Understanding Higher-Group Symmetries

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Outline

- Overview of generalized symmetries
- Strict and weak 2-group symmetries
- 2-group gauge theory
- Landau-Ginzburg model for strict 2-group symmetries and SSB
- 3-group symmetries

Motivations

- Symmetry is a central concept in physics

(1) Global symmetry: $\phi \to g \cdot \phi$, g is constant in spacetime.

(2) Local (gauge) symmetry: $\phi \to g(x) \cdot \phi$, $g(x)$ is spacetime dependent.

Examples of "ordinary symmetry"

- 0-form: acting on local operators
- invertible: the symmetry transformations are invertible

	Global symmetry	Local (gauge) symmetry
Spacetime	Lorentz symmetry, C, P, T	Diffeomorphism
Internal	Flavor symmetry, $U(1)_B$, $U(1)_L$	Gauge symmetry of SM

- By default we are talking about global internal symmetries.

Generalized Global Symmetries

- Generalize the ordinary (invertible 0-form) global symmetry

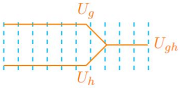
(1) 0-form $\rightarrow$ Higher-form symmetry acting on extended operators

(2) Group G $\rightarrow$ Higher-group, non-invertible categorical symmetries ...

	Local operator	Higher-dim. operators
Invertible	ordinary sym.	higher-form, higher-group sym.
Non-invertible	non-invertible sym.	higher-categorical sym.

Higher-form symmetry

- p-form symmetry with group G (Gaiotto, Kapustin, Seiberg, Willett 14')
- A p-form symmetry is generated by a $(d-p-1)$-dimensional topological operator $U(g, M^{(d-p-1)})$:



and acts on p-dimensional object(operator) $V^i(\mathcal{C}^{(p)})$.

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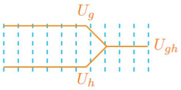
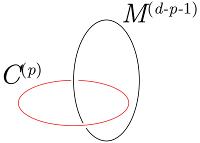


and acts on p-dimensional object(operator) $V^i(\mathcal{C}^{(p)})$.

- $U(g, M^{(d-p-1)})$ has non-trivial action on $V^i(\mathcal{C}^{(p)})$ when $M^{(d-p-1)}$ and $\mathcal{C}^{(p)}$ are non-trivially linked.

$$U(g, M^{(d-p-1)}) V^i(\mathcal{C}^{(p)}) = R^i{}_j(g) V^j(\mathcal{C}^{(p)}) . \quad (1)$$



Higher-form symmetry

- Examples:

(1) Pure 4D $U(1)$ Maxwell theory has $U(1)_e \times U(1)_m$ 1-form symmetry

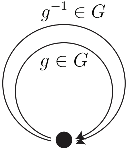
(2) Pure 4D $SU(N)$ Yang-Mills theory has $\mathbb{Z}_N$ 1-form symmetry

(3) 3D $U(1)_k$ Chern-Simons theory has $\mathbb{Z}_k$ 1-form symmetry

(4) 6D (2,0) theory has 2-form symmetries

Higher-group symmetry

- What is a higher-group symmetry?
- Step One: how to think a group as a 1-category (object+morphisms):



$$g \circ h = gh \tag{2}$$

- A group is a 1-category with only one object and invertible morphisms

- Generalizations

2-category: object, morphisms, 2-morphisms (between morphisms)

n-category: object, morphisms, 2-morphisms, . . . , n-morphisms

Higher-group symmetry

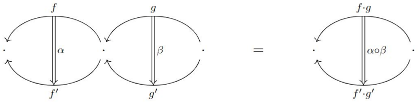
• An *n-group* is an *n*-category with only one object where all *k*-morphisms ($1 \leq k \leq n$) are invertible.

