BOOTSTRAPPING THE BFSS MODEL

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Outline

- Matrix Quantum Mechanics. (In progress, with Henry Lin)

Relevant literatures

[Anderson and Kruczenski, 2017]

[Lin, 2020]

[Han et al., 2020]

[Kazakov and Zheng, 2022]

[Kazakov and Zheng, 2023]

[Cho et al., 2022]

[Lin, 2023]

[Kazakov and Zheng, 2024]

[Li and Zhou, 2024]

Small pieces of Optimization theory

Basically bootstrap method is solving problems in theoretical physics by optimization theory.

- Quadratic programming:

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & y = x^2 + 3x + 1 \end{aligned} \tag{1}$$

- Linear programming:

$$\begin{aligned} \max \quad & 300x + 100y \\ \text{s.t.} \quad & 6x + 3y \leq 40 \\ & x - 3y \leq 0 \\ & x + \tfrac{1}{4}y \leq 4 \end{aligned} \tag{2}$$

Semi-definite Programming

- Semi-definite Programming:

$$\begin{aligned} \min \quad & 2x + 3y \\ \text{s.t.} \quad & \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0 \end{aligned} \tag{3}$$

- Linear programming and Quadratic programming are special situations of Semi-definite Programming(SDP).
- They all fall into the class of Convex Optimization.
- Generally we cannot solve large-scale non-convex optimization problem (NP hard).

BFSS MODEL

The Hamiltonian is chosen to be [Banks, Fischler, Shenker, Susskind 97]:

$$H = \frac{1}{2}\,\mathrm{Tr}\left(g^2 P_I^2 - \frac{1}{2g^2}\left[X_I, X_J\right]^2 - \psi_\alpha \gamma^I_{\alpha\beta}\left[X_I, \psi_\beta\right]\right) \tag{4}$$

Here:

$$[X_{ij}, P_{kl}] = i\delta_{il}\delta_{jk},\ \{\psi_{\alpha,ij}, \psi_{\beta,kl}\} = \delta_{\alpha\beta}\delta_{il}\delta_{kj} \tag{5}$$

Dual to the dynamics of the D0-brane.

The matrices are in multiples of the $SO(9)$ symmetry, with the supercharge:

$$Q_\alpha = g\,\mathrm{tr} P_I \gamma^I_{\alpha\beta}\psi_\beta - \frac{i}{2g}\mathrm{tr}[X^I, X^J]\gamma^{IJ}_{\alpha\beta}\psi_\beta \tag{6}$$

Why bootstrap

MC:

- physics simplifies at large N but the computation gets harder
- sign problem, finite volume truncation, finite N truncation

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Bootstrap:

- works in large N directly; gives rigorous bounds
- no sign problem, no finite volunmn or finite N truncation

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- works in large N directly; gives rigorous bounds
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Toy model

The Hamiltonian is chosen to be($g = 1$):

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}gx^2y^2 \tag{7}$$

This model is not trivially solved by numerical method/analytical method[Hoppe, 1980] [Simon, 1983][Komatsu et al., 2024]. It keeps certain key feature of the BFSS Hamiltonian:

$$H = \frac{1}{2}\,\mathrm{Tr}\left(g^2 P_I^2 - \frac{1}{2g^2}\left[X_I, X_J\right]^2 - \psi_\alpha \gamma^I_{\alpha\beta}\left[X_I, \psi_\beta\right]\right) \tag{8}$$

Loop equations

Our goal is to solve all the eigenvalues and all the expectations of the operators under different eigenstates.

