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Dissecting Quantum Many-body Chaos in the Krylov Space

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L.Chen, B.Mu, H.W, and P.Zhang, arXiv: 2404.08207

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Motivation

- Quantum chaos is important: chaos bound, black hole signature, etc
- Signature of quantum chaos: spectral properties, OTOC Lyapunov exponents, etc

$$F(t_1,t_2) = G(t_{12}) \mp \langle \hat{\mathcal{O}}(t_2)^\dagger \widehat{\mathcal{O}'}(0)^\dagger \hat{\mathcal{O}}(t_1) \widehat{\mathcal{O}'}(0) \rangle \sim f(t_{12}) e^{\varkappa T_{12}} \quad T_{12} = (t_1+t_2)/2, \qquad t_{12} = t_2 - t_1$$

- There are pros and cons…
- In some context (e.g. SYKs), OTOC admits alternative interpretation of operator-size.

$$F(t) \sim N(t),\ \psi(t) = \sum C_{i_1 i_2 \dots i_n} \psi^{i_1}\psi^{i_2} \dots \psi^{i_n},\ N(t) = \sum n\, |C_{i_1 \dots i_n}|^2$$

- Lyaponov exponent: operator spreading exponentially (scrambling)
- Measure of quantum chaos by operator-size growth?
- Subtlety: in general no intrinsic operator basis to define operator-size, such as $\psi^{i_1} \dots \psi^{i_n}$

Motivation

- A natural choice of operator basis: Krylov basis (1812.08657)

  initial operator O + dynamics H + operator norm $\langle . \rangle$ $\rightarrow$ dynamically generated operator basis

  - Recursively defined
  - $O_0 = O, \quad A_n = i[H, O_{n-1}] + b_{n-1} O_{n-2}, \quad O_n = b_n^{-1} A_n\,, \quad b_n = \langle A_n, A_n \rangle^{1/2}$
  - Lanczos coefficients $\{b_n\}$: to ensure orthonomality $\langle O_m, O_n \rangle = \delta_{mn}$
  - Natural choices of $\langle . \rangle$: (i) $\langle A, B \rangle = Tr\,(A^{+} B)$; (ii) $\langle A, B \rangle = Tr\,(e^{-\frac{\beta}{2}H} A^{+}\, e^{-\frac{\beta}{2}H} B)$
  - Non-Hermitian Hamiltonian dynamics: diagonal terms

- Krylov complexity: "average size" in Krylov basis

  - $O(t) = e^{iHt}\, O\, e^{-iHt} = \sum_n \varphi_n(t)\, O_n\,, \quad K(t) = \sum_n n\, |\varphi_n(t)|^2$
  - emergent "wave" dynamics: $\dot{\varphi_n}(t) = b_n \varphi_{n-1}(t) - b_{n+1} \varphi_{n+1}(t)$
  - $K(t)$ completely determined by $\{b_n\}$, $t \gg 1 \leftrightarrow n \gg 1$
  - $\{b_n\}$ determined by auto-correlation function $G(t) = \langle O(t), O \rangle$ by moments expansion

- Exponential growth $K(t) \propto e^{2\alpha t}$: signature of quantum chaos?

  - $\lim_{t\to\infty} K(t) \propto e^{2\alpha t} \quad \leftrightarrow \quad \lim_{n\to\infty} b_n \propto \alpha\, n$
  - QM: saturation for large n ; $b_n = 0$ for $n \geq \dim H_{op}$;

Motivation

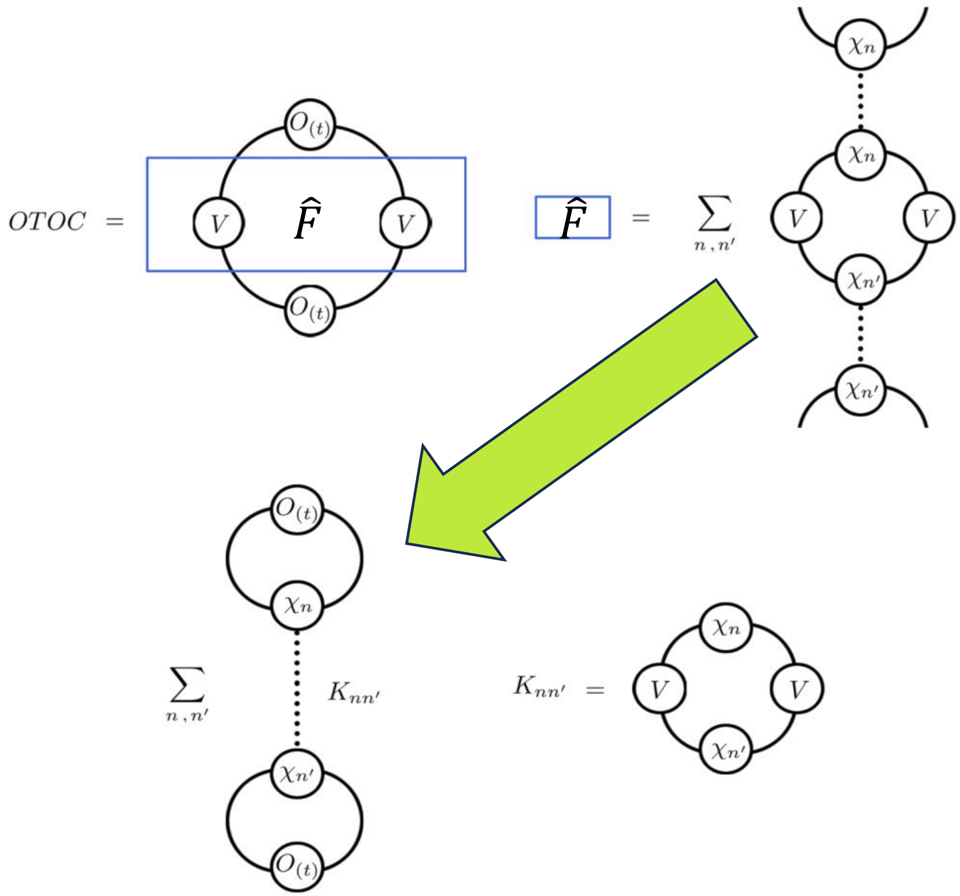
- Problems with Krylov complexity (as a measure of quantum chaos)
  - In general, it does not equal to the Lyapunov exponent: $\varkappa \leq 2\alpha$ (using trace norm)
  - Discrepancy at maximum: $\alpha > 0$ even for free CFTs $\varkappa = 0$
  - Question: to what extent does Krylov complexity reflect quantum chaos (defined by $\varkappa$)?
- Heuristic perspective:
  - Krylov complexity completely determined by auto-correlation functions: $G(t) = \langle O(t), O(0) \rangle$
  - Lanczos coefficients $\{b_n\}$ from spectral weight of $G(t)$: (i) finite sequence $\leftrightarrow$ discrete spectrum; (ii) saturation $\leftrightarrow$ bounded spectrum; (ii) linear growth $\leftrightarrow$ exponential decaying weight
  - They reflect properties of: (i) energy spectrum; (ii) matrix elements + norm
  - Relation to OTOC is obscure, if not independent (e.g. CFTs with finite β norm)
- Key insights: why does 2α overestimate $\varkappa$?
  - $K(t) = \sum n \, |\varphi_n(t)|^2$: assigning "chaotic measure" $= n$ for the Krylov basis O_n
  - In SYKs, $F(t) = \langle \widehat{N} \rangle_{O(t)} = \sum_n \widehat{N} \, (O_n, O_m) \, \varphi_n(t)^* \varphi_m(t)$, $\widehat{N}$ is the operator-size super-operator
  - $K(t) \;\; v.s. \;\; F(t) \;\; \longrightarrow \;\; \widehat{N}(O_n, O_m) \;\; v.s. \;\; n \, \delta_{mn}$
  - Intuition: overestimation in free theories because $\langle \widehat{N} \rangle_{O_n} = 1 < n$
  - Goal of the work, understand this comparison in more general terms

Outline

- General framework: Krylov metric
- Example: SYK model
- Example: Luttinger liquid
- Example: MBL
- Operator size distribution
- Conclusion and Outlook

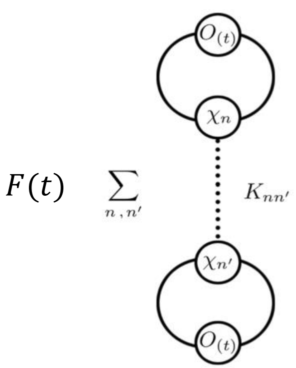
General framework: Krylov metric

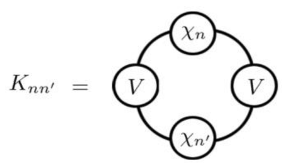
- OTOC as a super-operator

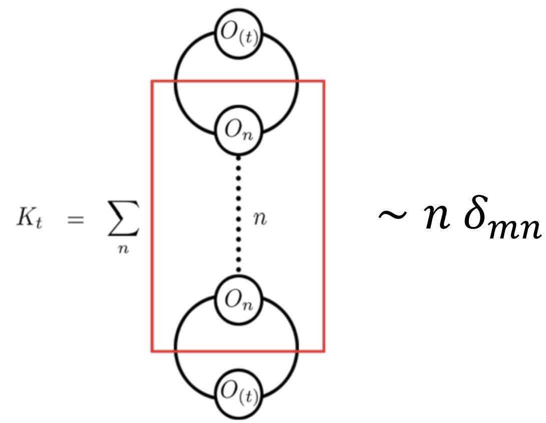


General framework: Krylov metric

- OTOC as a super-operator
- Krylov complexity as a super-operator







General framework: Krylov metric

- OTOC as a super-operator
- Krylov complexity as a super-operator
- Thus, OTOC in terms of the Krylov metric K_{mn}
- Krylov metric: "bridge" between Krylov dynamics and OTOC
- Encodes what is missed from the Krylov complexity

$$F(t_1, t_2) = \sum_{m,n} K_{mn}\, \varphi_m(t_1)^* \varphi_n(t_2)\,, \qquad K_{mn} = \langle O_m\, V\, O_n\, V \rangle$$

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- Krylov metric: "bridge" between Krylov dynamics and OTOC
- Encodes what is missed from the Krylov complexity
- Hypothesis for fast scrambler (quantum chaos):
  - Krylov complexity grows exponentially in time, i.e. $\alpha > 0$
  - Off-diagonal elements parametrically suppressed: $K_{mn} \ll K_{nn}$
  - Diagonal metric: power-law growth, i.e. $K_{nn} \propto n^h,\ \varkappa = 2\alpha h$
- Examine these hypothesis in models with known chaotic properties

General framework: Krylov metric

- Intuition for the hypothesis:
  - exponentially growing Krylov complexity: $\alpha > 0 \longrightarrow b_n \propto \alpha n$ for $n \gg 1$
  - Krylov dynamics with such b_n, wave eqn: $\dot{\varphi}_n(t) \approx 2\alpha n\, \partial_n \varphi_n(t) \to \varphi_n(t) \propto f(n - \mathrm{e}^{2\alpha t})$
  - At $t = 0$, $\varphi_n(0) = \delta(n)$, so roughly: $\varphi_n(t) \sim \delta(n - e^{2\alpha t})$
  - If the Krylov metric satisfy: (i) diagonally dominant; (ii) $K_{nn} \sim n^h$
  - $F(t) = \sum_{mn} K_{mn}\, \varphi_n(t)^* \varphi_m(t) \approx \sum_n K_{nn}\, \delta\left(n - e^{2\alpha t}\right) \sim e^{2\alpha h t}$

