}p_\mu l_\nu+2\eta_{\alpha\sigma}\eta_{\beta\rho}q_\mu q_\nu-3/2\eta_{\alpha\sigma}\eta_{\beta\rho}q_\mu l_\nu-3/2\eta_{\alpha\sigma}\eta_{\beta\rho}q_\nu l_\mu+2\eta_{\alpha\sigma}\eta_{\beta\rho}l_\mu l_\nu+\eta_{\alpha\rho}\eta_{\beta\sigma}\eta_{\mu\nu}p\cdot$$
$$p+\eta_{\alpha\rho}\eta_{\beta\sigma}\eta_{\mu\nu}p\cdot q-\eta_{\alpha\rho}\eta_{\beta\sigma}\eta_{\mu\nu}p\cdot l+\eta_{\alpha\rho}\eta_{\beta\sigma}\eta_{\mu\nu}q\cdot q-3/2\eta_{\alpha\rho}\eta_{\beta\sigma}\eta_{\mu\nu}q\cdot l+\eta_{\alpha\rho}\eta_{\beta\sigma}\eta_{\mu\nu}l\cdot l-$$
$$\eta_{\alpha\rho}\eta_{\beta\sigma}p_\mu p_\nu-2\eta_{\alpha\rho}\eta_{\beta\sigma}p_\mu q_\nu+2\eta_{\alpha\rho}\eta_{\beta\sigma}p_\mu l_\nu+2\eta_{\alpha\rho}\eta_{\beta\sigma}q_\mu q_\nu-3/2\eta_{\alpha\rho}\eta_{\beta\sigma}q_\mu l_\nu-3/2\eta_{\alpha\rho}\eta_{\beta\sigma}q_\nu l_\mu+$$
$$2\eta_{\alpha\rho}\eta_{\beta\sigma}l_\mu l_\nu-4\eta_{\beta\mu}\eta_{\sigma\rho}p_\alpha p_\nu-2\eta_{\beta\mu}\eta_{\sigma\rho}p_\alpha q_\nu+2\eta_{\beta\mu}\eta_{\sigma\rho}p_\alpha l_\nu-6\eta_{\beta\mu}\eta_{\sigma\rho}p_\nu q_\alpha+4\eta_{\beta\mu}\eta_{\sigma\rho}p_\nu l_\alpha+$$
$$4\eta_{\beta\mu}\eta_{\sigma\rho}q_\alpha q_\nu-2\eta_{\beta\mu}\eta_{\sigma\rho}q_\alpha l_\nu-2\eta_{\beta\mu}\eta_{\sigma\rho}q_\nu l_\alpha+4\eta_{\beta\mu}\eta_{\sigma\rho}l_\alpha l_\nu-4\eta_{\beta\sigma}\eta_{\mu\nu}p_\alpha p_\rho-2\eta_{\beta\sigma}\eta_{\mu\nu}p_\alpha q_\rho+$$
$$3\eta_{\beta\sigma}\eta_{\mu\nu}p_\alpha l_\rho-3\eta_{\beta\sigma}\eta_{\mu\nu}p_\rho q_\alpha+2\eta_{\beta\sigma}\eta_{\mu\nu}p_\rho l_\alpha-2\eta_{\beta\sigma}\eta_{\mu\nu}q_\alpha q_\rho+3\eta_{\beta\sigma}\eta_{\mu\nu}q_\alpha l_\rho+\eta_{\beta\sigma}\eta_{\mu\nu}q_\rho l_\alpha-$$

What happens when the number of external particles increases?

Yang-Mills: The number of Feynman diagrams increases sharply!

Gravity: A lot of terms even in four-graviton tree amplitude!

Recursion relations: BCFW on-shell recursion (Britto,Cachazo,Feng, 04; Britto,Cachazo,Feng,Witten,05),
Berends-Giele off-shell recursion (87)

Relations between color-ordered YM amplitudes: KK (Kleiss and H. Kuijf, 89), BCJ (Bern, Carrasco,Johansson, 08) relations

Relations between GR and YM: KLT relation (Kawai-Lewellen-Tye (1986))

Amplitude relations for color-ordered YM

Proofs: Del Duca, Dixon, Maltoni, 99; Bjerrum-Bohr, Damgaard, Vanhove, 09; Stieberger, 09; Feng, Huang, Jia, 10; Chen, Du, Feng, 11

KK relation: $$A(\beta_1,...,\beta_i,1,\alpha_1,...,\alpha_j,n) = \sum_{\shuffle} (-1)^i A(1,\boldsymbol{\beta}^T \shuffle \boldsymbol{\alpha}, n)$$

BCJ relation: $$\sum_{\shuffle} \sum_{l=1}^{i} \left[k_{\beta_l} \cdot X_{\beta_l}(\shuffle)\right] A(1,\boldsymbol{\alpha} \shuffle \boldsymbol{\beta}, n) = 0$$

$$\boldsymbol{\beta} \equiv \{\beta_1 ... \beta_i\}, \quad \boldsymbol{\alpha} \equiv \{\alpha_1 ... \alpha_j\}, \quad X^{\mu}_{\beta_i} = \sum_{a \prec \beta_i} k^{\mu}_a,$$

$$\{1,2\} \shuffle \{3,4\} = \big\{\{1,2,3,4\},\{1,3,2,4\}, \{1,3,4,2\}\{3,1,2,4\},\{3,1,4,2\},\{3,4,1,2\}\big\}$$

KK+BCJ $\Rightarrow$ Minimal basis expansion:

$$A(1,\boldsymbol{\beta},2,\alpha,n) = \sum_{\boldsymbol{\sigma}} C(\boldsymbol{\sigma})\, A(1,2,\boldsymbol{\sigma} \in \text{perms}\,(\boldsymbol{\beta}) \shuffle \boldsymbol{\alpha}, n)$$

KLT double-copy relation: GR as $(\mathrm{YM})^2$

KLT relation:

Closed string tree amplitudes	$\sim$	$(\text{Open string tree amplitudes})^2$
$\Downarrow$		$\Downarrow$
GR tree amplitudes	$\sim$	$(\text{YM tree amplitudes})^2$

KLT in field theory:

(Bern, De Freitas, Wong, 99; Bjerrum-Bohr, Damgaard, Feng, Sondergaard 10; Du, Feng, Fu, 11)

$$\boxed{M_n = \sum_{\sigma,\rho} A_n(\rho) S[\rho|\sigma] \widetilde{A}_n(\sigma)}$$