For an eigenstate with eigenvalue E, the corresponding loop equations are:

$$\langle [H, \mathcal{O}] \rangle = 0, \ \forall \mathcal{O} \tag{9}$$

$$\langle H\mathcal{O} \rangle = E\langle \mathcal{O} \rangle, \ \forall \mathcal{O} \tag{10}$$

together with the Ward identities:

$$\langle \mathcal{O}_g \rangle = \langle \mathcal{O} \rangle, \ \forall \mathcal{O} \tag{11}$$

These are all linear equations, we can expand any operators as:

$$\mathcal{O} = \sum \alpha_{mnts} p_x^m p_y^n x^t y^s \tag{12}$$

Positivity by inner product

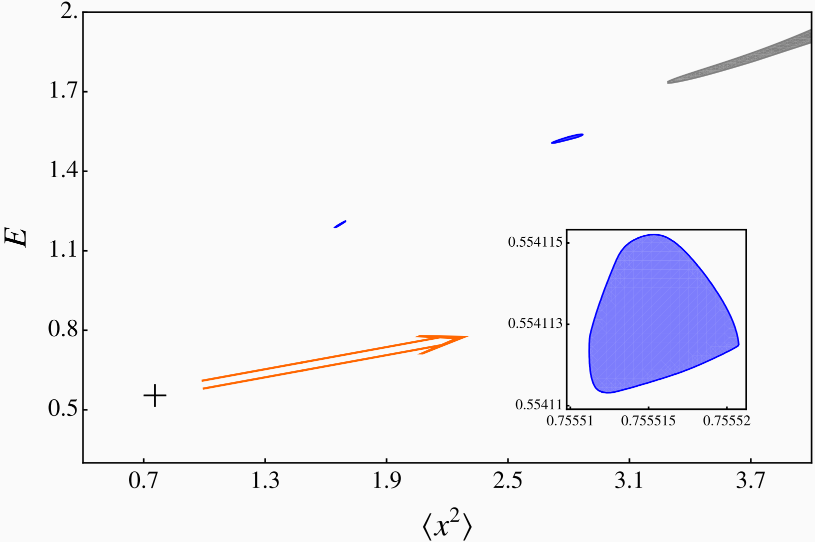
Generalization: Any inner products defined on the vector space of operators or its subspace could leads to positivity condition:

$$\langle \mathcal{O}|\mathcal{O}\rangle = \langle \mathcal{O}^\dagger \mathcal{O}\rangle = \alpha^{*\mathrm{T}}\mathcal{M}\alpha \geq 0,\ \forall \alpha \Leftrightarrow \mathcal{M} \succeq 0. \tag{13}$$

Here we do the expansion $\mathcal{O} = \sum \alpha_i \mathcal{O}_i$, $\mathcal{M}_{ij} = \langle \mathcal{O}_i^\dagger \mathcal{O}_j \rangle$.

$$\begin{array}{c} \\ 1 \\ x^2 \\ p^2 \\ px \\ \vdots \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & x^2 & p^2 & xp & \dots \end{array} \\ \begin{pmatrix} 1 & \langle x^2\rangle & \langle p^2\rangle & \langle xp\rangle & \dots \\ \langle x^2\rangle & \langle x^4\rangle & \langle x^2p^2\rangle & \langle x^3p\rangle & \dots \\ \langle p^2\rangle & \langle p^2x^2\rangle & \langle p^4\rangle & \langle p^2xp\rangle & \dots \\ \langle px\rangle & \langle px^3\rangle & \langle pxp^2\rangle & \langle px^2p\rangle & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \succeq 0.$$

Toy model($\Lambda = 12$)



Ground State Positivity

For the ground state, or more generally, any stationary state, the corresponding positivities are:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0,\ \forall \mathcal{O} \tag{14}$$