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  - **SYK models (coupled to heat-bath)**
  - **Luttinger liquids**
  - **Many-body localized (MBL) systems**

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  - **SYK models (coupled to heat-bath)**
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  - **Many-body localized (MBL) systems**
- Tricks for computing Krylov metric (valid in all examples)
  - Known results for OTOC: $F(t_1, t_2)$
  - Special form of the Krylov wave-functions: $\varphi_n(t) \propto h(t)\, y(t)^n$
  - $F(t_1, t_2) = F(y_1, y_2) = \sum_{mn} K_{mn}\, h(y_1)^* h(y_2)\, y_1^m\, y_2^n,\ y_{1,2} = y(t_{1,2})$
  - Krylov metric obtained from double series expansion of $F(y_1, y_2) h(y_1)^{-1} h(y_2)^{-1}$
  - Equivalently via contour integrals: $K_{mn} = \frac{1}{4\pi^2} \oint dy_1 \oint dy_2 \frac{F(y_1, y_2)}{y_1^{(m+1)} y_2^{(n+1)} h(y_1) h(y_2)}$

Outline

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SYK couple to bath

$$H = \sum_{i<j<k<l} J_{ijkl}\,\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k\hat{\chi}_l + \sum_{a<b<c<d} J'_{abcd}\,\hat{\psi}_a\hat{\psi}_b\hat{\psi}_c\hat{\psi}_d + \sum_{i<j}\sum_{a<b} u_{ijab}\,\hat{\chi}_i\hat{\chi}_j\hat{\psi}_a\hat{\psi}_b$$

$$M \gg N$$

$$\overline{J_{ijkl}^2} = \frac{6J^2}{N^3} \quad \overline{J'^2_{abcd}} = \frac{6J^2}{M^3} \quad \overline{u_{ijab}^2} = \frac{2u^2}{NM^2}$$

- The model has been studied under large- N and low energy limit with $\beta J \gg 1$
- For $\hat{\mathcal{O}} \sim \hat{\chi}_1, G(t) = (\cosh(\alpha t))^{-2\Delta}$, $\alpha = \pi/\beta, \Delta = 1/4$, independent u/J
- Krylov dynamics (finite β) coincides with original SYK: Krylov exponent α

$$b_n = \alpha\sqrt{n(n+2\Delta-1)}\,, \quad \varphi_n(t) = D_n \frac{\tanh^n(\alpha t)}{\cosh^{2\Delta}(\alpha t)}\,, \quad D_n = \sqrt{\frac{\Gamma(2\Delta+n)}{\Gamma(n+1)\Gamma(2\Delta)}}$$

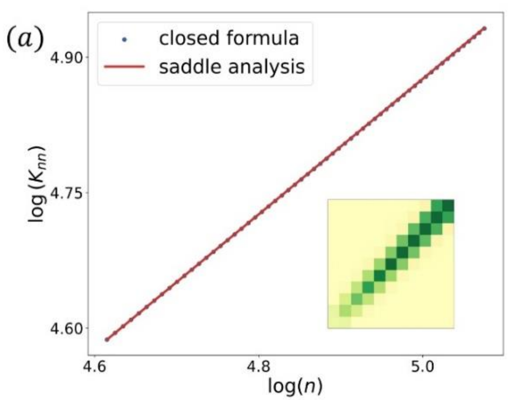
- $F(t_1,t_2) = f(t_{12})e^{\varkappa T_{12}} = C_0 \frac{e^{2\alpha h T_{12}}}{\cosh^{2\Delta+h}(\alpha t_{12})},\ h = \left(1 - \frac{\sqrt{k^4+4k^2}-k^2}{2}\right),\ k = u^2/J^2$
- Lyapunov exponent $\varkappa = 2\alpha h$
- As $k \to \infty$, $\varkappa \to 0$ system transits into non-chaotic dissipative phase

Example: SYK model

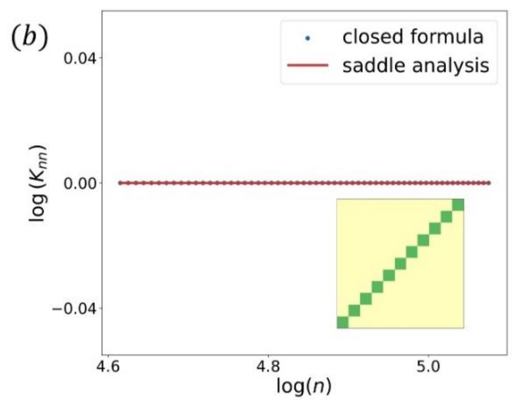
- Krylov metric can be computed analytically:
  - $\varphi_n(t) = D_n \frac{\tanh^n(\alpha t)}{\cosh^{2\Delta}(\alpha t)} = h(t)y(t)^n,\ y(t) = \tanh(\alpha t)$
  - $\cosh^{2\Delta}(\alpha t_1)\cosh^{2\Delta}(\alpha t_2)\, F(t_1, t_2) \propto \frac{(1+y_1)^h(1+y_2)^h}{(1-y_1 y_2)^{h+2\Delta}} = \sum_{mn} D_m D_n K_{mn} y_1^m y_2^n$
  - $K_{mn} = \sqrt{\frac{\Gamma(2\Delta)}{\Gamma(n+1)\Gamma(2\Delta+n)}}\sqrt{\frac{\Gamma(2\Delta)}{\Gamma(m+1)\Gamma(2\Delta+m)}}\Gamma(h+1)^2 \times \frac{{}_3F_2(-m,-n,h+2\Delta;h-m+1,h-n+1;1)}{\Gamma(h-m+1)\Gamma(h-n+1)}$
- Asymptotic behavior from saddle-point analysis:
  - Coefficients from contour integral: $K_{mn} = D_m^{-1} D_n^{-1} \oint \frac{dy_1}{y_1^{n+1}} \oint \frac{dy_2}{y_2^{m+1}} \frac{(1+y_1)^h(1+y_2)^h}{(1-y_1 y_2)^{h+2\Delta}}$
  - Asymptotic limit $m, n \gg 1$, semiclassical limit of effective action: $S_{eff} = (n+1)\ln y_1 + (\mathrm{m} + 1)\ln \mathrm{y}_2 + \cdots$
  - Leading order behavior from saddle-point approximation: finding (y_1^*, y_2^*) minimizing S_{eff}
  - $y_1^* = -\frac{h^2+2(n-m)(m+\Delta)\pm\sqrt{h^4+4(n-m)^2\Delta^2+4h^2(mn+(m+n)\Delta)}}{2(h+n-m)(m+2\Delta)}, y_2^* = -\frac{h^2+2(m-n)(n+\Delta)\pm\sqrt{h^4+4(n-m)^2\Delta^2+4h^2(mn+(m+n)\Delta)}}{2(h+m-n)(n+2\Delta)}$

Example: SYK model

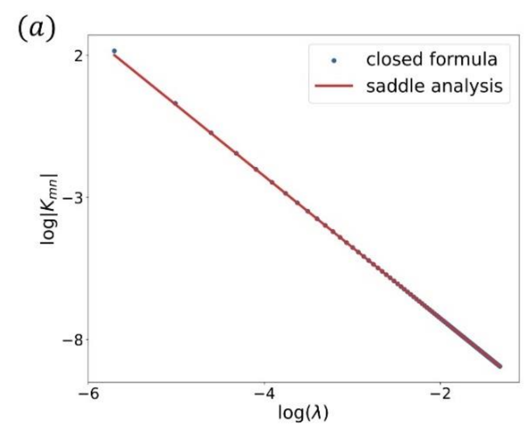
- Summary of asymptotic behaviors:
  - For diagonal $m = n \gg 1$: $\boldsymbol{K_{nn} \sim n^h}$
  - For (parametric) off-diagonal $n = L(1-\lambda), m = L(1+\lambda), L \gg 1$: $\boldsymbol{K_{n+m,n-m} \propto K_{nn} m^{-2h-1}}$
  - Intriguingly: $\frac{K_{n+1,n}}{K_{nn}} \propto h(1-h)$ as $h \to 1, 0$, Krylov metric approaches exactly diagonal there



$h = 3/4$



$h = 0$



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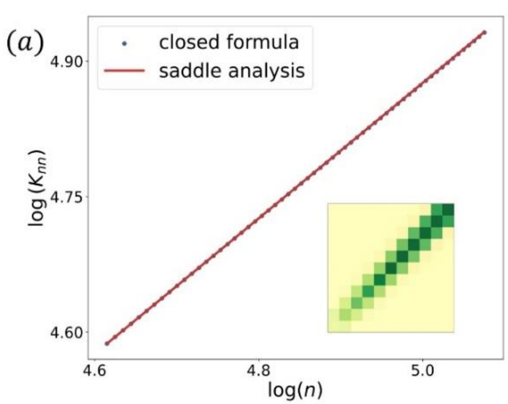
  - Intriguingly: $\frac{K_{n+1,n}}{K_{nn}} \propto h(1-h)$ as $h \to 1, 0$, Krylov metric approaches exactly diagonal there
- Checking against the hypothesis (chaotic for $h \neq 0$):
  - Krylov complexity: grows exponentially in time ✓
  - Off-diagonal elements: parametrically suppressed ✓
  - Diagonal metric: power-law growth ✓



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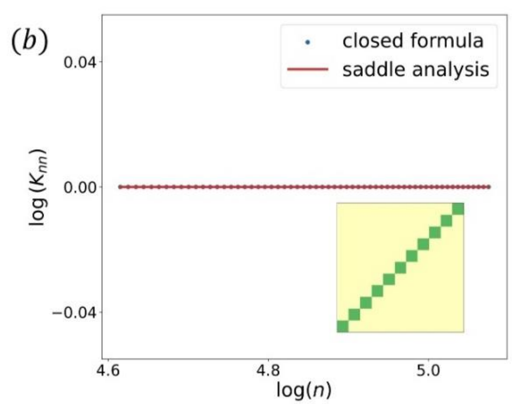

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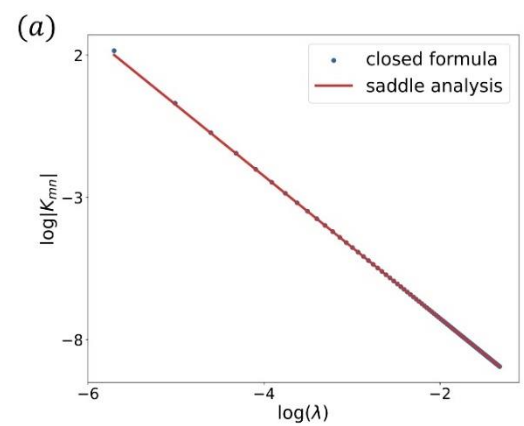

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Example: Luttinger liquid

Giamarchi, Quantum Physics in One Dimension

$$H = \frac{u}{2\pi}\int dx\left[\frac{1}{K}\left(\nabla\phi(x)\right)^2 + K\left(\pi\Pi(x)\right)^2\right] = \frac{u}{2\pi}\int dx\left[\frac{1}{K}\left(\nabla\phi(x)\right)^2 + K\left(\nabla\theta(x)\right)^2\right]$$

$$\left[\phi(x), \frac{1}{\pi}\nabla\theta(x')\right] = i\delta(x - x') \qquad \nabla\theta(x)/\pi = \Pi(x)$$