M_n :	GR	color-dressed YM
A_n :	YM	ϕ^3 scalar
$\widetilde{A}_n$:	YM	YM

Further developments of double-copy: BCJ and CHY formulas

The BCJ form of amplitudes

Bern, Carrasco, Johansson, 08, 10

BCJ form of YM amplitudes:

$$M^{\mathrm{YM}} = \sum_{\mathcal{G}} \frac{c_{\mathcal{G}}\, n_{\mathcal{G}}}{\prod_i D^i_{\mathcal{G}}}$$

$\mathcal{G}$: diagrams with cubic vertices

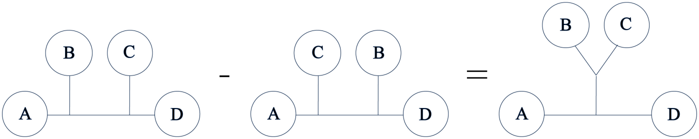
$D^i_{\mathcal{G}}$: propagators $c_{\mathcal{G}}$: color factors

$n_{\mathcal{G}}$: BCJ numerators satisfying

GR as (BCJ) double copy of YM

$$M^{\mathrm{GR}} = \sum_{\mathcal{G}} \frac{n_{\mathcal{G}}\, \widetilde{n}_{\mathcal{G}}}{\prod_i D^i_{\mathcal{G}}}$$

Jacobi identity



The CHY form of amplitudes

Cachazo, He, Yuan, 13, 14

$$M = \int d\mu \, \mathcal{I}_L \, \mathcal{I}_R$$

Theories	BS	YM	YMS	EYM	GR
Amplitudes	$A^{\text{tree}}_{\text{BS}}(\boldsymbol{\sigma}\|\boldsymbol{\rho})$	$A^{\text{tree}}_{\text{YM}}(\boldsymbol{\sigma})$	$A^{\text{tree}}_{\text{YMS}}(\boldsymbol{\sigma}_1;...;\boldsymbol{\sigma}_m\|\|\boldsymbol{G}\|\boldsymbol{\rho}_{\boldsymbol{S}\cup\boldsymbol{G}})$	$A^{\text{tree}}_{\text{EYM}}(\boldsymbol{\sigma}_1;...;\boldsymbol{\sigma}_m\|\|\boldsymbol{H})$	$M^{\text{tree}}_{\text{GR}}(1,...,n)$
$\mathcal{I}_{\text{L}}$	$\text{PT}(\boldsymbol{\sigma})$	$\text{PT}(\boldsymbol{\sigma})$	$\text{PT}(\boldsymbol{\sigma}_1)...\text{PT}(\boldsymbol{\sigma}_m)\,\mathcal{P}$	$\text{PT}(\boldsymbol{\sigma}_1)...\text{PT}(\boldsymbol{\sigma}_m)\,\mathcal{P}$	$\text{Pf}\,'[\Psi]$
$\mathcal{I}_{\text{R}}$	$\text{PT}(\boldsymbol{\rho})$	$\text{Pf}\,'[\Psi]$	$\text{PT}\,(\boldsymbol{\rho}_{\boldsymbol{S}\cup\boldsymbol{G}})$	$\text{Pf}\,'[\Psi]$	$\text{Pf}\,'[\Psi]$

Scattering equations $$\sum_{\substack{j=1\\ j\neq i}} \frac{k_i\cdot k_j}{z_{ij}} = 0$$

$$\text{PT}(1\cdots n) \equiv \frac{1}{z_{12}z_{23}\cdots z_{n1}} \qquad \text{Pf}\,'[\Psi] \equiv \frac{perm(ij)}{z_{ij}}\text{Pf}[\Psi^{ij}_{ij}]$$

The CHY form of amplitudes

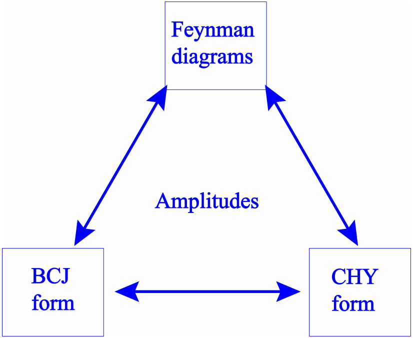
Ψ is a $2n \times 2n$ skew-symmetric matrix:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

The $n \times n$ submatrices A, B and C are

	A_{ab}	B_{ab}	C_{ab}
$a \neq b$	$\frac{s_{ab}}{z_{ab}}$	$\frac{2\epsilon_a \cdot \epsilon_b}{z_{ab}}$	$\frac{2\epsilon_a \cdot k_b}{z_{ab}}$
$a = b$	0	0	$-\sum_{c \neq a} \frac{2\epsilon_a \cdot k_c}{z_{ac}}$

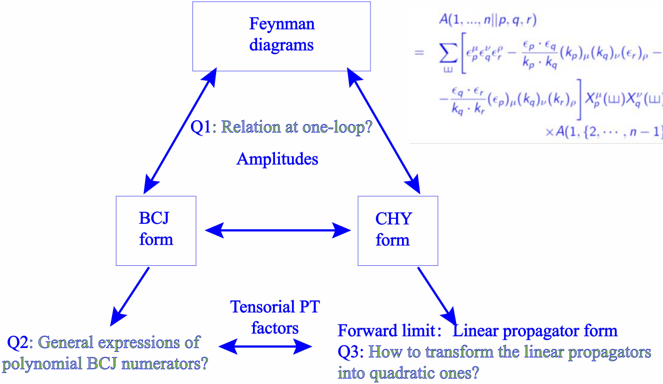
Relations between FD, BCJ and CHY at tree level



At tree level:

- ★ From CHY to polynomial BCJ (e.g. Du,Feng, Fu, Huang, 17; Du, Feng,Teng 17; Du, Teng, 17)
- ★ From FD to polynomial BCJ (e.g. Du, Wu, 21, 22)

Extensions to one-loop level



$$
\begin{aligned}
&A(1,\ldots,n||p,q,r)\\
=\;&\sum_{\shuffle}\Bigg[\epsilon_p^{\mu}\epsilon_q^{\nu}\epsilon_r^{\rho}-\frac{\epsilon_p\cdot\epsilon_q}{k_p\cdot k_q}(k_p)_\mu(k_q)_\nu(\epsilon_r)_\rho-\frac{\epsilon_p\cdot\epsilon_r}{k_p\cdot k_r}(k_p)_\mu(\epsilon_q)_\nu(k_r)_\rho\\
&-\frac{\epsilon_q\cdot\epsilon_r}{k_q\cdot k_r}(\epsilon_p)_\mu(k_q)_\nu(k_r)_\rho\Bigg]X_p^{\mu}(\shuffle)X_q^{\nu}(\shuffle)X_r^{\rho}(\shuffle)\\
&\qquad\times A(1,\{2,\cdots,n-1\}\shuffle\{p\}\shuffle\{q\}\shuffle\{r\},n)
\end{aligned}
$$

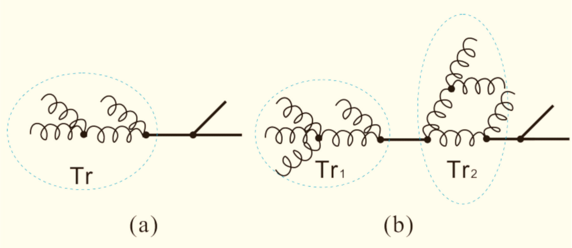
At one-loop level there are three related questions Q1, Q2, and Q3:

★ Once tensorial PT factors are constructed, Q2 and Q3 have been answered, but it is still lack of general construction of the tensorial PT factors. (Feng,He,Zhang,Zhang,22; Dong, Zhang, Zhang, 23)

★ In this talk, we try to answer Q1 and Q3, partly.