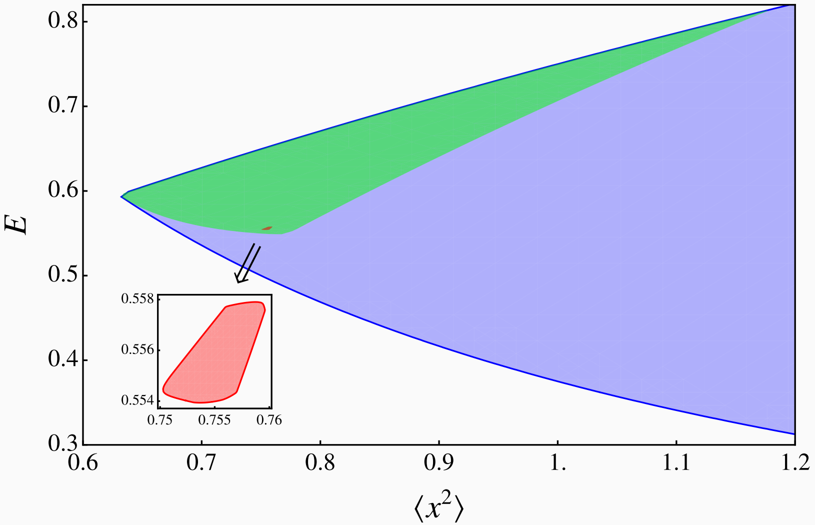
$$\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle \geq 0,\ \forall \mathcal{O} \tag{15}$$

The later positivity is specialized for the ground state. For more general thermal state with inverse temperature β [Fawzi, Fawzi and Scalet 23],

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \log \frac{\langle \mathcal{O}^\dagger \mathcal{O} \rangle}{\langle \mathcal{O} \mathcal{O}^\dagger \rangle} \leq \beta \langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle,\ \forall \mathcal{O} \tag{16}$$

Mathematically, these positivities together with the loop equations is necessary and sufficient.

Ground state($\Lambda = 4, 6, 8$)



One-matrix quantum mechanics

The Hamiltonian is chosen to be:

$$H = \mathrm{tr}(P^2 + X^2 + gX^4) \tag{17}$$

Here X is a large N Hermitian matrix:

$$[X_{ij}, P_{kl}] = i\delta_{il}\delta_{jk} \tag{18}$$

The ground state is known to be solvable.

[Han et al., 2020] [WIP w/ Henry Lin]

The corresponding loop equations are:

$$\langle [H, \mathcal{O}] \rangle = 0,\ \forall \mathcal{O} \tag{19}$$

$$\langle \mathrm{tr}(G\mathcal{O}) \rangle = 0,\ \forall \mathcal{O} \tag{20}$$

together with the cyclicity of $\mathrm{tr}\mathcal{O}$. $G = i[X, P] + I$ is the generator of the $SU(N)$ gauge symmetry.

Result: general words in P and X can be reduced to polynomials of $\mathrm{tr}X^m$.

$$\begin{aligned}\mathrm{tr}P^2X^2P^2X^4 = &\frac{12}{77}g^2\mathrm{tr}X^{14} - \frac{2}{3}g\mathrm{tr}X^2\mathrm{tr}X^6 - \frac{1}{5}g\mathrm{tr}X^8 + \frac{40}{231}g\mathrm{tr}X^{12} \\ &+ \frac{\mathrm{tr}X^2}{24} - \frac{1}{3}\mathrm{tr}X^2\mathrm{tr}X^4 - \frac{\mathrm{tr}X^6}{10} + \frac{\mathrm{tr}X^{10}}{21}\end{aligned} \tag{21}$$

Positivity

For the ground state, or more generally, any stationary state, the corresponding loop equations are:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0, \ \forall \mathcal{O} \tag{22}$$

$$\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle \geq 0, \ \forall \mathcal{O} \tag{23}$$

