From OPE to Anomalous dimensions

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Renormalization to higher loops

- Renormalization group flow describes the change of physical quantities with energies and scales. In high energy physics, renormalization is required to enhance the precision of perturbative computations. Some processes in LHC, like the decay of Higgs particle to gluons and bottom quarks, can be sensitive to the 5-loop beta functions of gauge coupling.
- The critical exponents of second order phase transition are related to the anomalous dimensions of operators in scalar field theories at the Wilson-Fisher fixed point:

$$\begin{aligned} \text{Quantum field theories}: &\quad \langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \frac{1}{|x-y|^{\triangle}} \\ \text{Statistical systems}: &\quad \langle \Phi(x)\rangle \sim |T-T_c|^{\triangle} \end{aligned} \tag{1}$$

- Resumation is required in order to make this comparison. So it is perferable to compute the anomalous dimensions to higher loops.

Higher loop UV divergence

- The degree of difficulty in the evaluation of Feynman integrals depends on the number of scales and loops.
- The scales includes mass m^2 and inner products of external momenta $p_i^2, p_i \cdot p_j$.

$$\int \frac{d^D l}{i\pi^D} \frac{1}{(l^2 - m^2)(l+p_1)^2(l+p_1+p_2)^2(l+p_1+p_2+p_3)^2} \tag{2}$$

- At higher loop level $(L \geq 4)$, only single scale integrals can be evaluated systematically. So any higher loop computations should be reduced to the computation of single scale integrals.

$$\begin{aligned} L &= 2 : 5 \text{ scales} \\ L &= 3 : 2 \text{ scales} \\ L &\geq 4 : \text{single scale} \end{aligned} \tag{3}$$

Massless propagator integrals

- Two types of single scale integrals were used in the evaluation of higher loop UV divergence.
- In the R^* operation method [Vladimirov 1980, Chetyrkin, Smirnov 1984, Herzog, Ruijl 2017], the UV divergence of L-loop integrals are reduced to the $L-1$ loop propagator type integrals:

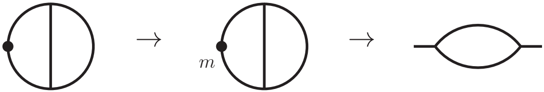

Figure 1: From vacuum integrals to propagator integrals.

- QCD 5-loop beta functions [Baikov, Chetyrkin, Kuhn 2016, Herzog, Ruijl, Uedaa, Vermaserena, Vogtc 2017], ϕ^3 5-loop [Kompaniets, Pikelner 2021], ϕ^Q anomalous dimension 5-loop [Jin, Li 2022]
- The R^* operation requires sophisticated subtraction of IR and UV sub divergences, and it is not efficient when dealing with tensor integrals. It can hardly be used in the computation of anomalous dimensions in gauge theories.

Massive vacuum integrals

- Alternatively, one can add the same mass to all propagators, and reduce the computation of UV divergence to the computation of massive vacuum integrals [Chetyrkin, Misiak, Munz 1997].
- The 4-loop beta function of Gross-Neveu model[Gracey, Luthe, Schroder 2016].
- The 5 loop beta function of QCD [Luthe, Maier, Marquard, Schroder 2017].
- The method is "global" in that one does not need to subtract sub divergences.
- The bottleneck of this method is the computation of master integrals. By now some of the 5-loop master integrals are still unknown [Luthe 2016].

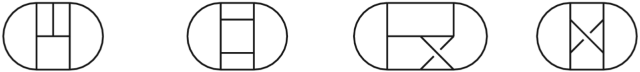

Figure 2: 5-loop vacuum graphs with 12 propagators.