II. Helpful tools 1: on-shell expansion formulas, graph-based relations at tree level

What is EYM?



EYM=YM+GR+Dilaton+B Fields

Typical Feynman diagrams in EYM

EYM recursive expansion for single-trace amplitudes

(Stieberger and T. R. Taylor, 16; Nandan, Plefka, Schlotterer, Wen, 16; de la Cruz, Kniss,Weinzierl, 16; Schlotterer, 16; Du, Feng, Fu, Huang, 17; Chiodaroli,Gunaydin,Johansson, 17; Teng, Feng, 17)

$$A(1,2,\ldots,r\|\mathrm{H}) = \sum_{\mathrm{H}\setminus\{h_a\}\to\mathrm{h}|\boldsymbol{h}} \epsilon_{h_a}\cdot F_{\rho_1}\cdots F_{\rho_i}\cdot Y_{\rho_i} \times A(1,\{2,\ldots,r-1\}\shuffle\{\rho_i,\ldots,\rho_1,h_a\},r\|\boldsymbol{h})$$

- $\mathrm{H}\setminus\{h_a\}\to\mathrm{h}|\boldsymbol{h}$: splittings of elements in $\mathrm{H}\setminus\{h_a\}$; Permutations $\boldsymbol{\rho}$ are summed over
- h_a: fiducial graviton; $Y_h = \sum_{j\in\{1,\ldots,r-1\}\,\mathrm{s.t.}\,j\prec h} k_j$; $F_i^{\mu\nu} = k_i^\mu\epsilon_i^\nu - k_i^\nu\epsilon_i^\mu$
- $\boldsymbol{A}\shuffle\boldsymbol{B}$: permutations with keeping the relative orders in each set

EYM recursion expansion: multi-trace case

EYM recursive expansion of multi-trace amplitudes

(Du, Feng, Teng, 17)

$$\text{Single-trace EYM} \xrightarrow{\text{Gravitons}\to\text{Gluon traces}} \text{Multi-trace EYM}$$

Type-I: Fiducial graviton $\rightarrow$ Graviton

Type-II: Fiducial graviton $\rightarrow$ Trace

From EYM to GR

Graviton amplitude can be expanded in terms of single-trace EYM amplitudes (Fu, Du, Huang, Feng, 17)

$$M_n^{\mathrm{GR}} = \sum_{\mathrm{H}\setminus\{h_1,h_2\}\to \mathrm{h}|\boldsymbol{h}} (-1)^{|\mathrm{h}|} \epsilon_{h_1}\cdot F_{\rho_1}\cdots F_{\rho_r}\cdot \epsilon_n A^{\mathrm{EYM}}(h_1,\rho,h_n\|\boldsymbol{h})$$

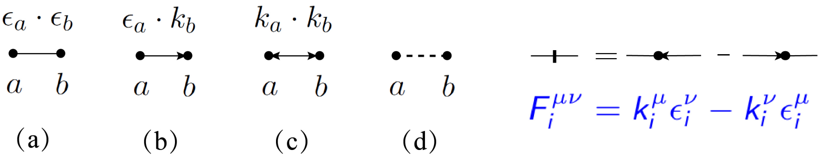
Here

- Sum over splittings of graviton set $\mathrm{H}\setminus\{h_1,h_2\}$ and sum over permutations ρ of elements in h
- $(F_i)^{\mu\nu}\equiv k_i^{\mu}\epsilon_i^{\nu}-k_i^{\nu}\epsilon_i^{\mu}$, gauge invariance conditions for gravitons (except for h_1 and h_n) are encoded in F_i
- In each term we turn gravitons in $\{h_1,h_n\}\cup \mathrm{h}$ to gluons

Expanding amplitudes in terms of graphs

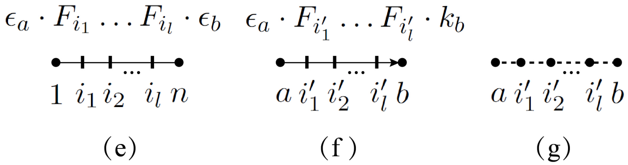
A graph: nodes (external particles) and lines (Lorentz contractions)

Line styles | **Strength tensors**

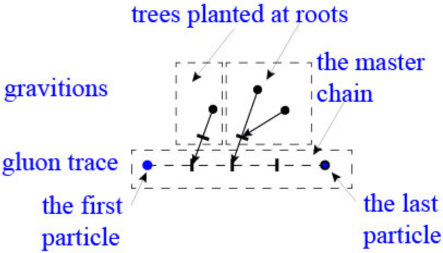


$$F_i^{\mu\nu} = k_i^{\mu} \epsilon_i^{\nu} - k_i^{\nu} \epsilon_i^{\mu}$$

Chains



Expanding amplitudes in terms of graphs



- Reference order:

$R = \{h_{\rho(1)}, \ldots, h_{\rho(s)}\}$

- Root set:

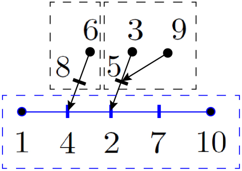
$\mathcal{R} = \{1, \ldots, r-1\}$

- Permutations: Shuffling the branches
- $C_{1,\boldsymbol{\sigma},n}$: Summing all graphs contributing $\boldsymbol{\sigma}$

$$A^{\text{EYM}}_{\text{SingleTrace}} = \sum_{\boldsymbol{\sigma}} C_{1,\boldsymbol{\sigma},n} A^{\text{YM}}(1, \boldsymbol{\sigma}, n)$$