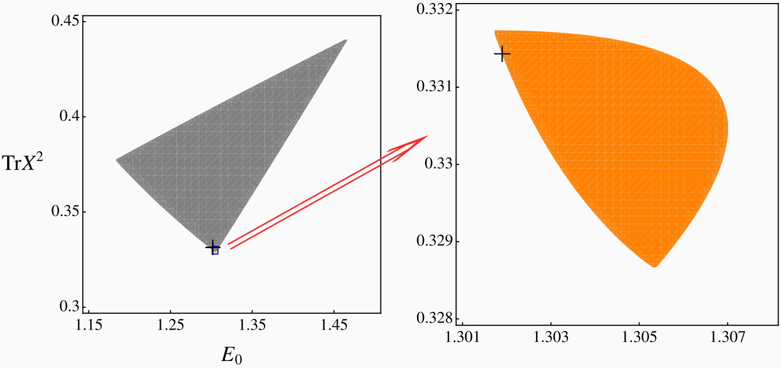
The later positivity is specialized for the ground state. For more general thermal state with inverse temperature β [Fawzi, Fawzi and Scalet 23],

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \log \frac{\langle \mathcal{O}^\dagger \mathcal{O} \rangle}{\langle \mathcal{O} \mathcal{O}^\dagger \rangle} \leq \beta \langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle, \ \forall \mathcal{O} \tag{24}$$

Mathematically, these positivities together with the loop equations is necessary and sufficient.

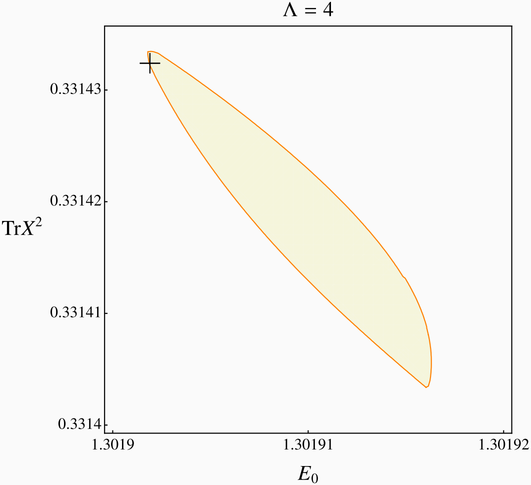
Convergence

The illustration of convergence, the left one is $\Lambda = 2$, whereas the right one corresponds to $\Lambda = 3$. The size of the SDP matrix is $2, 2, 2$ and $3, 3, 2, 3$, respectively,



Convergence

The size of the SDP matrices are $5, 4, 4, 4$.



BFSS model

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Here:

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Dual to the dynamics of the D0-brane.

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$$Q_\alpha = g\,\mathrm{tr} P_I \gamma^I_{\alpha\beta}\psi_\beta - \frac{i}{2g}\mathrm{tr}[X^I, X^J]\gamma^{IJ}_{\alpha\beta}\psi_\beta \tag{27}$$

BFSS model bootstrap [WIP w/ Henry Lin]

Loop equations:

$$\langle [H, \mathcal{O}] \rangle = 0 \tag{28}$$

$$\langle \{Q_\alpha, \mathcal{O}_\alpha\} \rangle = 0 \tag{29}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_2 \mathcal{O}_1 \rangle - \langle [\mathcal{O}_1, \mathcal{O}_2] \rangle \tag{30}$$

$$\langle \mathrm{tr}(C_{ij} \mathcal{O}_{ji}) \rangle = 0, \ \forall \mathcal{O} \tag{31}$$

This is the gauge singlet condition:

$$C_{ij} = -[X^I, P^I]_{ij} - \psi^\alpha_{ik} \psi^\alpha_{kj} - \mathbf{1}_{ij} \tag{32}$$

Positivities:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0, \ \forall \mathcal{O} \tag{33}$$

$$\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle \geq 0, \ \forall \mathcal{O} \tag{34}$$

BFSS model bootstrap [WIP w/ Henry Lin]

Loop equations:

$$\cancel{\langle [H, \mathcal{O}] \rangle = 0} \tag{35}$$

$$\langle \{ Q_\alpha, \mathcal{O}_\alpha \} \rangle = 0 \tag{36}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_2 \mathcal{O}_1 \rangle - \langle [\mathcal{O}_1, \mathcal{O}_2] \rangle \tag{37}$$

