The unity of colored scalars, pions & gluons: from combinatorics to real world scattering

何颂 (Institute of Theoretical Physics, CAS)

based on works w. N. Arkani-Hamed, Q. Cao (曹趣), J. Dong (董晋), C. Figueiredo: 2312.16282, 2401.00041, 2401.05483, to appear

& w. Q. Cao, J. Dong, C. Shi (施灿欣), 2403.08855, + F. Zhu (朱凡), 2406.03838

(see also Arkani-Hamed, Salvatori, Frost, Plamondon, Thomas: 2309.15913, 2311.09284)

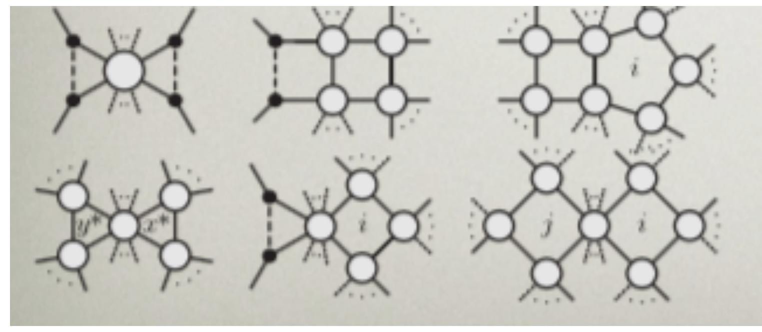
第五届“场论与弦论”学术研讨会

彭桓武高能基础理论研究中心

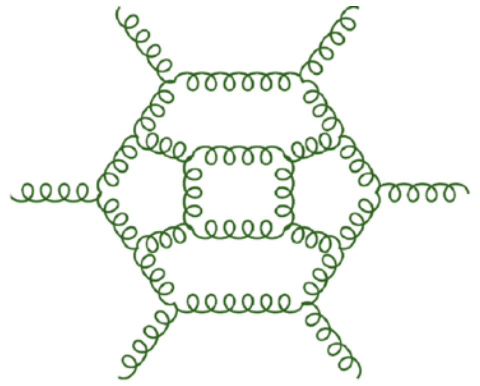
June 25, 2024

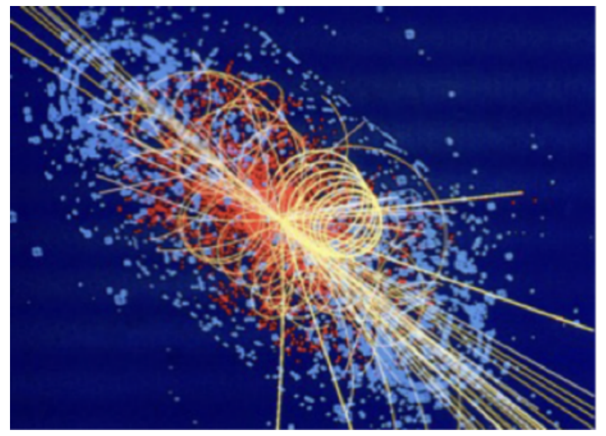
"Amplitudes"