New method based on OPE

- We develop a new, more efficient method based on OPE.

$$\phi(x)\Omega_I(0) = \sum_\alpha C_{I\alpha}(x) O_\alpha(0) \tag{4}$$

- The OPE coefficients contains the renormalization factors of Ω_I and O_α,

$$C_{I\alpha}(x) = Z_{IJ}\mathcal{T}_{J\beta}(x)(Z^{-1})_{\beta\alpha} \tag{5}$$

- At the same time, the OPE coefficients are UV finite quantities, and its UV finiteness can be used to fix Z_{IJ}, if Z_α are already known,

$$C_{I\alpha}(x) = \text{UV finite} \tag{6}$$

- The new method is "global". No need to subtract sub divergences.
- $\mathcal{T}_{J\beta}(x)$ are propagator type integrals. The IBP reduction is relatively simple, and their master integrals have been computed (analytically) to 5-loop [Georgoudis et al. 2021].

Operators and anomalous dimensions

- The most simple operators are scalar operators without Lorentz indices. For example, dimension-6 operators in the ϕ^3 theory are

$$\Omega_I = \left(\frac{g^2}{3!}\phi^3, \ \frac{g}{2}\partial^2\phi^2, \ g\phi\partial^2\phi, (\partial^2)^2\phi \right) \tag{7}$$

- The operators are renormalized by quantum corrections, $\Omega_I^R = Z_{IJ}\Omega_I^0$. Ω_I^0 are the bare operators

$$\Omega_I^0 = \Omega_I \Big|_{\phi\to\phi_0, \ g\to g_0} \tag{8}$$

- The renormalization flow of Ω_I^R is characterized by their anomalous dimensions γ_{IJ}:

$$\begin{aligned} \frac{d\Omega_I^R}{d\ln\mu} &= -\gamma_{IJ}\ \Omega_J^R, \\ \gamma_{IJ} &= -\frac{dZ_{IK}}{d\ln\mu}(Z^{-1})_{KJ} \end{aligned} \tag{9}$$

The Z factors in correlation functions

- The Z factors are determined by the UV finiteness of correlation functions:

$$\left\langle \Omega_I^R(p_0)\phi(p_1)\cdots\phi(p_n)\right\rangle = \text{UV finite} \tag{10}$$

- In Z factor dependence of correlation functions can be clear elucidated using the bare correlation functions. Since the bare Lagrangian only contain bare coupling constants and other bare parameters like bare mass, and the bare operators are also constructed from bare coupling constants, and bare fields. Therefore, the bare correlation functions is a function of bare coupling constants, and do not contain any Z factors,

$$G_{In}^{\text{full}}(g_0) = \left\langle \Omega_I^0(p_0)\phi_0(p_1)\cdots\phi_0(p_n)\right\rangle \tag{11}$$

- The correlation functions can be written as a function of g_0, times some overall Z factors

$$\left\langle \Omega_I^R(p_0)\phi(p_1)\cdots\phi(p_n)\right\rangle = Z_{IJ} Z_\phi^{-\frac{n}{2}} G_{In}^{\text{full}}(g_0) \tag{12}$$

The Z factors in correlation functions

- The full correlation function can be written as the product of amputated correlation function and 2 point correlation functions associated with the external legs:

$$\left\langle \Omega_I^R(p_0)\phi(p_1)\cdots\phi(p_n)\right\rangle = \left\langle \Omega_I^R(p_0)\phi(p_1)\cdots\phi(p_n)\right\rangle^{\text{amp}} \prod_{i=1}^{n}\left\langle \phi(p_i)\phi(-p_i)\right\rangle \tag{13}$$

- The bare correlations can be decomposed in the same way,

$$G_{In}^{\text{full}}(g_0) = G_{In}^{\text{amp}}(g_0)\prod_{i=1}^{n}\triangle(p_i, g_0) \tag{14}$$

in which the bare two point functions $\triangle(p_i, g_0)$ satisfies

$$\left\langle \phi(p)\phi(-p)\right\rangle = Z_\phi^{-1}\triangle(p, g_0) \tag{15}$$