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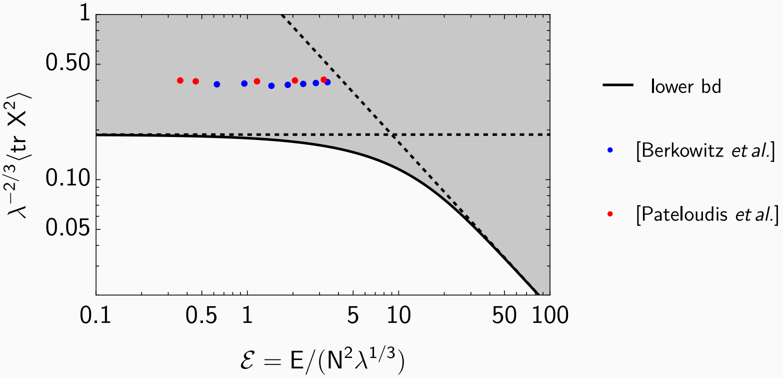
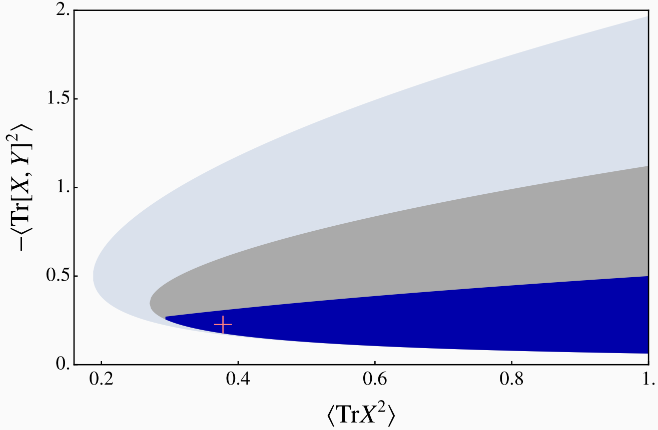
$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0, \ \forall \mathcal{O} \tag{40}$$

$$\cancel{\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle \geq 0}, \ \forall \mathcal{O} \tag{41}$$


1
0.50
0.10
0.05
$\lambda^{-2/3}\langle \mathrm{tr}\, X^2 \rangle$
0.1
0.5
1
5
10
50
100
$\mathcal{E} = E/(N^2\lambda^{1/3})$
lower bd
[Berkowitz *et al.*]
[Pateloudis *et al.*]

Numerical result

Numerical result



Pink cross + is the Monte Carlo result of [Pateloudis et al, 22].

Convergence

Records	$\langle \mathrm{tr} X^2 \rangle$
MC [Pateloudis et al, 2022]	~ 0.37
Primitive bootstrap[Lin, 2023]	≥ 0.1875
cutoff 6	≥ 0.294
cutoff 7	≥ 0.331

Even at level 7, the SDP problem scale is extremely small (at most 10×10 matrix). Which solves instantly on a laptop.

Questions?

Reference

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Loop Equations and bootstrap methods in the lattice.
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Cho, M., Gabai, B., Lin, Y.-H., Rodriguez, V. A., Sandor, J., and Yin, X. (2022).
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JHEP, 06:030.

Kazakov, V. and Zheng, Z. (2023).
Bootstrap for lattice Yang-Mills theory.
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Bootstrap for Finite N Lattice Yang-Mills Theory.

Komatsu, S., Martina, A., Penedones, J. a., Suchel, N., Vuignier, A., and Zhao, X. (2024).
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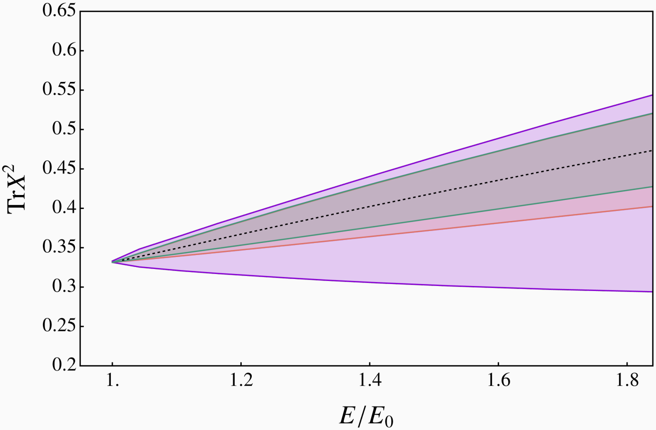
Li, Z. and Zhou, S. (2024).