on/off-shell, weak/strong coupling, …

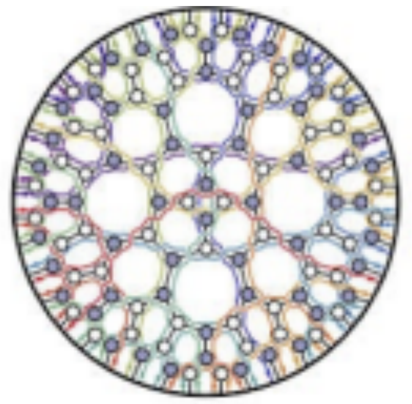


Formal QFT





Geometries, combinatorics, number theory, …

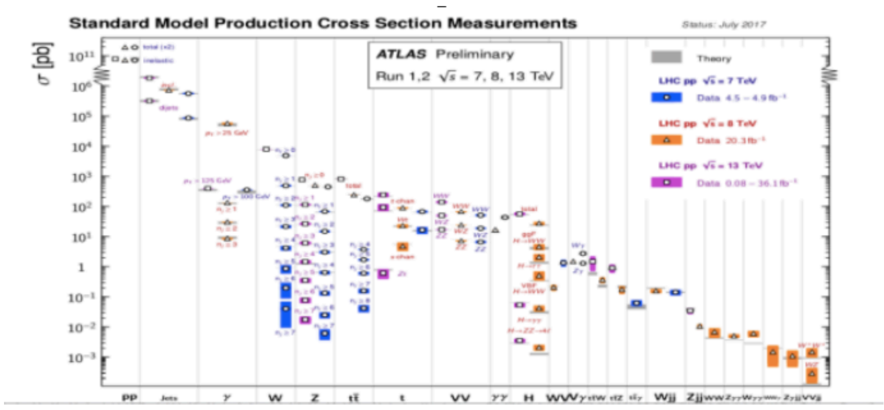


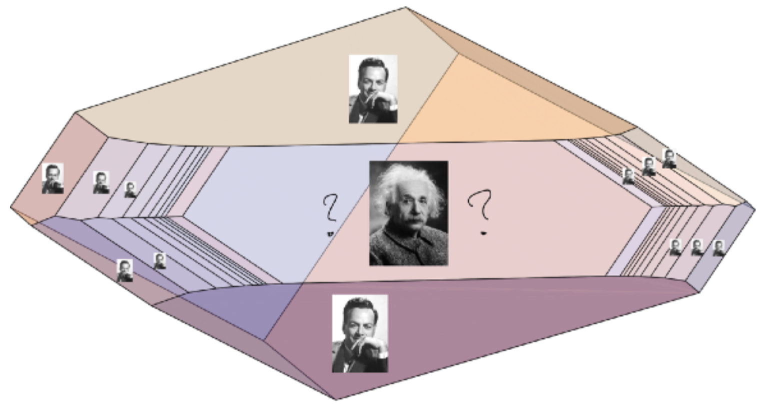
Mathematics

Amplitudes

precision frontier: loop integrands + integrals

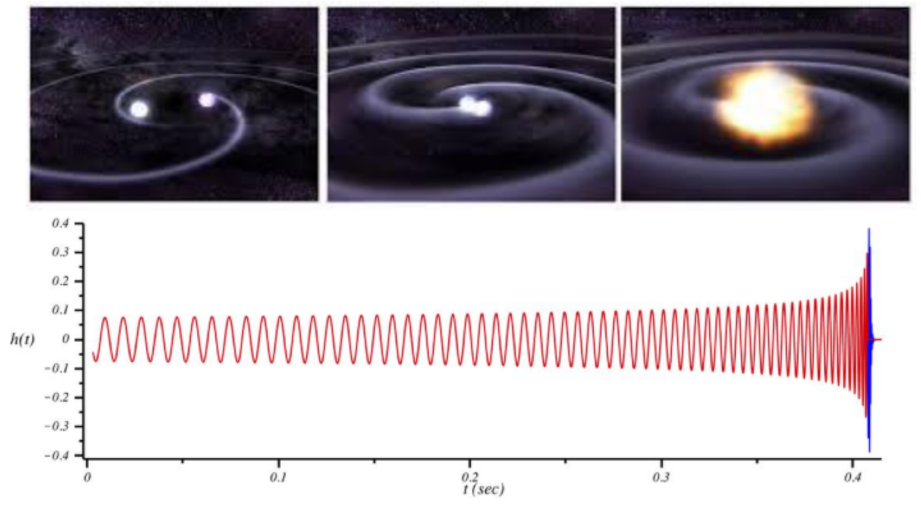
Collider Phenomenology



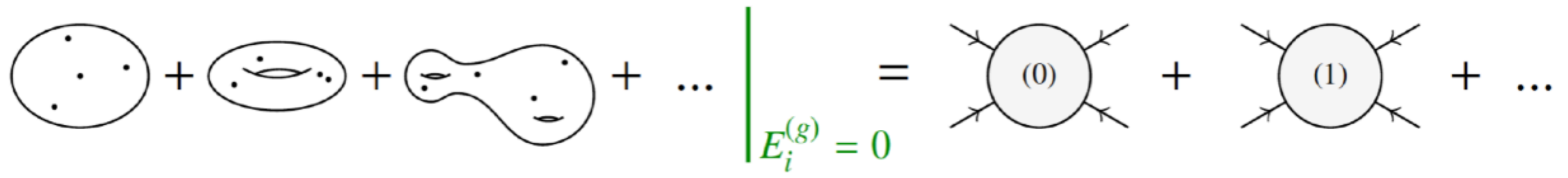


String Theory

Quantum/ Classical Gravity



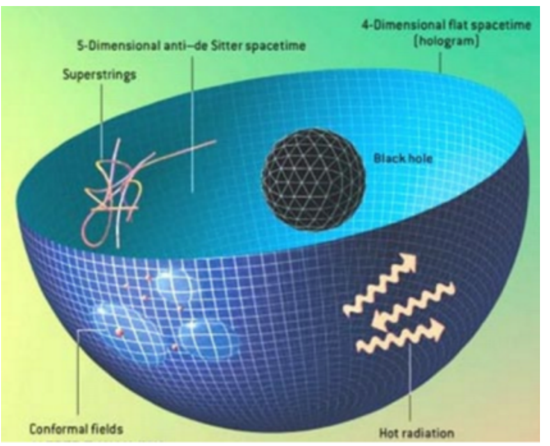
AdS/dS Cosmology

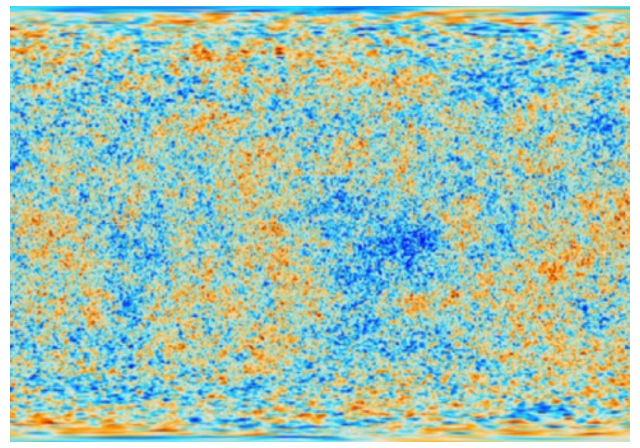


AdS/CFT, curved background, inflation, …

gravity amps, black holes & GW

moduli spaces, string perturbation + CFT…





Combinatorial Geometries (with ``factorizing bd.”) underlying scattering amplitudes & beyond (by 2023)

- moduli space $\mathscr{M}_{g,n}$ for conventional & (ambi-)twistor strings [Witten, Berkovits 04’; CHY 13’; Mason, Skinner 14’; …]
- positive Grassmannian $G_+(k, n)$, on-shell diagrams etc. for planar N=4 SYM [Arkani-Hamed 12’ et al]

- Amplituhedron: map from $G_+(k, n)$ →all-loop integrand of SYM in momentum twistor space [Arkani-Hamed, Trnka 13’ + Thomas;…] => momentum space [SH, Zhang 18’] & momentum amplituhedron [Ferro et al]