• The consistency relations are complicated, and there are different versions of *n*-groups with different associativity

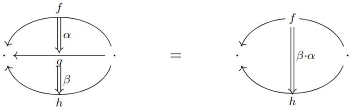
• We first talk about 2-groups, which are invertible 2-categories with only one object.
(1) Strict 2-group (strict 2-category): morphisms are associative
(2) Weak 2-group (bicategory): morphisms are not associative

Strict 2-group

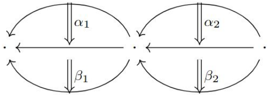
- One object $\bullet$, morphisms $\rightarrow$, 2-morphisms $\Rightarrow$
- All invertible, identity exists for morphisms and 2-morphisms
- Horizontal composition is associative:



- Vertical composition is associative:



- Compatibility:



Strict 2-group

- Equivalent algebraic formulation: crossed module $(G, H, \partial, \triangleright)$

(1) G and H are groups

(2) $\partial : H \to G$ is a homomorphism

(3) $\triangleright : G \to \mathrm{Aut}(H)$ is a group action of G on H

- Additional constraints for all $g \in G$, $h, h' \in H$:

$$\partial(g \triangleright h) = g(\partial h)g^{-1} \tag{3}$$

$$(\partial h) \triangleright h' = hh'h^{-1} \,. \tag{4}$$

- Remark: the original notion of 2-group in the construction of 2-gauge theory by (Baez, Schreiber 04')...

Weak 2-group

• However, strict 2-group is not often used in the generalized symmetry literature of 2-group symmetries. G and H are NOT actual symmetries!

• Physicists instead use weak 2-groups (Cordova, Dumitrescu, Intriligator 18')(Benini, Cordova, Hsin 18')...

• A weak 2-group is defined by the data $(\Pi_1, \Pi_2, \rho, \beta)$

(1) Π_1, Π_2 are groups

(2) $\rho : \Pi_1 \to \mathrm{Aut}(\Pi_2)$ is a group action

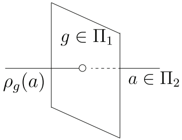
(3) $\beta \in H^3_\rho(B\Pi_1, \Pi_2) \equiv H^3_{\rho,grp}(\Pi_1, \Pi_2)$ is called the Postnikov class (element of twisted group cohomology)

Physical Meanings

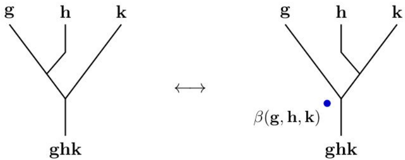
(1) Π_1: 0-form symmetry generated by cod-1 top. operator

(2) Π_2: 1-form symmetry generated by cod-2 top. operator

(3) $\rho : \Pi_1 \to \text{Aut}(\Pi_2)$:



(4) $\beta \in H^3_\rho(B\Pi_1, \Pi_2)$, $g, h, k \in \Pi_1$, $\beta(g, h, k) \in \Pi_2$:



- A weak 2-group $(\Pi_1, \Pi_2, \rho, \beta)$ is called split (non-split) if $\beta = 0$ ($\neq 0$).

Physical Examples

(1) 5d $SU(2)_0$ SCFT with $\Pi_1 = SO(3)$, $\Pi_2 = \mathbb{Z}_2$, and a non-split 2-group symmetry $(SO(3), \mathbb{Z}_2, \mathrm{id.}, \beta)$. $\beta \neq 0 \in H^3(BSO(3), \mathbb{Z}_2) = \mathbb{Z}_2$. (Apruzzi, Bhardwaj, Oh, Schafer-Nameki 21')(del Zotto, García-Extebarria, Schafer-Nameki 22')...

(2) 4d QED type examples: starting from 4d $U(1)_A^{(0)} \times U(1)_C^{(0)}$ global symmetry with mixed 't Hooft anomaly. Gauge $U(1)_C^{(0)} \to$ New $U(1)_B^{(1)}$ magnetic 1-form symmetry, and a non-split 2-group symmetry $(U(1)_A, U(1)_B, \mathrm{id.}, \kappa)$, $\kappa \in H^3(BU(1), U(1)) = \mathbb{Z}$. (Cordova, Dumitrescu, Intriligator 18')...