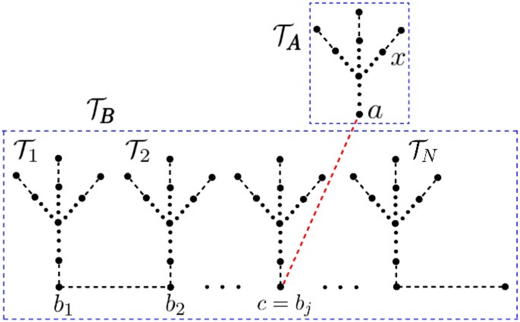
$$M^{\text{GR}} = \sum_{\boldsymbol{\sigma} \in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\text{YM}}(1, \boldsymbol{\sigma}, n)$$

$$A^{\text{YM}} = \sum_{\boldsymbol{\sigma} \in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\text{BS}}(1, \ldots, n | 1, \boldsymbol{\sigma}, n)$$



Graph-based BCJ relations

Du, Hou, 18, 19



$$\sum_{a\in\mathcal{T}_A}(-)^{|ax|}\sum_{c\in\mathcal{T}_{\boldsymbol{B}}}\sum_{\boldsymbol{\alpha}\in\mathcal{T}_{\boldsymbol{A}}|_a}\sum_{\boldsymbol{\beta}\in\mathcal{T}_{\boldsymbol{B}}|_b}\left[\sum_{\boldsymbol{\gamma}\in\boldsymbol{\alpha}\sqcup\boldsymbol{\beta}|_{c\prec a}} s_{ac}\,A(\boldsymbol{\gamma})\right]=0$$

From the perspective of CHY

Expansion relation between half integrands:

(Fu, Du, Huang, Feng, 17; Teng, Feng, 17; Du, Teng, 17; Du, Teng, Feng, 17)

$$\begin{aligned} \mathrm{PT}(\boldsymbol{\sigma}_1)\ldots\mathrm{PT}(\boldsymbol{\sigma}_m)\,\mathcal{P} &= \sum_{\boldsymbol{\sigma}} C_{1,\boldsymbol{\sigma},n}\,\mathrm{PT}(1,\boldsymbol{\sigma},n) \\ \mathrm{Pf}'[\Psi] &= \sum_{\boldsymbol{\sigma}\in S_{n-2}} n_{1,\boldsymbol{\sigma},n}\mathrm{PT}(1,\boldsymbol{\sigma},n) \end{aligned}$$

Graph-based relations for PT factors:

$$\sum_{a\in\mathcal{T}_A}(-)^{|ax|}\sum_{c\in\mathcal{T}_{\boldsymbol{B}}}\sum_{\boldsymbol{\alpha}\in\mathcal{T}_{\boldsymbol{A}}|_a}\sum_{\boldsymbol{\beta}\in\mathcal{T}_{\boldsymbol{B}}|_b}\left[\sum_{\boldsymbol{\gamma}\in\boldsymbol{\alpha}\shuffle\boldsymbol{\beta}|_{c\prec a}} s_{ac}\,\mathrm{PT}(\boldsymbol{\gamma})\right]=0$$

Summing over permutations ⇔ summing over graphs

Du, Hou, 18, 19

$$M = \underbrace{\sum_{\boldsymbol{\sigma}\in S_{n-2}}}_{\text{summing over permutations}} n_{1,\boldsymbol{\sigma},n} A(1,\boldsymbol{\sigma},n)$$

$$\Leftrightarrow M = \underbrace{\sum_{\mathcal{F}}}_{\text{summing over graphs}} C^{\mathcal{F}} \left[\underbrace{\sum_{\boldsymbol{\sigma}^{\mathcal{F}}}}_{\substack{\text{permutations}\\ \text{corresponding to a graph}}} A(1,\boldsymbol{\sigma}^{\mathcal{F}},n) \right]$$

III. Helpful tools 2: Off-shell expansions and relations for Berends-Giele currents

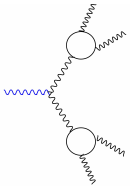
What is the Berends-Giele current?

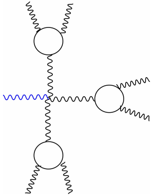
- A recursive approach to packaging Feynman diagrams
- Solution to classical equation of motion
  (approach along this line： Lee, Mafra, Schlotterer 15;E. Bridges, Mafra 19)
- On-shell limit $\Rightarrow$ amplitude

Berends,Giele, 87

B-G currents for YM

$$
\begin{aligned}
& J^{\rho}(1,\ldots,n-1) \\
= \quad & \frac{1}{s_{1\ldots n-1}}\Bigg[\sum_{1\leq i<n-1} V_3^{\mu\nu\rho} J_\mu(1,\ldots,i) J_\nu(i+1,\ldots,n-1) \\
& + \sum_{1\leq i<j<n-1} V_4^{\mu\nu\tau\rho} J_\mu(1,\ldots,i) J_\nu(i+1,\ldots,j) J_\tau(j+1,\ldots,n-1)\Bigg]
\end{aligned}
$$





Starting point: $J^\mu(a) = \epsilon_a$

Berends-Giele recursion for BS

B-G currents for BS Mafra, 16

$$\begin{aligned}
&\phi\big(1,...,n-1\big|\sigma_1,...,\sigma_{n-1}\big) \\
= \quad & \frac{1}{s_{1\dots n-1}}\sum_{i=1}^{n-2}\Big[\phi\big(1,...,i\big|\sigma_1,...,\sigma_i\big)\phi\big(i+1,...,n-1\big|\sigma_{i+1},...,\sigma_{n-1}\big) \\
&\qquad -\phi\big(1,...,i\big|\sigma_{n-i},...,\sigma_{n-1}\big)\phi\big(i+1,...,n-1\big|\sigma_1,...,\sigma_{n-i}\big)\Big]
\end{aligned}$$

Starting point $\phi(a|a) = 1$, $\phi(a|b)$ $(a \neq b)$

Expansion formula of YM

Wu, Du, 21

BG current in Feynman gauge

$$\underbrace{J^{\rho}(1,2,...,n-1)}_{\text{Feynman gauge}} = \underbrace{\widetilde{J}^{\rho}(1,2,...,n-1)}_{\text{BCJ gauge}} + \underbrace{K^{\rho}(1,2,...,n-1) + L^{\rho}(1,2,...,n-1)}_{\text{Vanish in the on-shell limit}}$$