- Finite temperature correlation functions between general vertex operator:
  - $I = \left\langle \prod_j e^{iA_j\phi(r_j)} \right\rangle_\beta = e^{\frac{1}{2}\sum_{i<j}[(A_iA_jK)F(r_i - r_j)]}$, $F(r) = \frac{1}{2}\log\left[\sinh^2\left(\frac{\pi x}{\beta u}\right) + \sin^2\left(\frac{\pi\tau}{\beta}\right)\right]$
- As a CFT, the auto-correlation functions are identical to SYK
- For $V_n(x,t) =: \exp\left(in\phi(x,t)\right)$, $\Delta = Kn^2/4$, OTOCs can be explicitly computed:

$$\boldsymbol{C(t_1, t_2) = \langle V_{-n}(t_1 - i3\,\beta/4)V_n(-i\,\beta/2)V_n(t_2 - i\,\beta/4)V_{-n}(0)\rangle_\beta}$$

$$\boldsymbol{= \left[\frac{\cosh(\alpha\, t_{12}) - i\sinh(\alpha\, T_{12})}{(\cosh(\alpha\, t_{12}) + i\sinh(\alpha\, T_{12}))\cosh(\alpha\, t_{12})}\right]^{2\Delta}}$$

$$\boldsymbol{\alpha = \pi/\beta\,, t_{12} = t_1 - t_2, T_{12} = t_1 + t_2}$$

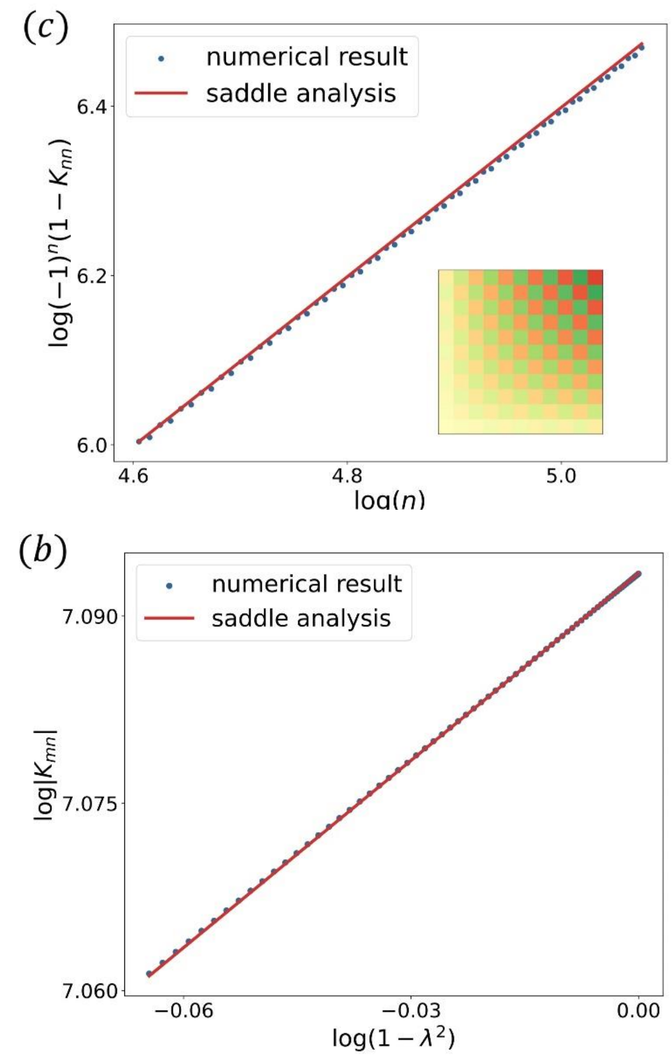
Example: Luttinger liquid

- Same Krylov wave function as before: $\varphi_n(t) = D_n \frac{\tanh^n(\alpha t)}{\cosh^{2\Delta}(\alpha t)} = h(t)y(t)^n$, the Krylov metric K_{mn} can be obtained by similar expansion technique:
  - $\sum_{m,n} \varphi_m(t_1)\varphi_n(t_2)K_{mn} = F(t_1,t_2) = \cosh^{-2\Delta}(\alpha t_{12})\left(1 - \left[\frac{\cosh(\alpha t_{12}) - i\sinh(2\alpha T_{12})}{\cosh(\alpha t_{12}) + i\sinh(2\alpha T_{12})}\right]^{2\Delta}\right)$
  - $K_{mn} = \delta_{mn} - D(m)^{-1}D(n)^{-1}\oint \frac{dy_1}{y_1^{m+1}}\oint \frac{dy_2}{y_2^{n+1}}\left[\frac{(1-iy_1)(1-iy_2)}{(1+iy_1)(1+iy_2)(1-y_1y_2)}\right]^{2\Delta}$
- Unfortunately, analytic expression for K_{mn} cannot be yielded.
- Saddle-point methods for extracting asymptotic behavior:
  - For generic $m, n \gg 1$: $\boldsymbol{K_{mn} \sim (-1)^{\frac{m+n}{2}}(mn)^{\Delta-1/2}}$
  - Along the diagonal: $\boldsymbol{K_{nn} \sim (-1)^n\, n^{2\Delta-1}}$
  - Along off-diagonal (orthogonal): $\boldsymbol{K_{n(1-\lambda),n(1+\lambda)} \sim K_{nn}\left(1-\lambda^2\right)^{\Delta-1/2}}$

Example: Luttinger liquid

Krylov metric results

- K_{nm} alternate in phases of $\pi/2$, like a "check box"
- Off-diagonal comparable to diagonal, i.e. no parametric suppression
- Lead to interference/cancellations, despite power-law increase along diagonal



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Krylov metric results

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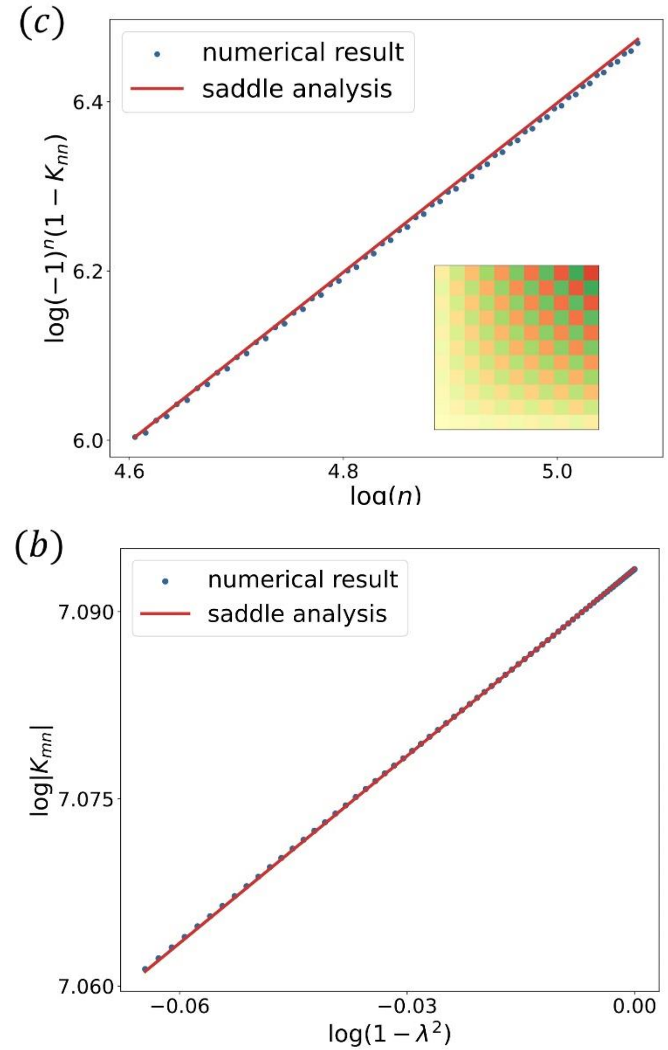
- Off-diagonal comparable to diagonal, i.e. no parametric suppression
- Lead to interference/cancellations, despite power-law increase along diagonal
- Checking against the hypothesis (integrable):
  - Krylov complexity: grows exponentially in time ✓
  - Off-diagonal elements: parametrically suppressed ✗
  - Diagonal metric: power-law growth ✓
- Distinction from free theory/dissipative SYK limit:
  - $K_{mn} \sim \delta_{mn}$



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Example: MBL

Effective Hamiltonian: Serbyn, Papic', Abanin, 2013, Huse, Nandkishore, Oganesyan, 2014, Vosk E. Altman, 2013

$$H = \frac{1}{2}\sum_{i\neq j} J_{ij}\sigma_z^i\sigma_z^j + \sum_i h_i\sigma_z^i, \quad i,j \in [-N/2+1, N/2], \quad \text{with } N \to \infty \qquad \langle J_{ij}^2\rangle = J^2 \exp\left(\frac{|i-j|}{\xi}\right), \ \langle h_i^2\rangle = h^2$$

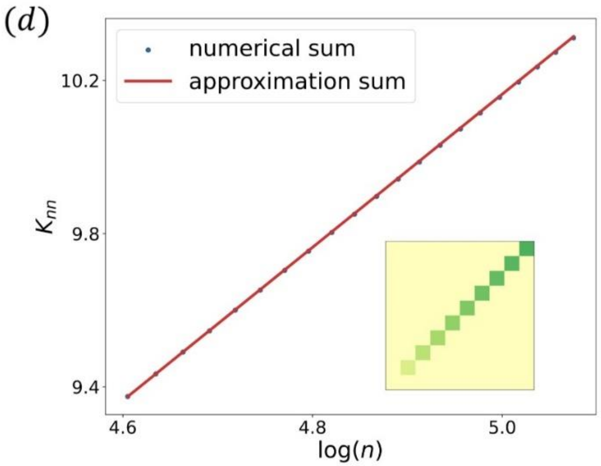
- σ_z^i: emergent conserved local charges
- Krylov basis constructed using $\langle A, B\rangle = \mathrm{Tr}\left(A^\dagger B\right)$, i.e. infinite temperature norm
- Auto-correlation function computed from multiple tricks of pauli matrix identities:
  - $C(t) = \langle \sigma_x^0(t), \sigma_x^0(0)\rangle = \mathrm{Tr}\left(e^{iHt}\sigma_x^0 e^{-iHt}\sigma_x^0\right) = \prod_{j\neq 0}\cos\left(2J_{0j}t\right)\cos(2h_0 t)$
  - Ensemble average: $\overline{C(t)} = \prod_{j\neq 0}\overline{\cos\left(2J_{0j}t\right)}\ \overline{\cos(2h_0 t)} = e^{-\frac{\gamma^2}{2}t^2}, \ \gamma^2 = 4J^2\sum_{j\neq 0} e^{-|j|/\xi} + 4h^2$
- Krylov dynamics for Gaussian auto-correlation function are also known analytically:
  - $\boldsymbol{C(t) = e^{-\frac{\gamma^2 t^2}{2}}} \quad \rightarrow \quad \boldsymbol{b_n = \gamma\sqrt{n}} \quad \rightarrow \quad \boldsymbol{\varphi_n(t) = e^{-\frac{\gamma^2 t^2}{2}}\frac{\gamma^n t^n}{\sqrt{n!}}}$
- Also of the form: $\varphi_n(t) \sim h(t) y(t)^n$; expansion technique for computing K_{mn} can still be applied, if OTOTC is known.