- The amputated correlation functions contain the following Z factors,

$$\left\langle \Omega_I^R(p_0)\phi(p_1)\cdots\phi(p_n)\right\rangle^{\text{amp}} = Z_{IJ}Z_\phi^{\frac{n}{2}}G_{Jn}^{\text{amp}}(g_0) \tag{16}$$

The Z factors in correlation functions

- The 1PI correlation functions satisfy the same relation:

$$\left\langle \Omega_I^R(p_0)\phi(p_1)\cdots\phi(p_n)\right\rangle^{1\mathrm{PI}} = Z_{IJ}Z_\phi^{\frac{n}{2}}\, G_{Jn}^{1\mathrm{PI}}(g_0) \tag{17}$$

- This relation also holds without the operator,

$$\left\langle \phi(p_1)\cdots\phi(p_n)\right\rangle^{1\mathrm{PI}} = Z_\phi^{\frac{n}{2}}\, G_n^{1\mathrm{PI}}(g_0) \tag{18}$$

- From now on, all correlation functions means 1PI correlation functions.

The Operator Product Expansion (OPE)

- The physical effects of two adjacent operators can be described by local operators[Wilson 1965] :

$$\Omega_I^R(x)\Omega_J^R(0) \sim \sum_\alpha C_{IJ}{}^\alpha(x)\, O_\alpha^R(0), \text{ when } x \to 0\ . \tag{19}$$

- Besides the application in conformal field theories, OPE is also vastly used in high energy physics and nuclear physics, to study processes like deep inelastic collision and heavy quark decay.
- Translate to momentum space,

$$\Omega_I^R(p)\Omega_J^R(k-p) \sim \sum_\alpha C_{IJ}{}^\alpha(p)\, O_\alpha^R(k), \text{ when } p \to \infty\ . \tag{20}$$

The OPE of an operator and a fundamental field

- We consider the OPE of an operator and a fundamental field:

$$\phi(p)\Omega_I^R(k-p) \sim \sum_\alpha C_{I\alpha}(p)\, O_\alpha^R(k) \tag{21}$$

- This gives the following relation between correlation functions:

$$\left\langle \phi(p)\Omega_I^R(k-p)\phi(k_1)\cdots\phi(k_n)\right\rangle \sim \sum_\alpha C_{I\alpha}(p)\left\langle O_\alpha^R(k)\phi(k_1)\cdots\phi(k_n)\right\rangle . \tag{22}$$

- In terms of bare correlation functions,

$$\begin{aligned}\left\langle \phi(p)\Omega_I^R(k-p)\phi(k_1)\cdots\phi(k_n)\right\rangle &= Z_\phi^{\frac{n+1}{2}} \sum_J Z_{IJ} G_J^\Omega(g_0)\\ \left\langle O_\alpha^R(k)\phi(k_1)\cdots\phi(k_n)\right\rangle &= Z_\phi^{\frac{n}{2}} \sum_\beta Z_{\alpha\beta} G_\beta^O(g_0)\end{aligned} \tag{23}$$

The large momentum expansion

- The LHS correlation function $G_J^{\Omega}(g_0)$ is an integral depending on a large momentum p, and some soft momenta k_i, and can be evaluated with the help of the large momentum expansion:

$$G_J^{\Omega}(p;k_i;g_0) \sim \sum_{\alpha} \mathcal{T}_{\alpha}\left[G_J^{\Omega}(p;0;g_0)\right] G_{\alpha}^{O}(k_i;g_0) \tag{24}$$

- $\mathcal{T}_{\alpha}$ is the Taylor expansion operator corresponding to the operator O_{α}. $\mathcal{T}_{\alpha}\left[G_J^{\Omega}(p;0;g_0)\right]$ only depend on a single scale p^2.
- From which we derive the following expression of the OPE coefficients

$$C_{I\alpha}(p) = Z_{\phi}^{\frac{1}{2}} \sum_{J,\beta} (Z^{-1})_{\beta\alpha} Z_{IJ} \mathcal{T}_{\beta}\left[G_J^{\Omega}(g_0)\right] \tag{25}$$