Above ground state

The dashed line is the thermal state with the corresponding energy expectation. Different colors correspond to $\Lambda = 8, 18, 26$.



One-matrix model[Lin, 2020][Kazakov and Zheng, 2022]

The partition function is chosen to be:

$$Z = \lim_{N\to\infty} Z_N = \lim_{N\to\infty} \int d^{N^2}M\,\mathrm{e}^{-N\mathrm{tr}V(M)}, \quad V(x) = \frac{1}{2}\mu x^2 + \frac{1}{4}gx^4, \qquad (42)$$

The integration is over Hermitian matrix.

The basis of operators are:

$$\mathcal{W}_k = \langle \mathrm{Tr}M^k \rangle = \lim_{N\to\infty} \int \frac{d^{N^2}M}{Z_N} \frac{1}{N} \mathrm{tr}M^k \mathrm{e}^{-N\mathrm{tr}V(M)}. \qquad (43)$$

And the Schwinger-Dyson equations:

$$\mu\mathcal{W}_{k+1} + g\mathcal{W}_{k+3} = \sum_{l=0}^{k-1} \mathcal{W}_l\ \mathcal{W}_{k-l+1},\ k = 1, 2, 3, \ldots \qquad (44)$$

Generalization: Any inner products defined on the vector space of operators or its subspace could leads to positivity condition:

$$\langle \mathcal{O}|\mathcal{O}\rangle = \langle \mathcal{O}^\dagger \mathcal{O}\rangle = \alpha^{*\mathrm{T}}\mathcal{M}\alpha \geq 0,\ \forall \alpha \Leftrightarrow \mathcal{M} \succeq 0. \tag{45}$$

Here we do the expansion $\mathcal{O} = \sum \alpha_i \mathcal{O}_i$, $\mathcal{M}_{ij} = \langle \mathcal{O}_i^\dagger \mathcal{O}_j \rangle$.

In the above case of Hermitian matrix integration, we were taking adjoint to be Hermitian conjugation:

$$\mathcal{O}^\dagger = \mathcal{O}^{*\mathrm{T}} = \mathcal{O} \tag{46}$$

Positivity

Considering the expectations of square of polynomials are always positive semi-definite:

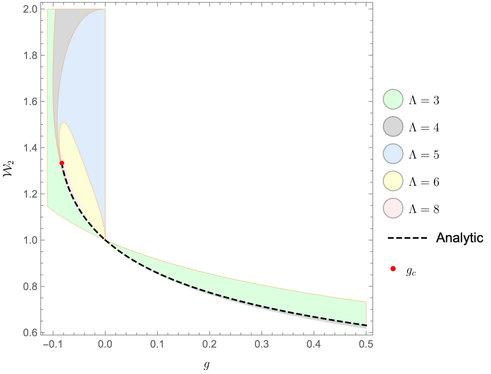
$$\frac{1}{Z}\int_{-\infty}^{\infty} dM\mathrm{Tr}(\sum \alpha_i M^i)^2 \exp(-N\mathrm{tr}V(M)) \geq 0,\ \forall\alpha \tag{47}$$

This is a quadratic form in α, its positivity is equivalent to:

$$\mathbb{W} = \begin{pmatrix} \mathcal{W}_0 & \mathcal{W}_1 & \mathcal{W}_2 & \dots \\ \mathcal{W}_1 & \mathcal{W}_2 & \mathcal{W}_3 & \dots \\ \mathcal{W}_2 & \mathcal{W}_3 & \mathcal{W}_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \succeq 0 \tag{48}$$

Bootstrapping Large N one-matrix model

This is the result of bootstrapping $\mu = 1$ and $\mathbb{Z}_2$ symmetry preserving solution$\mathcal{W}_1 = 0$. From the loop equation and symmetry assumption, all moments are polynomial functions of $\mathcal{W}_2$.