Example: MBL

- OTOC defined as: $F(t_1,t_2) = -\frac{1}{N}\sum_m \mathrm{Tr}[\sigma_x^0(t_1),\sigma_x^m][\sigma_x^0(t_2),\sigma_x^m]$
- Through (more) tricks of pauli matrix identities, can be computed:
  - $F(t_1,t_2) = C(T_{12}) + \sum_{m\neq 0}\cos(2J_{0m}T_{12})\prod_{j\neq 0,m}\cos\left(2J_{0j}t_{12}\right)\cos\left(2h_0(t_{12})\right)$
  - Ensemble average: $\overline{F(t_1,t_2)} = 2e^{-\frac{\gamma^2}{2}t_{12}^2} - \frac{2}{N}e^{-\frac{\gamma^2}{2}T_{12}^2} - \frac{2}{N}\sum_{m\neq 0}\exp\left(-8J^2e^{-|m|/\xi}t_1t_2\right)e^{-\frac{\gamma^2}{2}t_{12}^2}$
  - For late time $J^2t_1t_2/\xi \gg 1$, approximate $8J^2e^{-|m|/\xi}t_1t_2 \leq 1$ by 1; others by 0
  - $\sum_{m\neq 0}\exp\left(-8J^2e^{-|m|/\xi}t_1t_2\right) \approx (N-1) - 2\xi\log\left(8J^2t_1t_2\right)$
  - $\boldsymbol{\overline{F(t_1,t_2)} \approx \frac{2}{N}e^{-\frac{\gamma^2}{2}t_{12}^2} - \frac{2}{N}e^{-\frac{\gamma^2}{2}T_{12}^2} + \frac{2}{N}\xi\ln\left(8J^2t_1t_2\right)e^{-\frac{\gamma^2}{2}t_{12}^2}}$
- For MBL systems, OTOC grows logarithmically: $F(t) \sim \frac{4\,\xi}{N}\ln t$
- $\sum_{m,n} t_1^m t_2^n K_{mn} = \frac{\sqrt{m!n!}}{\gamma^{m+n}}e^{\frac{\gamma^2}{2}(t_1^2+t_2^2)}\,\overline{F(t_1,t_2)} \approx \frac{2\sqrt{m!n!}}{N\gamma^{m+n}}\left(e^{\gamma^2t_1t_2} - e^{-\gamma^2t_1t_2} + 2\xi\ln\left(8J^2t_1t_2\right)e^{\gamma^2t_1t_2}\right)$
- RHS depends only on $t_1t_2 \rightarrow\ K_{mn} = K_{nn}\delta_{mn}$

Example: MBL

- Diagonal elements only, obtained from a single contour integral:
  - $K_{nn} \approx \frac{2}{N} - \frac{2(-1)^n}{N} + \frac{2\xi n!}{N} \oint \frac{dx}{x^{n+1}} \ln\left(\frac{8J^2}{\gamma^2} x\right) e^x$
- Asymptotic behavior from saddle-point approximation:
  - $\boldsymbol{K_{nn} \sim \frac{2\xi}{N} \ln\left(\frac{8J^2}{\gamma^2} n\right)}$



Example: MBL

- Diagonal elements only, obtained from a single contour integral:
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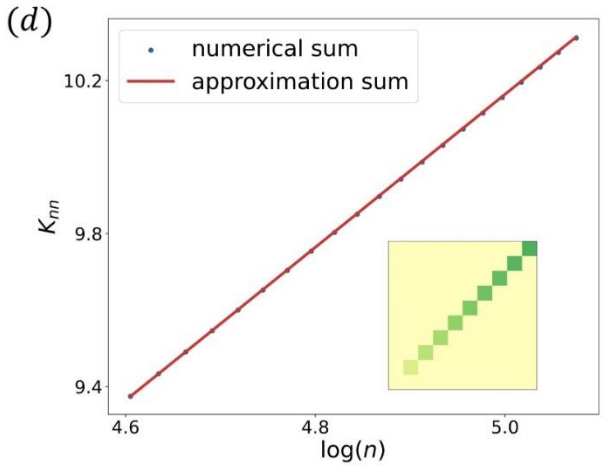
- Asymptotic behavior from saddle-point approximation:
  - $\boldsymbol{K_{nn} \sim \frac{2\xi}{N} \ln\left(\frac{8J^2}{\gamma^2} n\right)}$
- Checking against the hypothesis (non-scrambling):
  - Krylov complexity: grows exponentially in time ✗
  - Off-diagonal elements: parametrically suppressed ✓
  - Diagonal metric: power-law growth ✗



Outline

- General framework: Krylov metric
- Example: SYK model
- Example: Luttinger liquid
- Example: MBL
- Operator size distribution
- Conclusion and Outlook

Operator-size distribution

- In some models (SYK, MBL), the OTOCs describe operator-size growth:
  - $\chi_0(t) = \sum_{\vec{i}} \psi_{i_1,\ldots,i_n}(t)\, \chi_{i_1} \ldots \chi_{i_n} \to \ \sum_i \ Tr\, \{\chi_0(t),\ \chi_i\}\{\chi_0(t),\ \chi_i\} = \sum_{\vec{i}} n \left|\psi_{i_1 \ldots i_n}(t)\right|^2$
- In these models, Krylov metric = matrix elements of size super-operator $\hat{N}$:
  - $K_{mn} \ = \langle \mathcal{O}_m | \hat{N} | \mathcal{O}_n \rangle,\ \hat{N} |\, \mathcal{O}_n \,\rangle = \ n \ |\, \mathcal{O}_n \,\rangle,\ \ |\mathcal{O}_n\rangle = |\chi_{i_1} \ldots \chi_{i_n}\rangle \ or \ \left|\sigma^{i_1}_{\alpha_1} \ldots \sigma^{i_n}_{\alpha_n}\right\rangle,\ \ \alpha_i \in \{x, y, z\}$
- Hyper-fine structure of Krylov metric:
  - Resolve into distribution of operator-size ℓ: $I \to \sum_\ell \hat{P}(\ell)$
  - $K_{mn} = \sum_\ell K_{mn}(\ell)\,,\quad K_{mn}(\ell) = \langle \mathcal{O}_m |\ \hat{N} \hat{P}(\ell)\ | \mathcal{O}_n \rangle = \ell\ \hat{P}(\ell)_{mn}$
  - May shed more light on the relation between scrambling and Krylov dynamics
- Operator-size distribution from generating function
  - Define: $Z(t_1, t_2, \mu) = \left\langle \mathcal{O}(t_1) \right|\ e^{-\mu \hat{N}}\ \left| \mathcal{O}(t_2) \right\rangle = \sum_{l,m,n} e^{-\ell\mu} \hat{P}(\ell)_{mn}\ \ \varphi_m(t_1)\ \varphi_n(t_2)^*$
  - μ is "chemical potential" for operator-size
  - $\hat{P}(\ell)_{mn}$ from $Z(t_1, t_2, \mu)$ by: (i) conversion to Krylov basis; (ii) inverse Laplace transform on μ

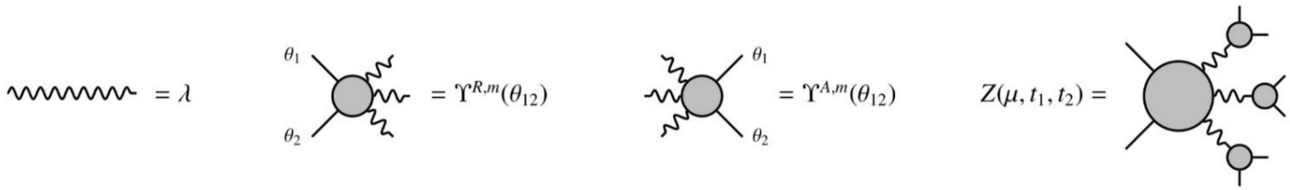
Operator size distribution

SYK TFD   Qi, Streicher, 2019

- Generating function at finite temperature (using finite β operator norm)
- Size-distribution in the operator ``smeared" by the thermal density matrix
  - $O \to \rho^{1/4} O \rho^{1/4}$
- Size super-operator $\widehat{N}$ defined as:
  - $\langle \chi | \widehat{N} | \gamma \rangle = \left\langle \chi\left(\frac{3i\beta}{4}\right) \left[\frac{1}{2} + \frac{1}{2}\sum_i \chi_i\left(\frac{i\beta}{2}\right)\chi_i(o)\right] \gamma\left(\frac{i\beta}{4}\right)\right\rangle_\beta$, out-of-time-ordered
- Generating function from re-summing multi-point OTOCs:
  - $Z(\mu, t_1, t_2) = e^{-\frac{\mu N}{2}} \sum_n \frac{1}{n!} \left\langle \mathcal{T} \chi_1\left(t_1 + \frac{3i\beta}{4}\right)\left[-\frac{\mu}{2}\sum_i \chi_i\left(\frac{i\beta}{2}\right)\chi_i(0)\right]^n \times \chi_1\left(t_2 + \frac{i\beta}{4}\right)\right\rangle_\beta$
- In generic context, computing this quantity is impossible!

Operator size distribution

Scramblon mode   Gu, Kitaev, Zhang, 2022



- For models with near-maximal chaos, the relevant dynamics captured by the effective description in terms of the "scramblon" mode:
  - scramblon mode propagator $\lambda = -\frac{e^{i\frac{\varkappa}{2}(\pi+\theta_3+\theta_4-\theta_1-\theta_2)}}{C}$, $\varkappa$ = Lyapunov exponent, $\varkappa \leq 1$
  - Advanced/retarded vertex function: $\Upsilon^{R/A,m}(\theta_{12}) = \Upsilon^{m}(\theta_{12})$
  - Explicit form of $\Upsilon^{m}(\theta_{12})$ known for maximally chaotic SYK model
  - Stringy corrected form of $\Upsilon^{m}(\theta_{12})$ can also be derived, with $\varkappa$<1 (D. Stanford, et al 2017)
- For $t \ll \log N$, dynamics dominated by single-scramblon propagation:

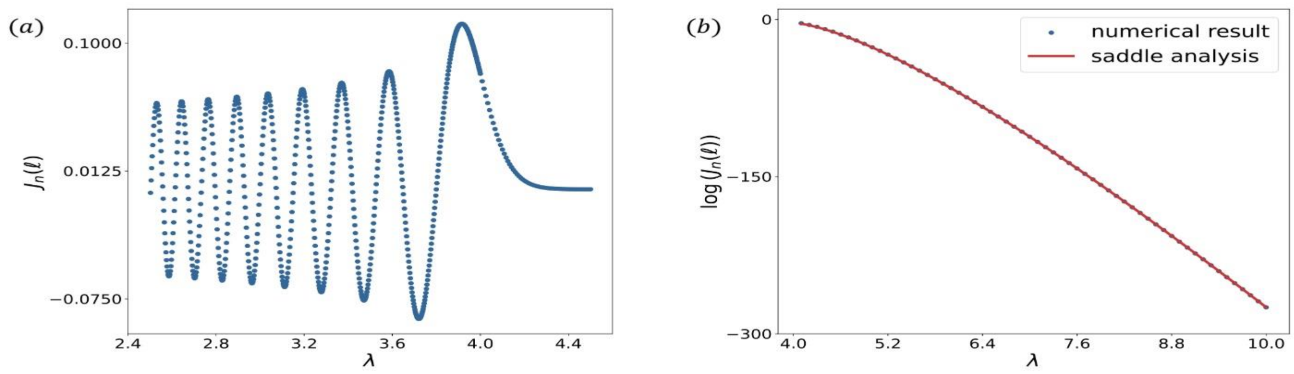
$$Z(t_1, t_2, \mu) = e^{-\mu N\left(\frac{1}{2}-G(\theta_{34})\right)} \sum_{m=0}^{\infty} \Upsilon^{m}(\theta_{12}) \left(\frac{\lambda \mu N}{2}\right)^{m} \frac{1}{m!} \Upsilon^{1}(\theta_{34})^{m}$$