(3) In condensed matter physics, non-split 2-group (non-zero β) $\to$ obstruction to symmetry fractionalization (Chen, Burnell, Vishwanath, Fidkowski 14')...

Strict and Weak 2-groups

- Recap: we have two versions of 2-groups

(1) Strict 2-group: $(G, H, \partial, \triangleright)$

(2) Weak 2-group: $(\Pi_1, \Pi_2, \rho, \beta)$

- Questions:

(1) How to relate them?

- $(G, H, \partial, \triangleright) \to (\Pi_1, \Pi_2, \rho, \beta)$: unique
- $(\Pi_1, \Pi_2, \rho, \beta) \xrightarrow{\textit{strictification}} (G, H, \partial, \triangleright)$: non-unique

(2) Why use the strict 2-group?

- The formulation of strict 2-group gauge theory in real space & path space is well established (Baez, Schreiber 04')...
- Easier to describe matter fields (2-matter) and SSB. (Liu, Luo, YNW 24')...

Strict to Weak

- Now we discuss the algebraic relations between strict and weak 2-groups
- Exact sequence:

$$1 \to \Pi_2 \xrightarrow{i} H \xrightarrow{\partial} G \xrightarrow{p} \Pi_1 \to 1\,. \tag{5}$$

(1) The “actual” 0-form symmetry: $\Pi_1 = G/\mathrm{im}(\partial)$
(2) The “actual” 1-form symmetry: $\Pi_2 = \ker(\partial)$
(3) Group action $\triangleright : G \to \mathrm{Aut}(H)$ naturally induces $\rho : \Pi_1 \to \mathrm{Aut}(\Pi_2)$
(4) Postnikov class β? (Brown 82')...

Strict to Weak

- Example: $G = H = \mathbb{Z}_4 = \{0,1,2,3\}$, $\partial : a \to 2a$, $i : a \to 2a$, $p :$ (mod 2)
- $\Pi_1 = \mathbb{Z}_4/\mathrm{im}(\partial) = \mathbb{Z}_2$, $\Pi_2 = \ker(\partial) = \mathbb{Z}_2$:

$$1 \to \mathbb{Z}_2 \xrightarrow{i:\times 2} \mathbb{Z}_4 \xrightarrow{\partial:\times 2} \mathbb{Z}_4 \xrightarrow{p:(mod 2)} \mathbb{Z}_2 \to 1\,. \tag{6}$$

- $\mathrm{Aut}(\mathbb{Z}_2) = 1$, hence ρ is trivial
- $H^3(B\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, two possibilities:

(1) Split 2-group, $\beta = 0 \in H^3(B\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$

(2) Non-split 2-group, $\beta \neq 0 \in H^3(B\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$

- Question: what determines β in the strict language?

Computation of β

$$1 \to \mathbb{Z}_2 \xrightarrow{i:\times 2} \mathbb{Z}_4 \xrightarrow{\partial:\times 2} \mathbb{Z}_4 \xrightarrow{p:(mod 2)} \mathbb{Z}_2 \to 1\,. \tag{7}$$

- Pick a cross-section function $s : \Pi_1 \to G$ s.t. $p \circ s =$id.

$$s(0) = 0\ ,\ \ s(1) = 1 \tag{8}$$

- Define $f : \Pi_1 \times \Pi_1 \to G$ with

$$s(g)s(h) = s(gh)f(g,h) \tag{9}$$

$$f(g,h) = \begin{cases} 2 & g = h = 1 \\ 0 & \text{other cases} \end{cases} \tag{10}$$

- Uplift f to $F : \Pi_1 \times \Pi_1 \to H$: $\partial F = f$

$$F(g,h) = \begin{cases} 1 & g = h = 1 \\ 0 & \text{other cases} \end{cases} \tag{11}$$