Z factors from OPE coefficients

- The UV finiteness of OPE coefficients give constraints to the combination $Z_\phi^{\frac{1}{2}}(Z^{-1})_{\beta\alpha}Z_{IJ}$.

$$C_{I\alpha}(p) = Z_\phi^{\frac{1}{2}} \sum_{J,\beta} (Z^{-1})_{\beta\alpha} Z_{IJ} \mathcal{T}_\beta \left[G_J^\Omega(g_0) \right] \tag{26}$$

- If Z_ϕ and $Z_{\alpha\beta}$ are already known, the relation gives constraints to Z_{IJ}.
- A properly chosen set of O_α completely fixes Z_{IJ}.

The large momentum expansion of Feynman integrals

- If a Feynman integrals contains two or more distinct scales, the asympotic expansion about the large scale can be performed using some graphcal rules [Chetyrkin 1983, Chetyrkin, Smirnov 1984].
- Consider the following Feynman integral, in which q_1 and q_2 are large momenta,

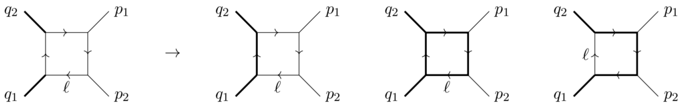

Figure 3: Large momentum expansion of Feynman integrals.

$$I_{\text{box}} = \frac{1}{\ell^2(\ell - q_1)^2(\ell + p_1 + p_2)^2(\ell + p_2)^2} \, . \tag{27}$$

- Split the Feynman diagram into the large momentum part(bold line in the RHS graph) and small momentum part, and perform a Taylor expansion to the large momentum part:

$$I_{\text{box}} = \left(\frac{1}{q_1^2} + \frac{2\ell \cdot q_1 - \ell^2}{(q_1^2)^2} + \cdots \right) \frac{1}{\ell^2(\ell + p_1 + p_2)^2(\ell + p_2)^2} \, , \tag{28}$$

The large momentum expansion of Feynman integrals

- The large momentum part becomes a propagator type integral, and the small momentum part has one less external legs than the original integral:

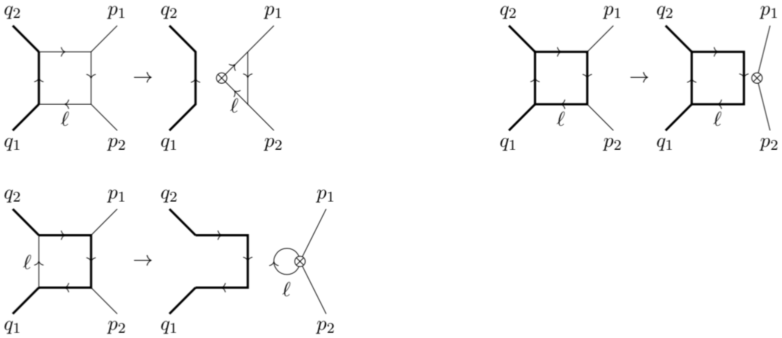

Figure 4: Large momentum expansion.

The large momentum expansion and OPE

- The original Feynman diagram correspond to the 4 point correlation function $\left\langle \Omega_I(q_1)\Omega_J(q_2)\phi(p_1)\phi(p_2) \right\rangle$

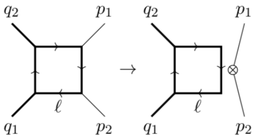

Figure 5: Large momentum expansion and OPE.

- After the expansion, the two graphs correspoind to OPE coefficients and $\left\langle O_\alpha(q)\phi(p_1)\phi(p_2) \right\rangle$, respectively.
- Therefore, the OPE coefficients can be obtained by evaluating propagator type integrals.