Operator size distribution

- Plugging in the explicit ingredients, the generating function can be re-summed:
  - $Z(t_1,t_2,\mu) = \frac{e^{-\mu N\left(\frac{1}{2}-G\right)G}}{\Gamma(2\Delta)} \int_0^\infty \frac{dy}{\varkappa}\, y^{\frac{2\Delta}{\varkappa}-1} \exp\left[-\mu K e^{\frac{\varkappa\pi(t_1+t_2)}{\beta}} y - \cosh\left(\frac{\pi t_{12}}{\beta}\right) y^{1/\varkappa}\right]$
  - Valid for generic $\varkappa \leq 1$, stringy effects: $y \to y^{1/\varkappa}$
- Inverse Laplace transform:
  - $\hat{P}(t_1,t_2,\ell) = \frac{G\,\ell^{\widetilde{2\Delta/\varkappa}-1}}{\varkappa K^{2\Delta/\varkappa}\Gamma(2\Delta)} \exp\left[-\frac{2\pi\Delta}{\beta}(t_1+t_2) - \frac{1}{2}\left(\frac{\tilde{\ell}}{K}\right)^{1/\varkappa}\left(e^{-\frac{2\pi t_1}{\beta}} + e^{\frac{-2\pi t_2}{\beta}}\right)\right] = Q_\ell(t_1) \times Q_\ell(t_2)$
  - $Q_\ell(t) = \frac{\tilde{\ell}^{\Delta/\varkappa-1/2}}{\sqrt{\varkappa}K^{\Delta/\varkappa}} \sqrt{\frac{G}{\Gamma(2\Delta)}}\, e^{-\frac{2\pi\Delta}{\beta}t - \frac{1}{2}\left(\frac{\tilde{\ell}}{K}\right)^{1/\varkappa} e^{-\frac{2\pi}{\beta}t}}$
- Factorized conversion into the Krylov basis: $K_{mn}(\ell) = \ell J_m(\ell) J_n(\ell)$
- Implication of factorization:
  - $\hat{P}(\ell)|O_m\rangle = J_m(\ell) \times |\Psi^\ell\rangle$: distinct O_n projects onto the same vector in $H_{op}(\ell)$
  - Intuitively, $|\Psi^\ell\rangle$ is the permutation invariant vector in $H_{op}(\ell)$
  - Consequence of permutation invariant Hamiltonian

Operator size distribution

- $J_n(\ell)$ = size wavefunction of the Krylov basis O_n
- Obtained from expansion trick in $Q_\ell(t)$:
  - $J_n(\ell) = \frac{\tilde{\ell}^{\Delta/\varkappa-1/2}}{\sqrt{\varkappa}K^{\Delta/\varkappa}}\sqrt{\frac{\Gamma(n+1)G}{\Gamma(2\Delta+n)}}\oint\frac{dy}{y^{n+1}}(1+y)^{-2\Delta}\,e^{-\frac{1}{2}\left(\frac{\tilde{\ell}}{K}\right)^{1/\varkappa}\left(\frac{1-y}{1+y}\right)}$
  - Asymptotic behavior from saddle-point approximation:
  - Define $(\ell/K)^{1/\varkappa}=\lambda n,\ \ J_n(\ell)\sim n^{-\varkappa/2}\left[e^{\frac{\sqrt{\lambda(\lambda-4)}}{2}}\left(\frac{\lambda-2-\sqrt{\lambda(\lambda-4)}}{2}\right)\right]^{-n}$
- Typical size of O_n at $\ell\sim Kn^{\varkappa}$
- Phase transition: (i) oscillatory for $\lambda<4$; (ii) exponential decay for $\lambda>4$



Operator size distribution

MBL

- Generating function at infinite temperature (trace norm)
  - $Z(t_1, t_2, \mu) = \mathrm{Tr}\left[\sigma_x^0(t_1) e^{-\mu \hat{N}} \sigma_x^0(t_1)\right]$
- Applying multiple tricks of pauli matrix identities, we get:
  - $Z(\mu, t_1, t_2) = e^{-\mu} \cos\left(2h_0(t_1 - t_2)\right) \prod_{j\neq 0}\left(\cos\left(2J_{0j}t_1\right)\cos\left(2J_{0j}t_2\right) + e^{-\mu}\sin\left(2J_{0j}t_1\right)\sin\left(2J_{0j}t_2\right)\right)$
- Ensemble average: $\overline{Z(\mu, t_1, t_2)} e^{\frac{\gamma^2}{2}(t_1^2 + t_2^2)} = e^{\gamma^2 t_1 t_2} e^{-\mu} \prod_{j\neq 0}\left(\frac{1+e^{-\mu}}{2} + \frac{1-e^{-\mu}}{2} e^{-8J^2 t_1 t_2 e^{-\frac{|j|}{\xi}}}\right)$
- RHS only depends on $t_1 t_2 \to K_{mn}(\ell) = K_{nn}(\ell)\delta_{mn}$
- Applying similar approximation: $\overline{Z(\mu, t_1, t_2)} e^{\frac{\gamma^2}{2}(t_1^2 + t_2^2)} \approx e^{\gamma^2 t_1 t_2} e^{-\mu}\left(\frac{1+e^{-\mu}}{2}\right)^{2\xi \ln\left(8J^2 t_1 t_2\right)}$
- $K_{nn}(\ell)$ follows the binomial distribution, which is asymptotically Gaussian:

$$K_{nn}(\ell) \sim \lambda \exp\left(-\frac{(\lambda - \xi)^2}{\xi} \ln n\right), \qquad \ell = \lambda \ln n$$

Operator size distribution

MBL

- Size distribution of O_n is gaussian with average and variance:
    - $\langle \ell \rangle \sim \xi \ln n\,, \ \ \langle \delta \ell^2 \rangle \sim \sqrt{\xi \ln n}$
- Implication of diagonality: $K_{mn}(\ell) = K_n \delta_{mn}$?
- For each $H_{op}(\ell)$, distinct $\{O_m, O_n\}$ project onto vectors that are orthogonal for all $m \neq n$
- A reflection of the underlying localization dynamics?

Outline

- General framework: Krylov metric
- Example: SYK model
- Example: Luttinger liquid
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- Operator size distribution
- Conclusion and Outlook

Conclusion and Outlook

- Krylov metric: bridge between quantum scrambling and Krylov dynamics
- Criteria for fast scrambler:
    1. Exponential growth in Krylov complexity
    2. Diagonal dominance of Krylov metric
    3. Asymptotic power-law growth along the diagonal
- Examined 3 examples:
    - SYK-coupled to heat-bath (chaotic) : satisfy all 1-3;
    - Luttinger-liquids (integrable): violates 2;
    - Many-body localized systems (non-scrambling): violates 1 and 3.
- For SYK and MBL: operator-size resolved Krylov metric

Conclusion and Outlook

- Criterion as sufficient condition, how necessary? More examples.
- Decomposition: $\varkappa = 2\alpha \times h$: different aspects of quantum chaos?
- Special status of Krylov basis: maximize “wave-like” dynamics?
- Extract more insights from the operator-size distribution
- Corresponding pictures in QFT, e.g. Luttinger liquids, relation between OTOCs and operator-size growth is obscure, alternative sign of Krylov metric a result of this?
- Novel aspects of quantum chaos probed by Krylov metric?

Thanks

Supplementary: saddle-point SYK

Asymptotic behavior

$$y_{1,2} = e^{\theta_{1,2}} \qquad K_{mn} = D(m)^{-1} D(n)^{-1} \int d\theta_1 d\theta_2 \; e^{-S(\theta_1,\theta_2)}$$

$$S(\theta_1,\theta_2) = m\theta_1 + n\theta_2 + (2\Delta + h)\ln\left(1 - e^{\theta_1+\theta_2}\right) - h\ln\left(1 + e^{\theta_1}\right) - h\ln\left(1 + e^{\theta_2}\right)$$

$$m = \frac{he^{\theta_1}}{1 + e^{\theta_1}} + \frac{(2\Delta + h)e^{\theta_1+\theta_2}}{1 - e^{\theta_1+\theta_2}} \qquad n = \frac{he^{\theta_1}}{1 + e^{\theta_2}} + \frac{(2\Delta + h)e^{\theta_1+\theta_2}}{1 - e^{\theta_1+\theta_2}}$$

$$e^{\theta_1^*} = -\frac{h^2 + 2(n-m)(m+\Delta) \pm \sqrt{h^4 + 4(n-m)^2\Delta^2 + 4h^2(mn + (m+n)\Delta)}}{2(h+n-m)(m+2\Delta)}$$

$$e^{\theta_2^*} = -\frac{h^2 + 2(m-n)(n+\Delta) \pm \sqrt{h^4 + 4(n-m)^2\Delta^2 + 4h^2(mn + (m+n)\Delta)}}{2(h+m-n)(n+2\Delta)}$$

Supplementary: saddle-point SYK

Asymptotic behavior

$$\boldsymbol{n = L(1+\lambda)\,,\, m = L(1-\lambda)\,,\, 0<\lambda<1\,,\ \ large\ \ L} \qquad \boldsymbol{K_{mn} \sim L^{-h-1}\lambda^{-2h-1}}$$

$$\theta_1^* = i\pi - \frac{2\Delta\lambda + h(1-\lambda) - \sqrt{h^2(1-\lambda^2)+4\Delta\lambda^2}}{2L\lambda(1-\lambda)} + \cdots \qquad \theta_2^* = i\pi - \frac{2\Delta\lambda - h(1+\lambda) + \sqrt{h^2(1-\lambda^2)+4\Delta\lambda^2}}{2L\lambda(1+\lambda)} + \cdots$$

$$K_{mn} \sim L^{-h+1}\lambda^{-2h} \times (\text{fluctuation})$$

Near (θ_1^*, θ_2^*), $S(\theta_1,\theta_2) = S^* + A_+ L^2 \delta\theta_+^2 + A_- L^2\lambda^2\delta\theta_-^2 + \sum_{p+q\geq 3} S_{pq}\ \delta\theta_+^p\ \delta\theta_-^q,\ \ S_{pq} \sim L^{p+q}\lambda^q$

$A_\pm$ are $\mathcal{O}(1)$ constant, $\delta\theta_\pm$ are eigen modes of the Hessian matrix. At $L \gg 1$,