Computation of β

$$1 \to \mathbb{Z}_2 \xrightarrow{i:\times 2} \mathbb{Z}_4 \xrightarrow{\partial:\times 2} \mathbb{Z}_4 \xrightarrow{p:(mod 2)} \mathbb{Z}_2 \to 1\,. \tag{12}$$

- Failure of cocycle condition for F:

$$(s(g) \triangleright F(h,k))F(g,hk) = i(\beta(g,h,k))F(g,h)F(gh,k)\,, \tag{13}$$

- $\beta : \Pi_1 \times \Pi_1 \times \Pi_1 \to \Pi_2$: Postnikov class

Computation of β

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- $\beta : \Pi_1 \times \Pi_1 \times \Pi_1 \to \Pi_2$: Postnikov class
- The choice of β is encoded in the group action $\triangleright : G \to \mathrm{Aut}(H)$!

(1) If the group action $\triangleright$ is trivial: $a \triangleright b = b$ $(a, b \in \mathbb{Z}_4)$, then $\beta(g,h,k) \equiv 0 \to$ split 2-group

(2) If the group action $\triangleright$ is non-trivial: $a \triangleright b = (2a+1)b$ $(a, b \in \mathbb{Z}_4)$, then

$$\beta(g,h,k) = ghk \quad (g,h,k = 0,1) \tag{14}$$

Non-split 2-group with non-trivial β!

Strictifications

- $(\Pi_1, \Pi_2, \rho, \beta) \xrightarrow{\textit{strictification}} (G, H, \partial, \triangleright)$: non-unique

(1) Strictification of $(\Pi_1, \Pi_2, \rho, \beta) = (\mathbb{Z}_2, \mathbb{Z}_2, \mathrm{id.}, \beta \neq 0)$:

$$1 \to \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{(mod 2)} \mathbb{Z}_2 \to 1\,. \tag{15}$$

with a non-trivial group action $a \triangleright b = (2a+1)b$

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- For a split 2-group $(\Pi_1, \Pi_2, \rho, \beta) = (\mathbb{Z}_2, \mathbb{Z}_2, \mathrm{id.}, \beta = 0)$, we can just use the trivial sequence

$$1 \to \mathbb{Z}_2 \xrightarrow{id.} \mathbb{Z}_2 \xrightarrow{\to 0} \mathbb{Z}_2 \xrightarrow{id.} \mathbb{Z}_2 \to 1\,. \tag{16}$$

Strictifications

(2) The case of 5d $SU(2)_0$ theory, $(\Pi_1, \Pi_2, \rho, \beta) = (SO(3), \mathbb{Z}_2, \mathrm{id.}, \beta \neq 0)$

$$1 \to \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{a \to e^{\pi i a} I} SU(2) \longrightarrow SO(3) \to 1\,. \tag{17}$$

- $G = SU(2)$, $H = \mathbb{Z}_4$, no need to use a non-trivial group action $\triangleright$

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- $G = SU(2)$, $H = \mathbb{Z}_4$, no need to use a non-trivial group action ▷
- CB description: $U(1)$ gauge theory$+$ matter:

$$\begin{array}{c|cc} & U(1)_{gauge} & U(1)_{flavor} \\ \hline f : \mathrm{Gauge\ W-boson} & -2 & 1 \\ e : \mathrm{Flavor\ W-boson} & 0 & -2 \end{array} \tag{18}$$

- Naive flavor symmetry rotating f and $e + f$ is $G = SU(2)$, however the center $\mathbb{Z}_2 \subset G = SU(2)$ is a part of gauge symmetry!
- The actual flavor symmetry is $\Pi_1 = G/\mathbb{Z}_2 = SO(3)$.
- What about $H = \mathbb{Z}_4$? All charged matter has integral charge under $\frac{1}{4}(U(1)_{gauge} + 2U(1)_{flavor})$. Crucial in identification of non-trivial 2-group from the exact sequence:(Apruzzi, Bhardwaj, Oh, Schafer-Nameki 21')

$$1 \to \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \to 1\,. \tag{19}$$