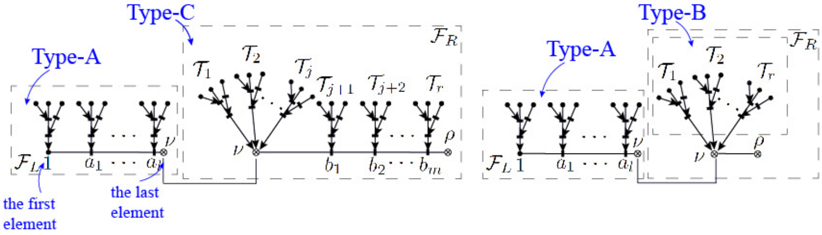
Effective current

$$\widetilde{J}^{\rho}(1,2,\ldots,n-1) = \sum_{\boldsymbol{\sigma}\in P(2,n-1)} N_A^{\rho}(1,\boldsymbol{\sigma})\phi(1,2,\ldots,n-1|1,\boldsymbol{\sigma})$$

Generalized strength tensor: $\widetilde{F}^{\nu\rho}(A) \equiv 2k_A^{\nu}\widetilde{J}^{\rho}(A) - 2k_A^{\rho}\widetilde{J}^{\nu}(A)$

$$\widetilde{F}^{\nu\rho}_{(1,n-1)} = \sum_{\boldsymbol{\sigma}\in P(1,n-1)} N_C^{\nu\rho}(\boldsymbol{\sigma})\phi(1,...,n-1|\boldsymbol{\sigma}) + \sum_{1\leq i<n-1} 2\left[\widetilde{J}^{\nu}_{(1,i)}\widetilde{J}^{\rho}_{(i+1,n-1)} - \widetilde{J}^{\rho}_{(i+1,n-1)}\widetilde{J}^{\nu}_{(1,i)}\right]$$

Three types of off-shell numerators



$$\boldsymbol{\sigma} = \{\sigma_2, ..., \sigma_{n-1}\} \in P(2, n-1), \boldsymbol{\sigma}_L = \{\sigma_2, ..., \sigma_{i-1}\}, \boldsymbol{\sigma}_R = \{\sigma_i, ..., \sigma_{n-1}\}$$

$$N_A^{\rho}(1,\boldsymbol{\sigma}) \quad = \quad \left[N_A(1,\boldsymbol{\sigma}_L)\cdot N_C(\boldsymbol{\sigma}_R) - N_A(1,\boldsymbol{\sigma}_L)N_B(\boldsymbol{\sigma}_R)\cdot 2k_{1,i-1}\right]^{\rho}$$

Gauge transformation terms

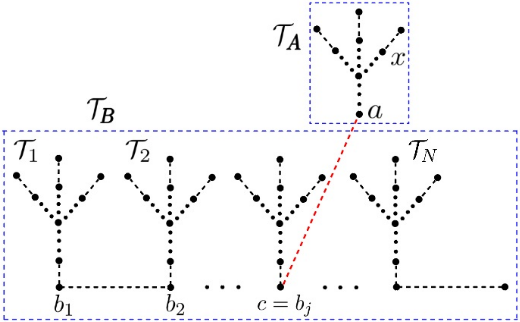
$$K^{\rho}(1,2,\ldots,n-1) = \frac{1}{s_{12\ldots n-1}}\, k^{\rho}_{1,n-1} \sum_{i=1}^{n-2} \widetilde{J}(1,\ldots,i)\cdot\widetilde{J}(i+1,\ldots,n-1)$$

$$L^{\rho}(1,2,\ldots,n-1) = \sum_{\{a_i,b_i\}\subset\{1,\ldots,n-1\}} (-1)^{l+1} J^{\rho}\Big(S_{1,a_1-1}, K_{(a_1,b_1)}, S_{b_1+1,a_2-1}, K_{(a_2,b_2)}, \ldots, K_{(a_l,b_l)}, S_{b_l+1,n-1}\Big)$$

On-shell limit: $s_{12\ldots n-1}\epsilon\cdot[K(1,2,\ldots,n-1)+L(1,2,\ldots,n-1)]\,|_{k_n^2\to 0} = 0$

Off-shell graph-based BCJ

Du, Wu, 22



$$\sum_{a\in\mathcal{T}_A}(-)^{|ax|}\sum_{c\in\mathcal{T}_{\boldsymbol{B}}}\sum_{\boldsymbol{\alpha}\in\mathcal{T}_{\boldsymbol{A}}|_a}\sum_{\boldsymbol{\beta}\in\mathcal{T}_{\boldsymbol{B}}|_b}\Big[\sum_{\boldsymbol{\gamma}\in\boldsymbol{\alpha}\sqcup\boldsymbol{\beta}|_{c\prec a}} s_{ac}\,\phi(\boldsymbol{\sigma}\,|\,\boldsymbol{\gamma})\Big]$$

$$=\sum_{\boldsymbol{\alpha}\in\mathcal{T}_{\boldsymbol{A}}|_x}\sum_{\boldsymbol{\beta}\in\mathcal{T}_{\boldsymbol{B}}|_b}\Big[\phi(\boldsymbol{\sigma}_{1,i}|\boldsymbol{\beta})\phi(\boldsymbol{\sigma}_{i+1,l}|\boldsymbol{\alpha})-\phi(\boldsymbol{\sigma}_{1,l-i}|\boldsymbol{\alpha})\phi(\boldsymbol{\sigma}_{l-i+1,l}|\boldsymbol{\beta})\Big]$$

IV. One-loop CHY formula and the problem of linear propagators

One-loop CHY formula

One loop CHY Geyer, L. Mason，15； He and Yuan, 15； Cachazo, He, Yuan, 16; EYM and YMS see Porkert,Schlotterer, 22）：

$$M_{n,\mathrm{B}\otimes\mathrm{C}}^{\text{1-loop}} = \int \frac{\mathrm{d}^D l}{l^2} \lim_{k_\pm \to \pm l} \int \mathrm{d}\mu_{n+2}^{\text{tree}} I_{\mathrm{B}}^{\text{1-loop}} I_{\mathrm{C}}^{\text{1-loop}}$$

Scattering equations at one-loop level:

$$\frac{l \cdot k_i}{z_i} + \sum_{\substack{j=1 \\ j \neq i}} \frac{k_i \cdot k_j}{z_{ij}} = 0$$