$\delta\theta_+ = \delta\theta_1 + \delta\theta_2\ \ \delta\theta_+ = \delta\theta_1 - \delta\theta_2\ \ \ \delta\theta_{1,2} = \theta_{1,2} - \theta_{1,2}^*$ Rescaling $\delta\theta_+ = \delta\tilde\theta_+/L \quad \delta\theta_- = \delta\tilde\theta_-/(L\lambda)$

$$\int_{-\pi}^{\pi} d\delta\theta_{+} \int_{-\pi}^{\pi} d\delta\theta_-\ \ e^{-\sum_{p,q} S_{pq}\ \delta\theta_+^p\delta\theta_-^q} = \frac{1}{L^2\lambda}\left(\int_{-\infty}^{\infty} d\,\delta\tilde\theta_+ \int_{-\infty}^{\infty} d\,\delta\tilde\theta_-\ e^{-\sum_{p,q}\tilde S_{pq}\ \delta\tilde\theta_+^p\delta\tilde\theta_-^q}\right)$$

$$\boldsymbol{n = m = L} \qquad\qquad \boldsymbol{K_{nn} \sim n^h}$$

$$\theta_1^* = \theta_2^* = -\frac{h+2\Delta}{2L} + \cdots \qquad K_{nn} \sim n^{h+1} \times (\text{fluctuation})$$

$S(y_1,y_2) = S^* + \sum_{p,q} S_{pq}\delta\theta_+^p\delta\theta_-^q\,,\quad S_{pq} \sim L^p$ Rescaling $d\theta_+ = d\tilde\theta_+/L$

$$\boldsymbol{K_{nn} \sim n^h,\qquad K_{mn} \sim K_{LL}|m-n|^{-2h-1},\qquad L = \frac{m+n}{2}}$$

Supplementary: saddle point Luttinger liquid

Saddle point approximation

$$\sum_{m,n} \varphi_m(t_1)\varphi_n(t_2)K_{mn} = F(t_1,t_2) = \cosh^{-2\Delta}(\alpha t_{12})\left(1-\left[\frac{\cosh(\alpha t_{12}) - i\sinh(2\alpha T_{12})}{\cosh(\alpha t_{12}) + i\sinh(2\alpha T_{12})}\right]^{2\Delta}\right)$$

$$\alpha = \pi/\beta\,, t_{12} = t_1 - t_2, T_{12} = t_1 + t_2$$

$$K_{mn} = \delta_{mn} - D(m)^{-1}D(n)^{-1}\oint \frac{dy_1}{y_1^{m+1}}\oint \frac{dy_2}{y_2^{n+1}}\left[\frac{(1-iy_1)(1-iy_2)}{(1+iy_1)(1+iy_2)(1-y_1y_2)}\right]^{2\Delta}$$

$$= \delta_{mn} - D(m)^{-1}D(n)^{-1}\int d\theta_1 d\theta_2\ e^{-S(\theta_1,\theta_2)}$$

$$m + \frac{2i\Delta}{\cosh\theta_1} + \frac{2\Delta}{1-e^{-(\theta_1+\theta_2)}} = 0 \qquad n + \frac{2i\Delta}{\cosh\theta_2} + \frac{2\Delta}{1-e^{-(\theta_1+\theta_2)}} = 0$$

In the limit of $m = L(1-\lambda),\ \ n = L(1+\lambda),\ \ L \gg 1$

$$\theta_1^* = -\frac{\pi i}{2} + \frac{2i\Delta}{1-\lambda}L^{-1} + \mathcal{O}\left(L^{-2}\right) \qquad \theta_2^* = -\frac{\pi i}{2} + \frac{2i\Delta}{1+\lambda}L^{-1} + \mathcal{O}\left(L^{-2}\right)$$

neglecting the subdominant δ_{mn} term

$$\boldsymbol{K_{mn} \sim (-1)^L L^{2\Delta+1}\left(1-\lambda^2\right)^{\Delta+1/2} \times \text{(fluctuation)}}$$

$$\boldsymbol{= (-1)^L L^{2\Delta-1}\left(1-\lambda^2\right)^{\Delta-1/2} = (-1)^{\frac{m+n}{2}}(mn)^{\Delta-1/2}}$$

alternating sign factor $(-1)^L$ comes from the imaginary leading order terms of $\theta_{1,2}^*$

Supplementary: saddle point MBL

Saddle point approximation

$$\sum_{m,n} t_1^m t_2^n K_{mn} = \frac{\sqrt{m!\,n!}}{\gamma^{m+n}} e^{\frac{\gamma^2}{2}(t_1^2+t_2^2)}\,\overline{F(t_1,t_2)} \approx \frac{2\sqrt{m!\,n!}}{N\gamma^{m+n}}\left(e^{\gamma^2 t_1 t_2} - e^{-\gamma^2 t_1 t_2} + 2\xi \ln\left(8J^2 t_1 t_2\right) e^{\gamma^2 t_1 t_2}\right)$$

$$K_{mn} = K_{nn}\delta_{mn}$$

Do single contour integral $x = \gamma^2 t_1 t_2$

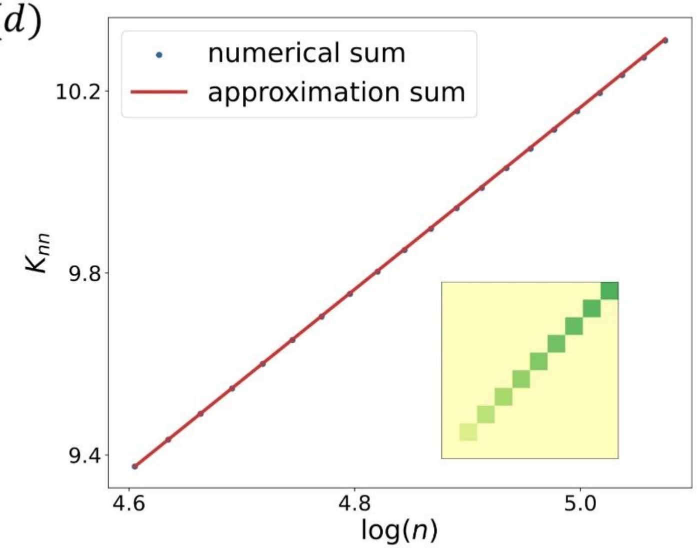
$$K_n \approx \frac{2}{N} - \frac{2(-1)^n}{N} + \frac{2\xi n!}{N}\oint \frac{dx}{x^{n+1}} \ln\left(\frac{8J^2}{\gamma^2}x\right) e^x$$

At $n \gg 1$ saddle point x^* satisfies

$$n + 1 = x^* + \ln\left(\frac{8J^2}{\gamma^2}x^*\right)^{-1} \rightarrow x^* \approx n$$

$$\boldsymbol{K_{nn} \approx \frac{2}{N} - \frac{2(-1)^n}{N} + \frac{2\xi}{N}\ln\left(\frac{8J^2}{\gamma^2}n\right) \approx \frac{2\xi}{N}\ln\left(\frac{8J^2}{\gamma^2}n\right)}$$

(d)



Supplementary: size-distribution SYK

Size generating function

Qi, Streicher, 2019

OTOCs are interpreted as operator spreading under time evolution

$$\hat{N}|\,\mathcal{O}_n\,\rangle = n\,|\,\mathcal{O}_n\,\rangle$$

$$|\chi_{i_1}\ldots\chi_{i_n}\rangle, \quad \left|\sigma^{i_1}_{\alpha_1}\ldots\sigma^{i_n}_{\alpha_n}\right\rangle, \quad \alpha_i \in \{x, y, z\}$$

$$F(t\text{'}\_1t\_2) = \langle\mathcal{O}|(t_1)\hat{N}|\mathcal{O}(t_2)\rangle \quad K_{mn} = \langle\mathcal{O}_m|\hat{N}|\mathcal{O}_n\rangle\text{'}$$

$$K_{mn} = \sum_{\ell} K_{mn}(\ell), \quad K_{mn}(\ell) = \langle\mathcal{O}_m|\,\hat{N}\hat{P}(\ell)\,|\mathcal{O}_n\rangle = \ell\,\hat{P}(\ell)_{mn}$$

$\hat{P}(\ell)$ is the super-projector into operator space with fixed operator size ℓ

$$Z(t_1, t_2, \mu) = \left\langle\mathcal{O}(t_1)\right|\, e^{-\mu\hat{N}}\,\left|\mathcal{O}(t_2)\right\rangle = \sum_{\mathrm{l},m,n} e^{-\ell\mu}\hat{P}(\ell)_{mn}\;\varphi_m(t_1)\;\varphi_n(t_2)^*$$

Supplementary: size-distribution SYK

SYK TFD

Qi, Streicher, 2019

$\{\chi_i, \chi_j\} = 2\delta_z$ In $H_L \otimes H_R$ $\hat{N} = \sum_i \frac{1}{2}(1 + i\chi_i^L \chi_i^R)$ $\{\chi_i^L, \chi_j^R\} = 0$

Using operator norm $\langle \alpha, \beta \rangle = \langle I |\, \alpha_L^\dagger \beta_L \, | I \rangle = \mathrm{Tr}(\alpha^\dagger \beta)$

$$|I\rangle \propto \Pi_{i=1}^N c_i^\dagger \, |\Omega\rangle, \quad c_i = \frac{1}{2}(\chi_i^L - i\chi_i^R)$$

$|\Omega\rangle = \Pi_{i=1}^N |\Omega\rangle_i^L \otimes |\Omega\rangle_i^R$ is the product state of the fermionic vacua for all $\chi_i^{L,R}$

$$(\chi_i^L + i\chi_i^R)|I\rangle = 2c_i^\dagger |I\rangle = 0 \qquad \hat{N}\chi_{i_1}^L \dots \chi_{i_n}^L |I\rangle = n\, \chi_{i_1}^L \dots \chi_{in}^L |I\rangle$$

$$|TFD\rangle \propto e^{-\frac{\beta}{4}(\hat{H}_L + \hat{H}_R)} \, |I\rangle$$

$$\langle \psi, \gamma_2 \rangle_{TFD} = \langle I | e^{-\frac{\beta}{8}(\hat{H}_L + \hat{H}_R)} \psi_L^\dagger e^{-\frac{\beta}{4}\hat{H}_L} e^{-\frac{\beta}{4}\hat{H}_L} \gamma_L e^{-\frac{\beta}{8}(\hat{H}_L + \hat{H}_R)} |I\rangle$$

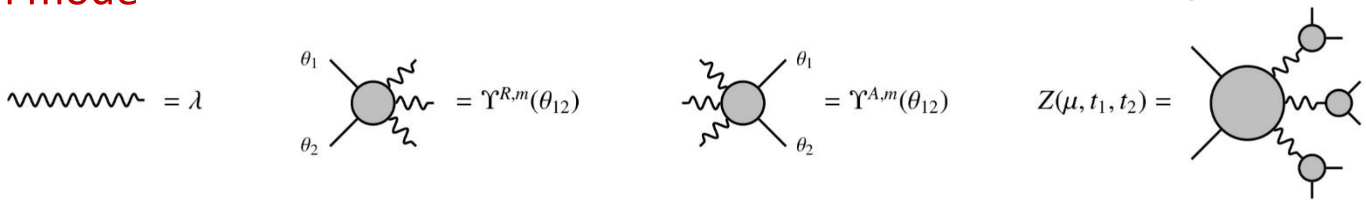
$$= \mathrm{Tr}(\rho^{1/4}\psi^\dagger \rho^{1/2}\gamma\rho^{1/4}), \quad \rho = e^{-\beta\hat{H}}$$

this can can be understood as measuring the original operator size in the operator ``smeared" by the thermal density matrix $O \to \rho^{1/4} O \rho^{1/4}$