Strictifications

(3) QED type model: $(\Pi_1, \Pi_2, \rho, \beta) = (U(1), U(1), \mathrm{id.}, \kappa \in \mathbb{Z})$

$$1 \to U(1) \xrightarrow{i} U(1) \times \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}.U(1) \xrightarrow{p} U(1) \to 1 \tag{20}$$

$$\partial(e^{2\pi i a}, b) = (b, 0) \,. \tag{21}$$

$$(a, e^{2\pi i b}) \triangleright (e^{2\pi i c}, d) = (e^{2\pi i (c+bd)}, d) \,. \tag{22}$$

- $\mathbb{Z}.U(1)$ is a central extension of $U(1)$ by $\mathbb{Z}$, group law:

$$(a, e^{2\pi i b}) + (c, e^{2\pi i d}) = \begin{cases} (a+c, e^{2\pi i(b+d)}) & (b+d<1) \\ (a+c+\kappa, e^{2\pi i(b+d)}) & (\text{otherwise}) \end{cases} \,. \tag{23}$$

(4) For general weak Lie 2-groups, strictification not necessarily exists

Strictification of gauge fields

- Derived the strictification of weak 2-group gauge fields $(a, b) \xrightarrow{\textit{strictification}} (A, B)$ (assuming $\ker(\partial)$ is abelian)
- Discrete 2-groups: putting on a triangulated space, reproduce the 0-form gauge transformation $\lambda \in G$:

$$A_{ij} \to \lambda_i A_{ij} \lambda_j^{-1} \ , \ B_{ijk} \to \lambda_i \triangleright B_{ijk} \tag{24}$$

- The 1-form gauge transformation $\Lambda \in H$:

$$A_{ij} \to A_{ij} \partial \Lambda_{ij} \ , \ B_{ijk} \to B_{ijk} (\delta_A \Lambda)_{ijk} \tag{25}$$

Physical Applications

- What can we do with strict 2-group symmetries?
- Landau-Ginzburg (LG) paradigm of SSB:

(1) Regular **0-form symmetry**, e.g. $G = U(1)$, introduce charged matter field ϕ under a representation of $U(1)$, i.e. $R : G \to \mathbf{Vect}$

$$S = \int d^d x((D_\mu \phi)^\dagger (D^\mu \phi) + V(\phi)) \ , \ V(\phi) = \mu|\phi|^2 + \lambda|\phi|^4 \qquad (26)$$

- Minimizing $V(\phi)$, breaks $G = U(1) \to 0$.

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- Minimizing $V(\phi)$, breaks $G = U(1) \to 0$.

(2) **1-form symmetry**

- Background gauge field: $B_{\mu\nu}$
- Charged matter? Wilson loops!

$$\Phi(C) = tr_R(\mathcal{P} \exp(\oint_C A)) \qquad (27)$$

- How to write down a Lagrangian LG model for 1-form symmetry?

Mean String Field Theory! (Iqbal, Mcgreevy 21')

Physical Applications

- Consider the path-space (loop-space) $\mathcal{P}(M)$, the partition function is (Iqbal, Mcgreevy 21')

$$Z = \int [\mathscr{D}B][\mathscr{D}\Phi] \exp\left(i \int_{\mathcal{P}(M)} \frac{1}{L(C)} \oint_C ds (\mathcal{D}\Phi)^\dagger (\mathcal{D}\Phi) + V(\Phi) \right) \quad (28)$$

- Covariant derivative with area derivative

$$\mathcal{D} = \frac{\delta}{\delta\sigma^{\mu\nu}} - B_{\mu\nu} \quad (29)$$

- Discussions of SSB for 1-form symmetry: similar to 0-form symmetry
- Area law: no SSB, Perimeter law: SSB
- Open question: how to derive the mean string field theory effective action for Yang-Mills theory?