One-loop CHY formula

Theories	BS	YM	YMS	EYM	GR
Amplitudes	$A_{\text{BS}}^{\text{1-loop}}(\boldsymbol{\sigma}\vert\boldsymbol{\rho})$	$A_{\text{YM}}^{\text{1-loop}}(\boldsymbol{\sigma})$	$A_{\text{YMS}}^{\text{1-loop}}(\boldsymbol{\sigma}_1;...;\boldsymbol{\sigma}_m\Vert\boldsymbol{G}\vert\boldsymbol{\rho}_{\boldsymbol{S}\cup\boldsymbol{G}})$	$A_{\text{EYM}}^{\text{1-loop}}(\boldsymbol{\sigma}_1;...;\boldsymbol{\sigma}_m\Vert\boldsymbol{H})$	$M_{\text{GR}}^{\text{1-loop}}(1,...,n)$
$I_{\text{L}}^{\text{1-loop}}$	$\text{PT}(+,\boldsymbol{\sigma},-)$ $+\text{cyc}(\boldsymbol{\sigma})$	$\text{PT}(+,\boldsymbol{\sigma},-)$ $+\text{cyc}(\boldsymbol{\sigma})$	$\text{PT}(+,\boldsymbol{\sigma}_1,-)...$ $\text{PT}(\boldsymbol{\sigma}_m)\,\mathcal{P}+\text{cyc}(\boldsymbol{\sigma}_1)$	$\text{PT}(+,\boldsymbol{\sigma}_1,-)...$ $\text{PT}(\boldsymbol{\sigma}_m)\,\mathcal{P}+\text{cyc}(\boldsymbol{\sigma}_1)$	$\text{Pf}'[\Psi_{n+2}]$
$I_{\text{R}}^{\text{1-loop}}$	$\text{PT}(+,\boldsymbol{\rho},-)$ $+\text{cyc}(\boldsymbol{\rho})$	$\text{Pf}'[\Psi_{n+2}]$	$\text{PT}\,(+,\boldsymbol{\rho}_{\boldsymbol{S}\cup\boldsymbol{G}},-)$ $+\text{cyc}(\boldsymbol{\rho}_{\boldsymbol{S}\cup\boldsymbol{G}})$	$\text{Pf}'[\Psi_{n+2}]$	$\text{Pf}'[\Psi_{n+2}]$

One-loop level CHY integrands for BS, YM, YMS, EYM and GR In the single-trace case $m=1$, there is only one PT factor with $n+2$ points and the $\mathcal{P}$ becomes $\text{Pf}\,[\Psi]_{\text{H;H}}$.

One-loop CHY formula

For BS, the CHY integral gives:

$$
\begin{aligned}
m^{\rm BS} &= \lim_{k_\pm \to \pm l} \int \mathrm{d}\mu^{\rm tree}_{n+2} I^{\text{1-loop}}_{\rm B} I^{\text{1-loop}}_{\rm C} \\
&= \sum_{\substack{(A_1A_2\ldots A_i)=\boldsymbol{\sigma}\\ (\widetilde{A}_1\widetilde{A}_2\ldots\widetilde{A}_i)=\boldsymbol{\rho}\\ A_j=\widetilde{A}_j}} \frac{1}{l^2}\frac{1}{s_{A_1,l}}\frac{1}{s_{A_1A_2,l}}\cdots\frac{1}{s_{A_1A_2\cdots A_{i-1},l}}\phi_{A_1|\widetilde{A}_1}\phi_{A_2|\widetilde{A}_2}\cdots\phi_{A_i|\widetilde{A}_i}
\end{aligned}
$$

For other theories, e.g, YMS, the CHY integral gives:

$$
\frac{1}{l^2}\int \mathrm{d}\mu^{\rm tree}_{n+2} I^{\text{1-loop}}_{\rm L} I^{\text{1-loop}}_{\rm R}
$$

$$
= \sum_{\mathcal{F}} \mathcal{C}^{\mathcal{F}}\Bigg[\sum_{\substack{(A_1A_2\ldots A_i)=\boldsymbol{\sigma}\sqcup\boldsymbol{\rho}^{\mathcal{F}}\\ (\widetilde{A}_1\widetilde{A}_2\ldots\widetilde{A}_i)=\boldsymbol{\gamma}\\ A_j=\widetilde{A}_j}} \frac{1}{l^2}\frac{1}{s_{A_1,l}}\frac{1}{s_{A_1A_2,l}}\cdots\frac{1}{s_{A_1A_2\cdots A_{i-1},l}}\phi_{A_1|\widetilde{A}_1}\phi_{A_2|\widetilde{A}_2}\cdots\phi_{A_i|\widetilde{A}_i}
$$

$$
+(\text{cyclic of }\boldsymbol{\sigma})\Bigg] + (\text{cyclic of }\boldsymbol{\gamma})
$$

One-loop CHY formula

One-loop CHY $\Rightarrow$ Linear loop propagators $\frac{1}{s_{A,l}} = \frac{1}{2k_A \cdot l + k_A^2}$

$\Downarrow$ Graphic rule Xie, Du, 24

Feynman diagrams $\Rightarrow$ Quadratic loop propagators $\frac{1}{l_A^2} = \frac{1}{(l + k_A)^2}$

The crucial relation between linear and quadratic propagators:

$$\frac{N(l)}{l^2 s_{A_1,l} s_{A_{12},l} \ldots s_{A_{12\ldots m-1},l}} + \frac{N(l+k_{A_m})}{l^2 s_{A_m,l} s_{A_{m1},l} \ldots s_{A_{m1\ldots m-2},l}} + \cdots + \frac{N(l+k_{A_{23\ldots m}})}{l^2 s_{A_2,l} s_{A_{23},l} \ldots s_{A_{23\ldots m},l}} \cong N(l)\, gon\,(A_1, A_2, ..., A_m)$$

where $gon(A_1, A_2, ..., A_m) \equiv \frac{1}{l^2 l_{A_1}^2 l_{A_{12}}^2 \ldots l_{A_{12\ldots m-1}}^2}$

V. From linear to quadratic propagators 1: YMS and YM

The main idea

Single-trace YMS

Cutting graphs $\Leftrightarrow$ Feynman diagrams

$\Downarrow$

Generalize $U(1)$-decoupling cancels disconnected subgraphs

$\Downarrow$

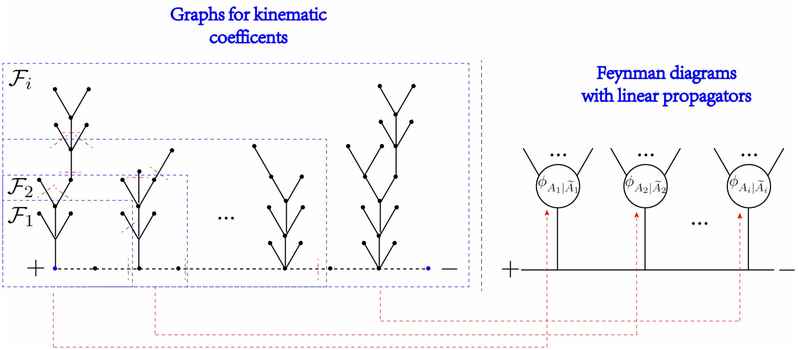
Off-shell BCJ, X pattern cancel nonlocal terms