Supplementary: size-distribution SYK

Scramblon mode

Gu, Kitaev, Zhang, 2022



scramblon mode propagator $\lambda = -\frac{e^{i\frac{\varkappa}{2}(\pi+\theta_3+\theta_4-\theta_1-\theta_2)}}{C}$

retarded/advanced vertex functions $\Upsilon^{R/A,m}(\theta_{12})$ $\theta_{ij} = \theta_i - \theta_j$ $\theta = \frac{2\pi}{\beta}(\tau + it)$

assuming time-reflection symmetry $\Upsilon_m^R = \Upsilon_m^A = \Upsilon_m$

at maximal chaos $\varkappa = 1$ $\Upsilon^m(\theta_{12}) = \int_0^\infty dy\ y^m\ h(y, \theta_{12})$

$$h(y,\theta_{12}) = \frac{G}{\Gamma(2\Delta)} y^{2\Delta-1} e^{-\Theta_{12} y},\ \ G = \frac{1}{2}\cos^{2\Delta}\left(\frac{\pi v}{2}\right),\ \ \Theta_{12} = \cos\left[\frac{v(\pi-\theta_{12})}{2}\right]$$

$$\Delta = 1/q \quad \frac{\pi v}{\cos\left(\frac{\pi v}{2}\right)} = \beta J$$

focus on the strong coupling limit $v \to 1$, $OTOC = \sum_{m=0}^{\infty} \Upsilon^m(\theta_{12}) \frac{(\lambda)^m}{m!} \Upsilon^m(\theta_{34})$

$$\theta_1 = \frac{2\pi i}{\beta} t + \frac{\pi}{2},\ \ \theta_2 = \frac{2\pi i}{\beta} t + \frac{3\pi}{2},\ \ \theta_3 = \pi,\ \ \theta_4 = 0$$

Supplementary: size-distribution SYK

$t \ll \log N$, $\lambda \propto e^{\varkappa t}/N \ll 1$ OTOC dominated by single scramblon exchange at $m = 1$

$$Z(\mu, t_1, t_2) = \left\langle \chi_1(t_1), e^{-\mu\hat{N}} \chi_1(t_2) \right\rangle_{TFD}$$

$$= e^{-\frac{\mu N}{2}} \left\langle I \left| e^{-\frac{\beta}{8}(\hat{H}_L+\hat{H}_R)} \chi_1^L(t_1) e^{-\frac{\beta}{4}\hat{H}_L} \sum_n \frac{1}{n!} \left(-\frac{i\mu}{2} \sum_i \chi_i^L \chi_i^R \right)^n \times e^{-\frac{\beta}{4}\hat{H}_L} \chi_1^L(t_2) e^{-\frac{\beta}{8}(\hat{H}_L+\hat{H}_R)} \right| I \right\rangle$$

transform χ^R into $\chi^L = \chi$, using $\left(\widehat{H_R} - \widehat{H_L}\right)|I\rangle = 0$

$$\left(\sum_i \chi_i^L \chi_i^R\right)^n e^{-\frac{\beta}{4}\hat{H}_L} \chi_1^L(t_2) e^{-\frac{\beta}{8}(\widehat{H_L}+\widehat{H_R})} \left| I \right\rangle$$

$$= (-1)^{\frac{n(n-1)}{2}} \sum_{i_1, i_2, \ldots, i_n} \left(\chi_{i_1}^L \ldots \chi_{i_n}^L\right)\left(\chi_{i_n}^R \ldots \chi_{i_1}^R\right) e^{-\frac{\beta}{4}\hat{H}_L} \chi_1^L(t_2) e^{-\frac{\beta}{8}(\widehat{H_L}+\widehat{H_R})} \left| I \right\rangle$$

$$= (-1)^{\frac{n(n+1)}{2}} \sum_{i_1, i_2, \ldots, i_n} \left(\chi_{i_1}^L \ldots \chi_{i_n}^L\right) e^{-\frac{\beta}{4}\hat{H}_L} \chi_1^L(t_2) e^{-\frac{\beta}{8}\hat{H}_L} \left(\chi_{i_n}^R \ldots \chi_{i_1}^R\right) e^{-\frac{\beta}{8}\hat{H}_R} \left| I \right\rangle$$

$$= (-i)^n \sum_{i_1, i_2, \ldots, i_n} \left(\chi_{i_1}^L \ldots \chi_{i_n}^L\right) e^{-\frac{\beta}{4}\hat{H}_L} \chi_1^L(t_2) e^{-\frac{\beta}{4}\hat{H}_L} \left(\chi_{i_1}^L \ldots \chi_{i_n}^L\right) \left| I \right\rangle$$

$$Z(\mu, t_1, t_2) = e^{-\frac{\mu N}{2}} \sum_n \frac{1}{n!} \left\langle \mathcal{T} \chi_1\left(t_1 + \frac{3i\beta}{4}\right) \left[-\frac{\mu}{2} \sum_i \chi_i\left(\frac{i\beta}{2}\right) \chi_i(0) \right]^n \times \chi_1\left(t_2 + \frac{i\beta}{4}\right) \right\rangle_\beta \qquad \langle \ldots \rangle = \mathrm{Tr}(\rho \ldots)$$

$\mathcal{T}$ denotes time-ordering in the imaginary time

Supplementary: size-distribution SYK

$$Z(t_1,t_2,\mu)=e^{-\mu N\left(\frac{1}{2}-G(\theta_{34})\right)}\sum_{m=0}^{\infty}\Upsilon^m(\theta_{12})\left(\frac{\lambda\mu N}{2}\right)^m\frac{1}{m!}\Upsilon^1(\theta_{34})^m \qquad G(\theta_{12})=\frac{1}{2}\left[\frac{\cos\left(\frac{\pi v}{2}\right)}{\Theta_{12}}\right]^{2\Delta}$$

included $e^{\mu N G(\theta_{34})}$ to factor out the disconnected contribution to the size operator $\hat{N}$

$\Upsilon^1(\theta_{34})$related to the size operator only contains those associated with single scramblon emissions
m = 1 , this is the leading order contribution in $t \ll \log N$

At maximal chaos $\varkappa = 1$

$$\theta_1=\frac{2\pi i}{\beta}t_1+\frac{\pi}{2},\quad \theta_2=\frac{2\pi i}{\beta}t_2+\frac{3\pi}{2},\quad \theta_3=0,\quad \theta_4=\pi$$

$$OTOC_{max}=\int_0^{\infty}dy_1\int_0^{\infty}dy_2 h(y_1,\theta_{12})h(y_2,\theta_{34})e^{-\lambda y_1 y_2}$$

Assign $t_1=t_2=t,\ t_3=t_4=0$

$$=\int_0^{\infty}dy_1\int_0^{\infty}dy_2 h(y_1,\theta_{12})h(y_2,\theta_{34})e^{\frac{e^t}{C}y_1 y_2}$$

At sub-maximal chaos $\varkappa < 1$

$$\widetilde{OTOC}=\int_0^{\infty}dy_1\int_0^{\infty}dy_2\ h(y_1,\theta_{12})h(y_2,\theta_{34})e^{\frac{e^{\varkappa t}}{C}(y_1 y_2)^{\varkappa}}$$

$$=\int_0^{\infty}dy_1\int_0^{\infty}dy_2\ \tilde{h}(y_1,\theta_{12})\tilde{h}(y_2,\theta_{34})e^{-\tilde{\lambda}y_1 y_2}$$

$$\tilde{\lambda}=-\frac{e^{i\frac{\varkappa}{2}(\pi+\theta_3+\theta_4-\theta_1-\theta_2)}}{C},\quad \tilde{h}(y,\theta_{ij})=\frac{y^{1/\varkappa-1}}{\varkappa}h\left(y^{1/\varkappa},\theta_{ij}\right)$$

Supplementary: size-distribution SYK

$$Z(t_1,t_2,\mu) = e^{-\mu N\left(\frac{1}{2}-G(\theta_{34})\right)}\int_0^\infty dy\ \tilde{h}(y,\theta_{12})\exp\left(\frac{\tilde{\lambda}\mu N\Upsilon^1(\theta_{34})}{2}y\right)$$

$$\Theta_{12} = \cosh\left(\frac{\pi t_{12}}{\beta}\right),\ \ t_{12} = t_1 - t_2,\ \ \Theta_{34} = 1$$

$$\lambda = -\frac{1}{C}e^{\frac{\pi\varkappa}{\beta}(t_1+t_2)},\ \ \Upsilon^1(\theta_{34}) = \frac{\Gamma(2\Delta+\varkappa)}{\Gamma(2\Delta)}G,\ \ G(\theta_{34}) = G$$

$$Z(t_1,t_2,\mu) = \frac{e^{-\mu N\left(\frac{1}{2}-G\right)}G}{\Gamma(2\Delta)}\int_0^\infty \frac{dy}{\varkappa}\ y^{\frac{2\Delta}{\varkappa}-1}\exp\left[-\mu K e^{\frac{\varkappa\pi(t_1+t_2)}{\beta}}y - \cosh\left(\frac{\pi t_{12}}{\beta}\right)y^{1/\varkappa}\right]$$

$$K = \frac{\Gamma(2\Delta+\varkappa)\mu N G}{2\Gamma(2\Delta)C}$$

$$P(t_1,t_2,\ell) = \frac{1}{2\pi i}\oint_\Gamma d\mu\ e^{\mu\ell}Z(\mu,t_1,t_2) \quad = \frac{G}{\Gamma(2\Delta)}\int_0^\infty \frac{dy}{\varkappa}\ \delta\left(\tilde{\ell} - K e^{\frac{\pi\varkappa}{\beta}(t_1+t_2)}y\right)\ y^{\frac{2\Delta}{\varkappa}-1}e^{-\cosh\left(\frac{\pi t_{12}}{\beta}\right)y^{1/\varkappa}}$$

$$= \frac{G\ \widetilde{\ell^{2\Delta/\varkappa-1}}}{\varkappa K^{2\Delta/\varkappa}\Gamma(2\Delta)}\exp\left[-\frac{2\pi\Delta}{\beta}(t_1+t_2) - \frac{1}{2}\left(\frac{\tilde{\ell}}{K}\right)^{1/\varkappa}\left(e^{-\frac{2\pi t_1}{\beta}} + e^{\frac{-2\pi t_2}{\beta}}\right)\right]$$

$$\tilde{\ell} = \ell - N\left(\frac{1}{2} - G\right)$$

$\left(\frac{\tilde{\ell}}{K}\right)^{1/\varkappa}$ as an effective operator-size for $\varkappa < 1$

Supplementary: size-distribution SYK

$$P(t_1, t_2, \ell) = Q_\ell(t_1) \times Q_\ell(t_2) \qquad Q_\ell(t) = \frac{\tilde{\ell}^{\Delta/\varkappa - 1/2}}{\sqrt{\varkappa} K^{\Delta/\varkappa}} \sqrt{\frac{G}{\Gamma(2\Delta)}} \, e^{-\frac{2\pi\Delta}{\beta} t - \frac{1}{2}\left(\frac{\tilde{\ell}}{K}\right)^{1/\varkappa} e^{-\frac{2\pi}{\beta} t}}$$

$$K_{mn}(\ell) = \ell J_m(\ell) J_n(\ell)$$

$$\ell^{-1} K_{mn}(\ell) = \left[L(\ell)^\dagger \cdot L(\ell)\right]_{mn} = J_m(\ell) J_n(\ell), \ [L(\ell)]_{m\alpha} = \langle O_m, \alpha \rangle, \ \ \alpha \in H_\ell$$