The 5-loop anomalous dimensions of ϕ^Q operators in $O(N)$ ϕ^4 model

- Using this method, we compute the 5-loop anomalous dimensions of ϕ^Q operators in $O(N)$ ϕ^4 model.
- The ϕ^Q operators refer to the symmetric traceless operators constructed by Q $O(N)$ scalar fields ϕ_i, and they are equivalent to

$$\phi^Q \sim \frac{1}{Q!}\varphi^Q, \tag{29}$$

  in which $\varphi^Q = q_i\phi_i$, and q_i satisfies $q_i^2 = 0$.
- ϕ^Q does not mix to any other operators, therefore its Z factor can be fixed by a single OPE coefficient:

$$\phi^Q(0)\phi(x) \sim \frac{C(x)}{x^2}\phi^{Q-1}(0) \tag{30}$$

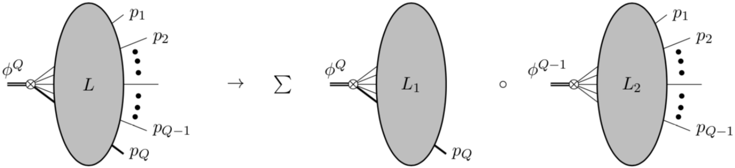

Figure 6: Reduce ϕ^Q to ϕ^{Q-1}.

The 5-loop anomalous dimension in 4-D $O(N)$ ϕ^4 model

- We computed the propagator integrals using Graphical function [Schnetz 2014], and reproduced the previous known result using R^* operations [Jin, Li 2022].
- The major computational time was spent on the evaluation of Graphical function, which runs for several hours on a personal computer. In comparison, the R^* operation approach required super computer clusters.

The 4-loop anomalous dimensions in 6-D $O(N)$ ϕ^3 model

- 6-D $O(N)$ ϕ^3 models is given by the following Lagrangian :

$$L = \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2}(\partial\sigma)^2 - \frac{g}{2}\sigma\phi_i^2 - \frac{h}{6}\sigma^3 \tag{31}$$

- Following the same routine, we computed the 4-loop anomalous dimensions of ϕ^Q in this model:

$$\begin{aligned}\gamma_4 = {} & g^2h^6\left[Q^2\left(\frac{29239}{559872} - \frac{\pi^4}{8640} - \frac{29\zeta_3}{1728}\right) + Q\left(\frac{-69419}{839808} + \frac{\pi^4}{6480} + \frac{35\zeta_3}{1296}\right)\right] \\ & + g^3h^5\left[Q^2\left(\frac{6665}{7776} + \frac{\pi^4}{240} - \frac{157\zeta_3}{144}\right) + Q^3\left(\frac{-487}{1728} - \frac{\pi^4}{1080} + \frac{79\zeta_3}{216}\right)\right. \\ & \left. + Q\left(\frac{-94597}{186624} - \frac{\pi^4}{270} + \frac{97\zeta_3}{144}\right)\right] + 152 \text{ more terms}\end{aligned} \tag{32}$$

- Our results are consistent with the know results in the large N limit [Vasiliev, Pismak, Khonkonen 1981].

Operator mixing

- More OPE coefficients are required to fix the Z factors in the presence of operator mixing. Let us go back to the following operators in ϕ^3 model :

$$\Omega_I = \left(\frac{g^2}{3!}\phi^3,\ \frac{g}{2}\partial^2\phi^2,\ g\phi\partial^2\phi, (\partial^2)^2\phi\right) \tag{33}$$

- The OPE is given by

$$\begin{aligned}\phi(p)\Omega_I^R(k-p) =& C_1(r)p^2\phi(k) + C_2(r)p^\mu\partial_\mu\phi(k) + C_3(r)\partial^2\phi(k) + C_4(r)\phi_R^2(k) \\ &+ C_5(r)\frac{p^\mu p^\nu}{p^2}\left[\partial_{\mu\nu}\phi(k) - \frac{1}{D}\eta_{\mu\nu}\partial^2\phi(k)\right]_R + \cdots\end{aligned} \tag{34}$$

  in which $r = \ln\frac{-p^2}{\mu^2}$.