$\Downarrow$

Remaining terms $\sim$ cyclic sum $\Rightarrow$ Quadratic propagators

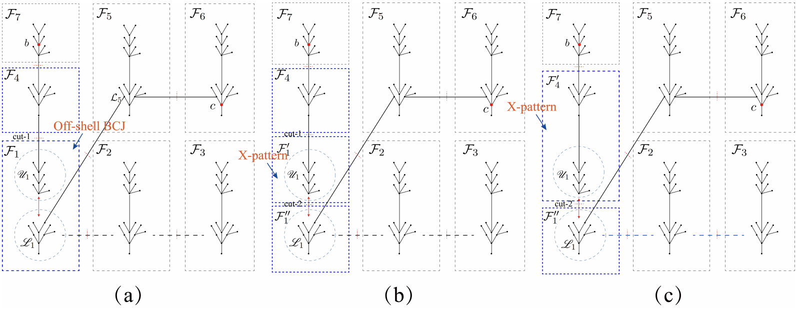
Single-trace YMS $\Rightarrow$ YM, multi-trace YMS

Cancellations of disconnected subgraphs



Disconnected subgraphs vanish: $\rightarrow \sum_{\sqcup} \phi_{A \sqcup B|C} = 0$ (Generalized $U(1)$-decoupling)

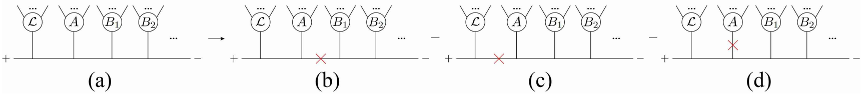
Further cancellations between BCJ pattern and X pattern



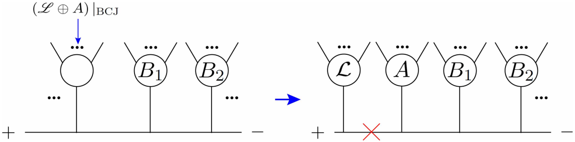
(a) → Off-shell BCJ relation → Cancels with X-pattern (b) → Cancels with other (a larger) X pattern e.g. (c)... The red nodes b, c are the highest-weight nodes on the corresponding branches

Cancellations between BCJ pattern and X pattern

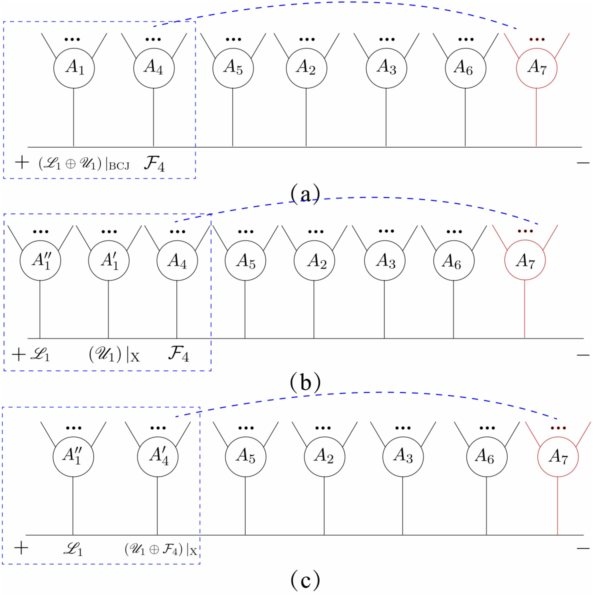
An X-pattern A reduces as:



A BCJ-pattern A reduces as:



Cancellations between BCJ pattern and X pattern

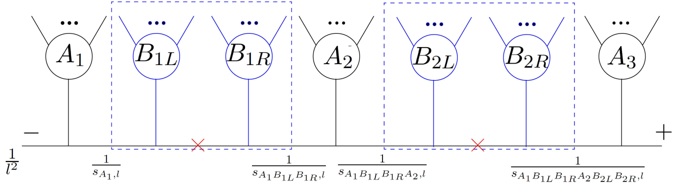


Further cancellations between BCJ pattern and X pattern

After performing all possible cancellations between BCJ and X patterns, the “non-local” terms (Terms involving contractions between A_4 and A_7 but A_4 and A_7 are separated by linear propagators) all cancel out!

Linear v.s. quadratic propagators at 1-loop

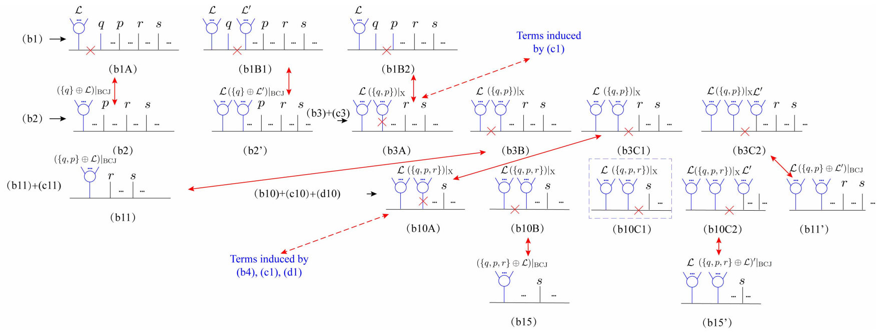
The remaining terms do not contain nonlocal structure, for example:



$$+\text{Cyc}\ [A_1, (B_{1L}B_{1R}), A_2, (B_{2L}B_{2R}), A_3]$$

$\Rightarrow$ quardratic propagators $\frac{1}{l^2}\frac{1}{l^2_{A_1}}\frac{1}{l^2_{A_1B_{1L}B_{1R}}}\frac{1}{l^2_{A_1B_{1L}B_{1R}A_2}}\frac{1}{l^2_{A_1B_{1L}B_{1R}A_2B_{2_L}B_{2_R}}}$.