Physical Applications

(3) **2-group symmetry**

- Background gauge fields (strict): A_μ, $B_{\mu\nu}$
- Charged matter? 2-representations of 2-group $\mathcal{G}$: a functor $\mathcal{G} \to$ **2Vect**

("Higher-charges" (Bhardwaj, Schafer-Nameki 23'))

- Higher-representation theory is subject to active research in mathematics!
- 2-vector space (Kapranov, Voevodsky 94' "Tetrahedron equations")
- 2-reps. for weak 2-groups (Elgueta 07')
- Physics: (Bartsch, Bhardwaj, Bottini, Bullimore, Decoppet, Delcamp, Ferrari, Grigoletto, Pearson, Schafer-Nameki, Tiwari, Yu. . .)

- What is a good notion of 2-representations to describe SSB of strict 2-group symmetry?

Automorphism 2-representations

- Consider an algebra Υ, we can define the following automorphism 2-group $\mathcal{A}ut(\Upsilon)$ (Kristel, Ludewig, Waldorf 22', 23'):

$$1 \to Z(\Upsilon^\times) \xrightarrow{i} \Upsilon^\times \xrightarrow{\partial: a \to adj_a} \mathrm{Aut}(\Upsilon) \xrightarrow{p} \mathrm{Out}(\Upsilon) \to 1 \tag{30}$$

- $G = \mathrm{Aut}(\Upsilon)$: Automorphism group of Υ
- $H = \Upsilon^\times$: invertible elements of Υ
- $\Pi_1 = \mathrm{Out}(\Upsilon)$: Outer automorphism group of Υ
- $\Pi_2 = Z(\Upsilon^\times)$: Center of $\Upsilon^\times$
- We use the automorphism 2-representation (as a strict intertwiner) $\mathcal{G} \to \mathcal{A}ut(\Upsilon)$ for some algebra Υ

Automorphism 2-representations

- 2-matter of 2-groups take value in Υ! Υ naturally has some physical meaning!
- Take the example of $(G, H, \partial, \triangleright) = (\mathbb{Z}_4, \mathbb{Z}_4, \times 2, \triangleright)$, we can take $\Upsilon = \mathbb{C}[\mathbb{Z}_4]$, the group algebra of $\mathbb{Z}_4$.
- An element of Υ has the form of a combination of Wilson loop operators

$$(a, b, c, d) \leftrightarrow aW_0 + bW_1 + cW_2 + dW_3 \ , \ \ W_i \cdot W_j = W_{i+j \ (mod \ 4)} \qquad (31)$$

- Multiplication rule of $\Upsilon = \mathbb{C}[\mathbb{Z}_4]$:

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 d_2 + c_1 c_2 + d_1 b_2 \\ a_1 b_2 + b_1 a_2 + c_1 d_2 + d_1 c_2 \\ a_1 c_2 + b_1 b_2 + c_1 a_2 + d_1 d_2 \\ a_1 d_2 + b_1 c_2 + c_1 b_2 + d_1 a_2 \end{pmatrix} . \qquad (32)$$

Automorphism 2-representations

- Action of $g \in G = \mathbb{Z}_4$ on $\Phi = (a, b, c, d)$:

$$(a, b, c, d) \to \begin{cases} (a, b, c, d) & (g = 0, 2) \\ (a, d, c, b) & (g = 1, 3) \end{cases} \tag{33}$$

- Action of $h \in H = \mathbb{Z}_4$ on $\Phi = (a, b, c, d)$:

$$(a, b, c, d) \to h \circ (a, b, c, d) = (a, i^h b, (-1)^h c, (-i)^h d)\,. \tag{34}$$

LG model for 2-group symmetry

- 2-group gauge fields $\mathcal{A}_{(A,B)}$ defined in $\mathcal{P}(M)$
- Path-space effective action with 2-matter $\Phi \in \mathbb{C}[\mathbb{Z}_4]$

$$Z = \int [\mathscr{D}A][\mathscr{D}B][\mathscr{D}\lambda][\mathscr{D}\phi] \exp\left\{ i \int_{\mathcal{P}(M)} \mathrm{Tr}_{\Upsilon}\left[-\frac{1}{2g^2}|\mathcal{F}_{\mathcal{A}}|^2 + \frac{1}{L(C)}(\mathbf{d}_{\mathcal{A}}\Phi)^\dagger(\mathbf{d}_{\mathcal{A}}\Phi) + V(\Phi)\right] + i\int_M \lambda^{(d-2)} \wedge (\partial(B) - F_A) \right\} \tag{35}$$