H_ℓ is the operator space sector with size ℓ

$L(\ell)$ is of rank one

$$\langle O_m, \alpha \rangle = [L(\ell)]_{m\alpha} = J_m(\ell) \times \Psi_\alpha^\ell \rightarrow \hat{P}(\ell)|O_m\rangle = J_m(\ell) \times \left|\Psi^\ell\right\rangle, \ \ \left|\Psi^\ell\right\rangle = \sum_{\alpha \in \mathcal{H}_\ell} \Psi_\alpha^\ell |\alpha\rangle$$

distinct $\mathcal{O}_m$ share the same normalized projection $\left|\Psi^\ell\right\rangle$ into each operator space sector H_ℓ

Hamiltonian is symmetric under the permutation $P(n)$ .the action is ``ergodic" in each H_ℓ

We expect $\left|\Psi^\ell\right\rangle$ to be the most permutation-symmetric operator state in H_ℓ

$J_m(\ell)$ can be viewed as the wave-function in operator-size of $\mathcal{O}_m$

Supplementary: size-distribution SYK

$$\varkappa = 1, \qquad J_n(\ell) = \frac{\tilde{\ell}^{\Delta/\varkappa - 1/2}}{\sqrt{\varkappa} K^{\Delta/\varkappa}} \sqrt{\frac{\Gamma(n+1) G}{\Gamma(2\Delta + n)}} \oint \frac{dy}{y^{n+1}} (1+y)^{-2\Delta} e^{-\frac{1}{2}\left(\frac{\tilde{\ell}}{K}\right)^{1/\varkappa}\left(\frac{1-y}{1+y}\right)}$$

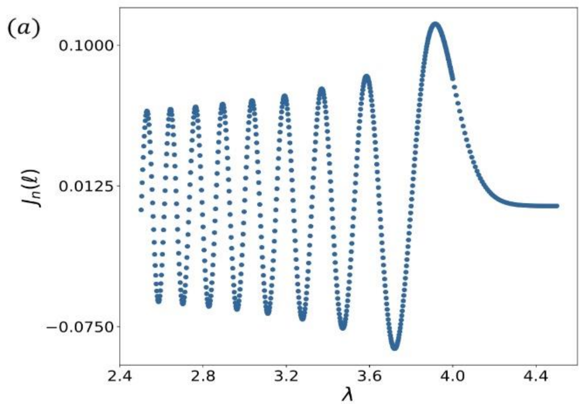
Rescale $(\ell/K)^{1/\varkappa} = \lambda n$

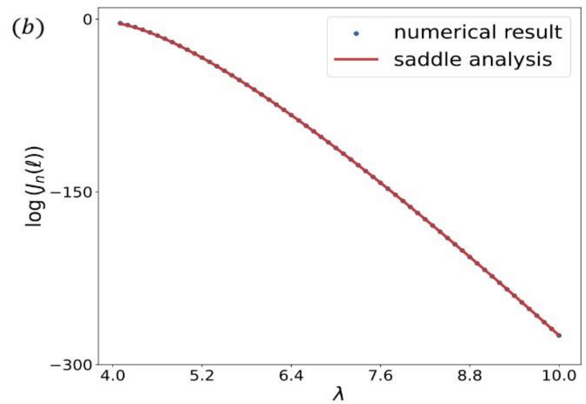
$$n + \frac{2\Delta y}{1+y} - \frac{\lambda n y}{(1+y)^2} = 0, \quad y_\pm^* = \frac{n(\lambda - 2) - 2\Delta \pm \sqrt{(n\lambda - 2\Delta)^2 - 4n^2\lambda}}{2(n + 2\Delta)}$$

physical saddle corresponds to y_-^*

$$J_n(\tilde{\ell}) \sim \frac{\tilde{\ell}^{\Delta - 1/2}}{K^\Delta} \sqrt{\frac{\Gamma(n+1)}{\Gamma(2\Delta + n)}} e^{-S^*} \times (\text{fluctuation}) \sim n^{-1/2} \left[e^{\frac{\sqrt{\lambda(\lambda-4)}}{2}} \left(\frac{\lambda - 2 - \sqrt{\lambda(\lambda - 4)}}{2} \right) \right]^{-n}$$

typical operator size of $\mathcal{O}_n$ to be of order $\tilde{\ell} \sim K n^\varkappa$





Supplementary: size-distribution MBL

MBL

$$\sigma_x^0(t) = e^{iHt}\sigma_x^0 e^{-iHt} = \sigma_x^0 \prod_{j\neq 0}\left(\cos(2J_{0j}t) + i\sin(2J_{0j}t)\sigma_z^0\sigma_z^j\right)\left(\cos(2h_0 t) + i\sin(2h_0 t)\,\sigma_z^0\right)$$

$$Z(t_1,t_2,\mu) = \mathrm{Tr}\left[\sigma_x^0(t_1)e^{-\mu\hat{N}}\sigma_x^0(t_1)\right]$$

$$= \mathrm{Tr}\left\{\sigma_x^0 \prod_{j\neq 0}\left(\cos(2J_{0j}t_1) + i\sin(2J_{0j}t_1)\sigma_z^0\sigma_z^j\right)\left(\cos(2h_0 t_1) + i\sin(2h_0 t_1)\,\sigma_z^0\right)\right.$$

$$\left.\times\, e^{-\mu\hat{N}}\sigma_x^0 \prod_{k\neq 0}\left(\cos(2J_{0k}t_2) + i\sin(2J_{0k}t_2)\sigma_z^0\sigma_z^k\right)\left(\cos(2h_0 t_2) + i\sin(2h_0 t_2)\,\sigma_z^0\right)\right\}$$

each $\cos(2J_{0j}t)$ corresponds to an identity operator on *j*, each $\sin(2J_{0j}t)$ indicates σ_z^j

$\cos(2h_0 t)$ *or* $\sin(2h_0 t)$ *differ by interchanging* σ_x^0 *and* σ_y^0. Taking trace pairs up $\sigma_x^0(t_{1,2})$ Pauli strings

neglect all cross terms between sine and cosine

$$Z(\mu,t_1,t_2) = e^{-\mu}\cos(2h_0(t_1-t_2)) \prod_{j\neq 0}\left(\cos(2J_{0j}t_1)\cos(2J_{0j}t_2) + e^{-\mu}\sin(2J_{0j}t_1)\sin(2J_{0j}t_2)\right)$$

first $e^{-\mu}$ comes from the zeroth site. Additional $e^{-\mu}$ appears along σ_z^j

$$\overline{Z(\mu,t_1,t_2)} = e^{-\mu}\,\overline{\cos(2h_0(t_1-t_2))} \prod_{j\neq 0}\left[\frac{1-e^{-\mu}}{2}\overline{\cos\left(2_{0j}(t_1+t_2)\right)} + \frac{1+e^{-\mu}}{2}\overline{\cos\left(2_{0j}(t_1-t_2)\right)}\right]$$

$$= e^{-\mu}e^{-2h^2(t_1-t_2)^2} \prod_{j\neq 0}\left(\frac{1-e^{-\mu}}{2}e^{-2J^2(t_1+t_2)^2 e^{-\frac{|j|}{\xi}}} + \frac{1+e^{-\mu}}{2}e^{-2J^2(t_1-t_2)^2 e^{-\frac{|j|}{\xi}}}\right)$$

Supplementary: size-distribution MBL

MBL

$$\overline{Z(\mu, t_1, t_2)} e^{\frac{\gamma^2}{2}(t_1^2+t_2^2)} = e^{\gamma^2 t_1 t_2} e^{-\mu} \prod_{j\neq 0} \left(\frac{1+e^{-\mu}}{2} + \frac{1-e^{-\mu}}{2} e^{-8J^2 t_1 t_2 e^{-\frac{|j|}{\xi}}} \right)$$

$$\gamma^2 = 4J^2 \sum_{j\neq 0} e^{-\frac{|j|}{\xi}} + 4h^2$$

this is only a function of $t_1 t_2$,so $K_{mn}(\ell\,)$ is exactly diagonal

$8J^2 t_1 t_2 e^{-\frac{|j|}{\xi}} \gg 1$ gives $(1+e^{-\mu})/2$

Indicating equal probability between σ_z^j and I

$8J^2 t_1 t_2 e^{-\frac{|j|}{\xi}} \ll 1$ gives 1

$$\overline{Z(\mu, t_1, t_2)} e^{\frac{\gamma^2}{2}(t_1^2+t_2^2)} \approx e^{\gamma^2 t_1 t_2} e^{-\mu} \left(\frac{1+e^{-\mu}}{2} \right)^{2\xi \ln(8J^2 t_1 t_2)}$$

$$= e^{\gamma^2 t_1 t_2} 2^{-M(t_1 t_2)} \sum_{l} B(M(t_1 t_2), \ell) e^{-\mu(\ell+1)}$$

$$= \sum_{l} e^{-\mu\ell} \ell^{-1} \sum_{n} \frac{(\gamma^2 t_1 t_2)^n}{n!} K_{nn}(\ell)$$

$M(t_1 t_2) = 2\xi \ln(8J^2 t_1 t_2)$ and $B(M, l)$ is the binomial coefficient

Supplementary: size-distribution MBL

MBL

$$K_{nn}(\ell) \approx \frac{n!\,\ell}{2\pi i} \oint \frac{dx}{x^{n+1}} \left[\frac{B\left(M\left(x/\gamma^2\right), \ell - 1\right)}{2^{M\left(x/\gamma^2\right)}} \right] e^{x} \qquad x = \gamma^2 t_1 t_2$$

For (M, ℓ) both large and of the same order $\frac{B(M,\ell)}{2^M} \approx (\pi M/2)^{-1/2} e^{-\frac{(\ell - M/2)^2}{M/2}}$

$$K(\ell) \approx \frac{n!\,\ell}{2\pi i} \oint dx \ \exp\left(-(n+1)\ln x + x - \frac{\left(\ell - \xi \ln\left(\frac{8J^2}{\gamma^2} x\right)\right)^2}{\xi \ln\left(\frac{8J^2}{\gamma^2} x\right)} - \frac{1}{2}\ln\left(\pi\xi \ln\left(\frac{8J^2}{\gamma^2} x\right)\right)\right)$$

Rescale $\ell = \lambda \ln n$, $x^* = n + ..$

$$K_{nn}(\ell) \sim \lambda \exp\left(-\frac{(\lambda - \xi)^2}{\xi} \ln n \right)$$

operator-size distribution of $\mathcal{O}_n$ is Gaussian, $\bar{\ell} \sim \xi \ln n$

$$K_{mn}(\ell) = \ell \langle \mathcal{O}_m |\ \hat{P}(\ell)\ | \mathcal{O}_n \rangle = \ell \langle \hat{P}(\ell)\mathcal{O}_m, \hat{P}(\ell)\mathcal{O}_n \rangle \propto \delta_{mn}$$