Multi-matrix bootstrap[Kazakov and Zheng, 2022]

Here we propose to study the following two-matrix model:

$$Z = \lim_{N\to\infty} \int d^{N^2}A\, d^{N^2}B\, \mathrm{e}^{-N\mathrm{tr}\left(-h[A,B]^2/2+A^2/2+gA^4/4+B^2/2+gB^4/4\right)} \tag{49}$$

The integration is over Hermitian matrix. To the best of our knowledge, this model with general g and h value, is not solvable!

$$\mathrm{Tr}A^2,\ \mathrm{Tr}A^4,\ \mathrm{Tr}A^2B^2,\ \mathrm{Tr}ABAB,\ \mathrm{Tr}A^6,\ \mathrm{Tr}A^4B^2,\ \mathrm{Tr}A^3BAB,\ \mathrm{Tr}A^2BA^2B,\ \mathrm{Tr}A^8,$$
$$\mathrm{Tr}A^6B^2,\ \mathrm{Tr}A^5BAB,\ \mathrm{Tr}A^4BA^2B,\ \mathrm{Tr}A^4B^4,\ \mathrm{Tr}A^3BA^3B,\ \mathrm{Tr}A^3BAB^3,\ \mathrm{Tr}A^3B^2AB^2,$$
$$\mathrm{Tr}A^2BABAB^2,\ \mathrm{Tr}A^2BAB^2AB,\ \mathrm{Tr}A^2B^2A^2B^2,\ \mathrm{Tr}ABABABAB\ \ldots \tag{50}$$

$\beta = (\mathrm{Tr}A^2)^2$:

$$
\begin{aligned}
1 &= \mathrm{Tr}A^2 + g\mathrm{Tr}A^4 - h(-2\mathrm{Tr}A^2B^2 + 2\mathrm{Tr}ABAB) \\
0 &= -2\mathrm{Tr}A^2 + \mathrm{Tr}A^4 - h(2\mathrm{Tr}A^3BAB - 2\mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^6 \\
0 &= -\mathrm{Tr}A^2 + \mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2BA^2B + 2\mathrm{Tr}A^3BAB - \mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^4B^2 \\
0 &= -h(2\mathrm{Tr}A^2BA^2B - 2\mathrm{Tr}A^3BAB) + g\mathrm{Tr}A^3BAB + \mathrm{Tr}ABAB \\
\beta &= -2\mathrm{Tr}A^4 + \mathrm{Tr}A^6 - h(2\mathrm{Tr}A^5BAB - 2\mathrm{Tr}A^6B^2) + g\mathrm{Tr}A^8 \\
\beta &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^4B^2 - h(-\mathrm{Tr}A^3B^2AB^2 + 2\mathrm{Tr}A^3BAB^3 - \mathrm{Tr}A^4B^4) + g\mathrm{Tr}A^6B^2 \\
0 &= -2\mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2B^2A^2B^2 + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^6B^2 \\
0 &= -\mathrm{Tr}A^4 + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^4B^4 - h(-\mathrm{Tr}A^4BA^2B + 2\mathrm{Tr}A^5BAB - \mathrm{Tr}A^6B^2) \\
0 &= \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^2BAB^2AB - \mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3BAB^3) + g\mathrm{Tr}A^5BAB - \mathrm{Tr}ABAB \\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^5BAB - 2\mathrm{Tr}ABAB - h(-2\mathrm{Tr}A^2BABAB^2 + 2\mathrm{Tr}ABABABAB) \\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^3BAB^3 - h(-\mathrm{Tr}A^3BA^3B + 2\mathrm{Tr}A^4BA^2B - \mathrm{Tr}A^5BAB) \\
0 &= g\mathrm{Tr}A^3BA^3B + \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^3B^2AB^2 - 2\mathrm{Tr}A^3BAB^3) \\
0 &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^2BA^2B - h(-\mathrm{Tr}A^2BAB^2AB + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + g\mathrm{Tr}A^4BA^2B \\
\beta &= \mathrm{Tr}A^2BA^2B + g\mathrm{Tr}A^3B^2AB^2 - h(2\mathrm{Tr}A^3BA^3B - 2\mathrm{Tr}A^4BA^2B).
\end{aligned} \tag{51}
$$