- e.g. for $\Upsilon = \mathbb{C}[\mathbb{Z}_4]$, we can write down some $V(\Phi)$ that's invariant under the action of G and H
- For simplicity turn off the gauge fields $\mathcal{A}$

LG model for 2-group symmetry

- For example $V(\Phi) = r|\Phi|^2 + u|\Phi|^4$, SSB patterns:

(1) The entire 2-group symmetry is preserved:

$$\Phi = \sqrt{\frac{-r}{u}}(1,0,0,0) \tag{36}$$

(2) $\Pi_1 = \Pi_2 = \mathbb{Z}_2$ is preserved, but the Postnikov class β is trivialized:

$$\Phi = \sqrt{\frac{-r}{u}}(a,0,b,0)\ ,\ \ |a|^2 + |b|^2 = 1\ ,\ \ a,b \neq 0\,. \tag{37}$$

- $H = \mathbb{Z}_4 \to \mathbb{Z}_2$
- After an SSB, non-split 2-group $\to$ split 2-group!

(3) Only preserves the 0-form symmetry $\Pi_1 = \mathbb{Z}_2$

$$\Phi = \sqrt{\frac{-r}{u}}(a,b,c,b)\ ,\ \ |a|^2 + 2|b|^2 + |c|^2 = 1\ ,\ \ a,b,c \neq 0\,. \tag{38}$$

3-groups

- 3-group: invertible 3-category with only one object

- Weak 3-group: $\mathcal{G}_3 = (\Pi_1, \Pi_2, \Pi_3, \rho, \beta, \gamma)$

(1) Π_1, Π_2 and Π_3 are 0-form, 1-form and 2-form symmetries
(2) $\rho : \Pi_1 \to \mathrm{Aut}(\Pi_2)$, $\Pi_1 \to \mathrm{Aut}(\Pi_3)$.
(3) $\beta \in H^3(B\Pi_1, \Pi_2)$, $\gamma \in H^4(B\mathcal{G}_2, \Pi_3)$ are Postnikov classes,
$\mathcal{G}_2 = (\Pi_1, \Pi_2, \rho, \beta) \subset \mathcal{G}_3$ is the sub-2-group.

- Strictification? Only to a semi-strict 3-group, i.e. a 2-crossed module!

3-groups

- Semi-strict 3-group: $(G, H, L, \partial_1, \partial_2, \triangleright, \{-,-\})$.

(1) G, H, L are groups

(2) $\partial_1 : H \to G$ and $\partial_2 : L \to H$ are homomorphisms

(3) $\triangleright : G \to \mathrm{Aut}(H)$, $G \to \mathrm{Aut}(L)$ are group actions

(4) $\{-,-\} : H \times H \to L$ is called the Pieffer lifting (not necessarily bilinear)

- Subject to many consistency conditions

- In our work, studied the strictification of weak 3-groups with either $\Pi_1 = 0$ or $\Pi_2 = 0$, as well as the gauge fields (Liu, Luo, YNW 24')...

- Algebraic strictification of a general weak 3-group is still unknown

Summary and Outlook

- Discussed the strictification of weak 2-groups/3-groups
- Formulated the strictification of 2-group/3-group gauge theory
- Formulated 2-matter of 2-group with automorphism 2-representation
- Constructed LG model for 2-group and discussed its SSB

- Future directions:
- Further investigate the algebraic formulation of 3-representations . . .
- Strictification of general 3-groups and 4-groups . . .
- How to quantatitvely write down the LG model of higher-form/higher-group symmetry for a given QFT?
- Realization of higher-matter in lattice models
- Thank you for the attention!