- It turns out that C_1 to C_4 are enough to completely fix Z_{IJ}.

Operator mixing

- All OPE coefficients of operators with higher dimensions and/or higher tensor ranks contains $\frac{1}{p^2}$ poles, which corresponds to sub-divergences.
- In general, we only need to consider the operators with

$$\triangle_O + R_O \leq \triangle_\Omega - 2 \tag{35}$$

in which R_O is the tensor rank of O. Their Z factors are easier to compute than that of Ω_I.

- Start with simple, lower dimensional operators, we can compute the anomalous dimensions of higher dimensional operators recursively.

From OPE to beta functions

- The beta functions in ϕ^4 model can be obtained from the OPE of two fundamental fields,

$$\phi(x)\phi(0) \sim C(x)\phi_R^2(0) + \cdots . \tag{36}$$

- This gives the following relation between correlation functions :

$$\Big\langle \phi(0)\phi(x)\phi(y_1)\phi(y_2) \Big\rangle \sim C(x) \Big\langle \phi_R^2(0)\phi(y_1)\phi(y_2) \Big\rangle + \cdots . \tag{37}$$

- The LHS correlation function has the overall factor of

$$Z_\phi^2 g_0 = Z_g g \tilde{\mu}^{2\epsilon} \tag{38}$$

The RHS correlation function has the overall factor of $Z_{\phi^2} Z_\phi$.

From OPE to beta functions

- The OPE coefficient is given by

$$C(p) = \frac{Z_g g \tilde{\mu}^{2\epsilon}}{Z_{\phi^2} Z_\phi} G_4(p, -p, 0, 0; g_0) \tag{39}$$

- Z_g can be obtained from the finiteness of the OPE coefficient, provided that Z_{ϕ^2} and Z_ϕ are already known.
- The story is even simpler in ϕ^3 model. The OPE is,

$$\phi(x)\phi(0) \sim C(x)\phi(0) + \cdots . \tag{40}$$

- The OPE coefficient is given by

$$C(p) = \frac{Z_g g \tilde{\mu}^{\epsilon}}{Z_\phi} G_3(p, -p, 0; g_0) \tag{41}$$

Apply to gauge theories

- In gauge theories, the fundamental field A_μ^a is not a gauge invariant operator. One way to preserve the gauge symmetry is using the background gauge. One expand the gauge field into the background part A and the fluctuation part $\mathbf{A}$

$$A \to A + \mathbf{A}\ . \tag{42}$$

  and impose background gauge to $\mathbf{A}$. The gauge symmetry of background field is still explicitly presever.

- The background field A corresponds to the soft field, and $\mathbf{A}$ corresponds to the hard field. The OPE of hard fields produces gauge invariant soft operators,

$$\mathbf{A}_\mu(x)\mathbf{A}_\nu(0) = \sum_\alpha C_\alpha(x) O_{\alpha,\mu\nu}(0) \tag{43}$$

- The soft operator basis contains Wilson-lines which are only gauge invariant when operators with different dimensions are combined together,

$$W[x,0] = \mathbf{P}\exp\left[ig\int_0^1 dsx^\mu A_\mu(sx)\right] = igx^\mu A_\mu(0) + \cdots \tag{44}$$

Conclusions

- We develop a new method of computing renormalization factors using OPE coefficients, and computed the 4 and 5 loop anomalous dimensions of ϕ^Q operators.
- The method is more efficient than traditional R^* operations and massive vacuum integral method, and can be applied to integrals with more scales and higher tensor ranks.
- We hope to compute the anomalous dimensions of higher dimensional operators in gauge theories in the future.
- Thank you for your attention!