- ABJM amplituhedron: reduction to D=3 from SYM amplituhedron —> all-loop integrand of ABJM! [SH, Kuo, Li, Zhang, 22’ + Huang 23’,…]

- kinematic associahedron (bi-adjoint ϕ^3 tree) & worldsheet associahedron [Arkani-Hamed, Bai, SH, Yan, 17’]
- cosmological polytopes +”kinematic flow” DE for tree-level wave function of universe [Arkani-Hamed et al 17’ , 23’,…]
- surfacehedra + binary geometries for surfaces… => “strings & particles without worldsheet”[w. Arkani-Hamed et al 20-]
- more applications of tropical geometry e.g. tropical Grassmannian for “symbology”, Feynman integrals, etc…

All-loop geometry for (4pt) correlator! [w. Y. Huang, C. Kuo 2405.20292]

canonical form ⟷ physical observables

Correlation function:

half BPS operator k=2

$\mathcal{O}_k(x,y)$

4-pt (loop) correlation function

$$\mathcal{G}^{(\ell)}_{2222} = (2x^2_{12}x^2_{13}x^2_{14}x^2_{23}x^2_{24}x^2_{34})\underline{R_{1234}} \times \mathcal{H}^{(\ell)}_{2222}(x_i), \quad \ell \geq 1$$

factor out y-dependent

Conjecture

$\mathcal{G}^{(\ell)}_{2222}$ for $\ell \geq 1$ related to the canonical form defined in Correlahedron.

n=4 L-loop geom:

Kinematic

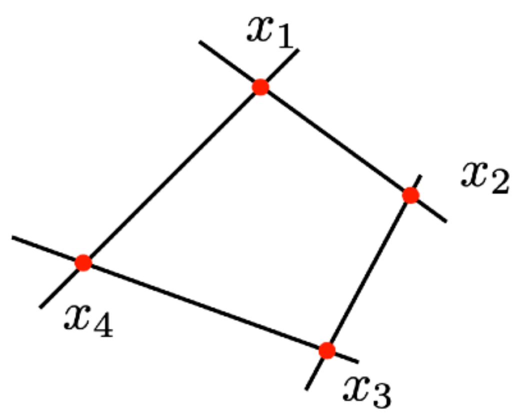
$$Y \in \mathrm{Gr}(4,8),\ \mathrm{X}_i \in \mathrm{Gr}(2,8)$$

$$\langle Y \mathrm{X}_i \mathrm{X}_j \rangle > 0 \ \text{ for } i,j = 1,2,3,4.$$