CUTOFF=4: POSITIVITY

For example, the block for the even-odd words reads:

$$\begin{pmatrix} \mathrm{Tr}A^2 & \mathrm{Tr}A^4 & \mathrm{Tr}A^2B^2 & \mathrm{Tr}ABAB & \mathrm{Tr}A^2B^2 \\ \mathrm{Tr}A^4 & \mathrm{Tr}A^6 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^4B^2 \\ \mathrm{Tr}A^2B^2 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^2BA^2B \\ \mathrm{Tr}ABAB & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^2BA^2B & \mathrm{Tr}A^3BAB \\ \mathrm{Tr}A^2B^2 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^2BA^2B & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^4B^2 \end{pmatrix} \succeq 0 \tag{52}$$

All the constraints are convex except the quadratic loop equations!

Relaxation

Our general strategy: we treat the quadratic terms in the loop equations as independent variable, and replace the algebraic equality by the convex inequality:

$$Q = xx^{\mathrm{T}} \tag{53}$$

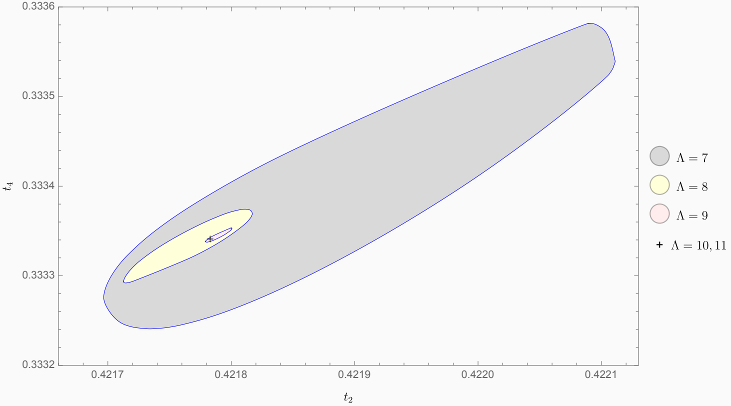
to:

$$\mathcal{R} = \begin{pmatrix} 1 & x^{\mathrm{T}} \\ x & Q \end{pmatrix} \succeq 0. \tag{54}$$

For the previous situation, we have a simple matrix:

$$\mathcal{R} = \begin{pmatrix} 1 & \mathrm{Tr}A^2 \\ \mathrm{Tr}A^2 & \beta \end{pmatrix} \succeq 0. \tag{55}$$

Results



$$\Lambda = 11,\ g = h = 1 : \begin{cases} 0.421783612 \leq \langle \mathrm{Tr} A^2 \rangle \leq 0.421784687 \\ 0.333341358 \leq \langle \mathrm{Tr} A^4 \rangle \leq 0.333342131 \end{cases} \tag{56}$$

Compare with MC

Compared to the MC study of the same model 2111.02410 (Jha), we are convinced that for this model bootstrap is at least two order of magnitude more efficient than MC.

- MC: 80-85 hours for N=800 simulation to get 4.5 digits.
- Bootstrap: less than 1 hour to get 6 digits.