A part of the cancellation map in YMS with four gluons



The final result for YMS

$$\sum_{\substack{\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_I = \boldsymbol{\sigma} \\ H_1 \cup \dots \cup H_I = H}} \frac{1}{I^2} J\left[A_1\right] \frac{1}{s_{A_1,I}} J\left[A_2\right] \frac{1}{s_{A_1 A_2, I}} J\left[A_3\right] \times \dots \times \frac{1}{s_{A_1 A_2 \dots A_{I-1}, I}} J\left[A_I\right]$$

$$+\text{cyc}[A_1, A_2, ..., A_I].$$

where we introduced the tree level currents $J\left[A_j\right]$ expressed by graphic rules

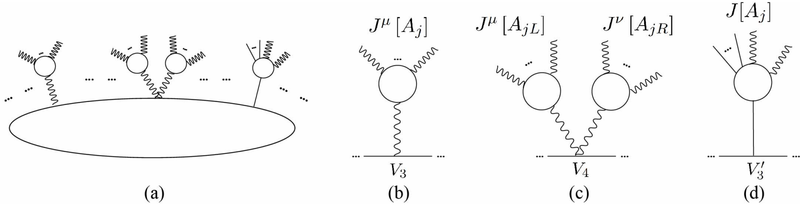
$$J\left[A_j\right] = \sum_{\mathcal{F}_j} \sum_{\sqcup} \mathcal{C}^{\mathcal{F}_j} \phi_{\boldsymbol{\sigma}_j \sqcup \boldsymbol{\rho}^{\mathcal{F}_j} | \widetilde{A}_j} \qquad \text{(if } A_j \text{ contains scalars)}$$

$$J\left[A_j\right] = \sum_{\mathcal{F}_j} \sum_{\sqcup} \mathcal{C}^{\mathcal{F}_j} \cdot \left(l + k_{A_1} + \dots + k_{A_{j-1}}\right) \phi_{\boldsymbol{\rho}^{\mathcal{F}_j} | \widetilde{A}_j} \quad \leftarrow \text{BG currents } J^{\mu} = \widetilde{J} + K + L$$

$$+ \sum_{\mathcal{F}_{jL}, \mathcal{F}_{jR}} \mathcal{C}^{\mathcal{F}_{jL}} \cdot \mathcal{C}^{\mathcal{F}_{jR}} \left[\sum_{\boldsymbol{\rho}^{\mathcal{F}_{jL}}, \boldsymbol{\rho}^{\mathcal{F}_{jR}}} \phi_{\boldsymbol{\rho}^{\mathcal{F}_{jL}} | \widetilde{A}_{jL}} \phi_{\boldsymbol{\rho}^{\mathcal{F}_{jR}} | \widetilde{A}_{jR}} \right]$$

(if A_j does not contain scalar)

The relation to Feynman rules in YMS



1 gluon-2 scalar vertices, 2-gluon, 2-scalar vertices, three-scalar vertices

From YMS to YM

YM is obtained from the forward limits of tree level YM (and its relation with YMS):

$$
\begin{aligned}
I_{\text{YM}}^{\text{1-loop}} &= (D-2) \sum_{A_1\cup\ldots\cup A_l=\{1,\ldots,n\}\setminus\{i_1,\ldots,i_l\}} gon\,(A_1, A_2, \ldots, A_l) \prod_{j=1}^{l} J\,[A_j] \\
&+ \sum_{l=2}^{n} (-1)^l \sum_{\{i_1,i_2,\ldots,i_l\}\in \mathrm{S}_l\setminus \mathrm{Z}_l} \mathrm{Tr}\,[F_{i_1}\cdot F_{i_2}\cdot \ldots \cdot F_{i_l}] \\
&\times \sum_{\substack{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_l = i_1 i_2\ldots i_l \\ H_1\cup\ldots\cup H_l=\{1,\ldots,n\}\setminus\{i_1,\ldots,i_l\}}} gon\,(A_1, A_2, \ldots, A_l) \prod_{j=1}^{l} J\,[A_j]
\end{aligned}
$$

VI. From linear to quadratic propagators 2: A taste of double-YMS, EYM and GR

(work in progress)

CHY: GR $\sim$ EYM $\sim$ double-YMS

$$\mathcal{I} \equiv \int d\mu \; [I_L + \mathrm{cyc}(\boldsymbol{\sigma})] \; [I_R + \mathrm{cyc}(\boldsymbol{\rho})]$$

$$I_L \equiv \mathrm{PT}(+, \boldsymbol{\sigma}, -)\mathrm{Pf}\,[\Psi_H] \qquad I_R \equiv \mathrm{PT}(+, \boldsymbol{\rho}, -)\mathrm{Pf}\,[\Psi_{\widetilde{H}}]$$

Define X,Y,Z,W as scalar, left-gluon, right-gluon, graviton sets

$$X \equiv \{1, 2, \ldots, i\}, \qquad Y \equiv \{a_1, a_2, \ldots, a_j\},$$
$$Z \equiv \{b_1, b_2, \ldots, b_k\}, \qquad W \equiv \{c_1, c_2, \ldots, c_l\}$$

$\boldsymbol{\sigma}$: perms $(X \cup Z)$, H : $Y \cup W$, $\boldsymbol{\rho}$: perms $(X \cup Y)$, $\widetilde{H}$: $Z \cup W$

Quadratic propagators in double-YMS

The main idea:

Cancellations in the left part

$\Downarrow$

Cancellations in the right part

$\Downarrow$

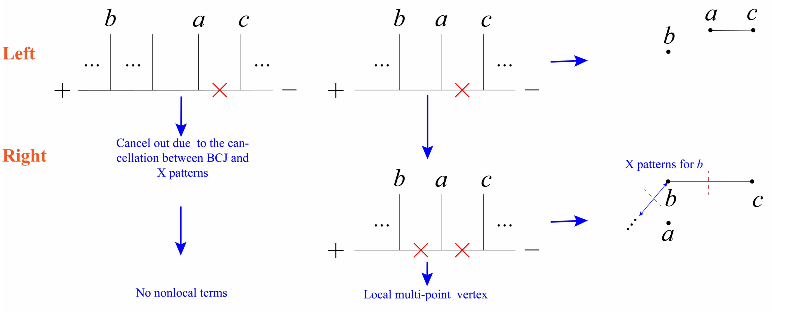
Cyclic sums

$\Downarrow$

Quadratic propagators

Quadratic propagators in double-YMS

A simple example: $|Y| = |Z| = |W| = 1$ with $(\epsilon_a \cdot \epsilon_c)$ and $(\widetilde{\epsilon}_b \cdot \widetilde{\epsilon}_c)$, reference order $a \prec c$, $b \prec c$ and we focus on the subgraph with the propagator between a, c is cut when cancellation on the left is performed



Diagrams obtained by cylcic permutations of the substructures are also allowed by the graphic rule $\rightarrow$ Quadratic propagator form

Comments on the more complicated cases

- ★ First left, then right $\sim$ First right, then left
- ★ Multi-point vertices naturally arise from the cancellation process
- ★ The right part cancellation can borrow factors $k \cdot k$ from the left part factors.

VII. Further discussions

Further discussions

★ How to express integrands via tensorial PT factors?

★ A general construction of one-loop BCJ?

★ The full connection to Feynman diagrams in GR?

★ On the reduction of one-loop integrals (disucssions with Hu, Xie and Zhou)

Main References

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