Loop/ AB space

$$\frac{\langle Y (\mathrm{AB})_a \mathrm{X}_i \rangle}{\langle Y (\mathrm{AB})_a \mathrm{X}_1 \rangle} > 0, \quad \frac{\langle Y (\mathrm{AB})_a (\mathrm{AB})_b \rangle}{\langle Y (\mathrm{AB})_a \mathrm{X}_1 \rangle \langle Y (\mathrm{AB})_b \mathrm{X}_1 \rangle} > 0$$

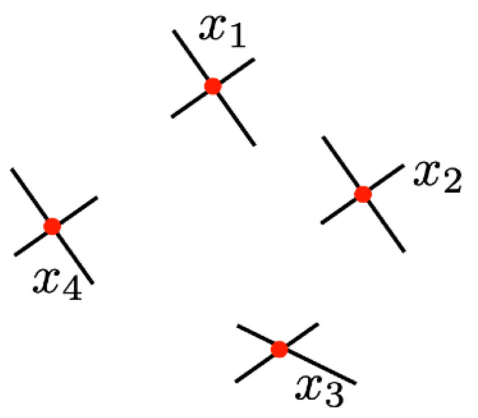
Amplitudes



$p_i^2 = 0$

ordering (cyclic)

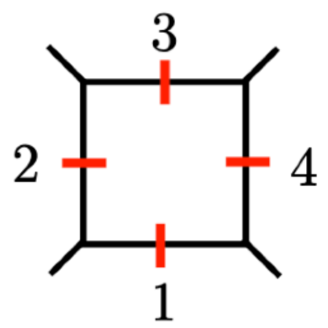
Correlators



$x^2_{i,j} \neq 0$

permutation

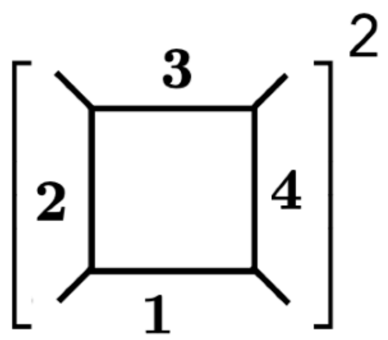
$\Delta^2 > 0:$ 4-mass box integrand



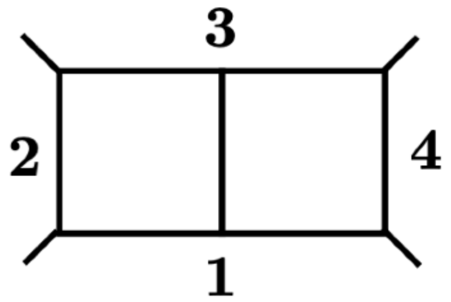
$$\frac{\pm \Delta}{\langle (AB) X_1 \rangle \langle (AB) X_2 \rangle \langle (AB) X_3 \rangle \langle (AB) X_4 \rangle} d\mu_{AB}$$

$$\Delta \equiv \langle X_1 X_3 \rangle \langle X_2 X_4 \rangle \sqrt{(1-v-w)^2 - 4vw}$$

$$\Delta^2 > 0: \quad I^{\ell=2}_{\pm} = \left(\frac{\Delta^2}{2} g^2_{1234} \pm \Delta \left(h_{12,34} + 5\,\text{perms} \right) \right)$$



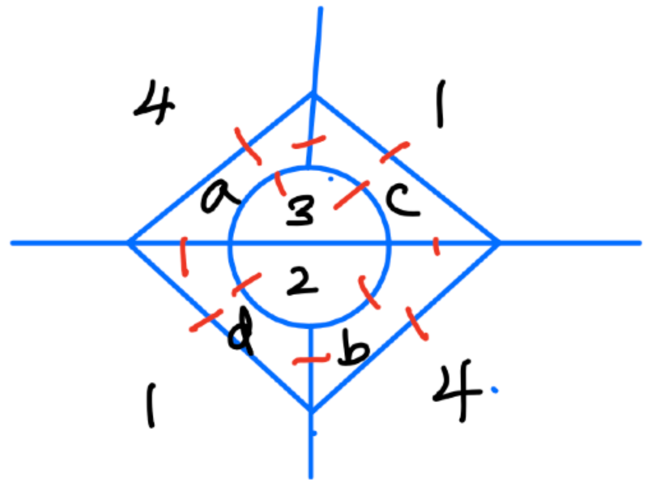
g^2_{1234}

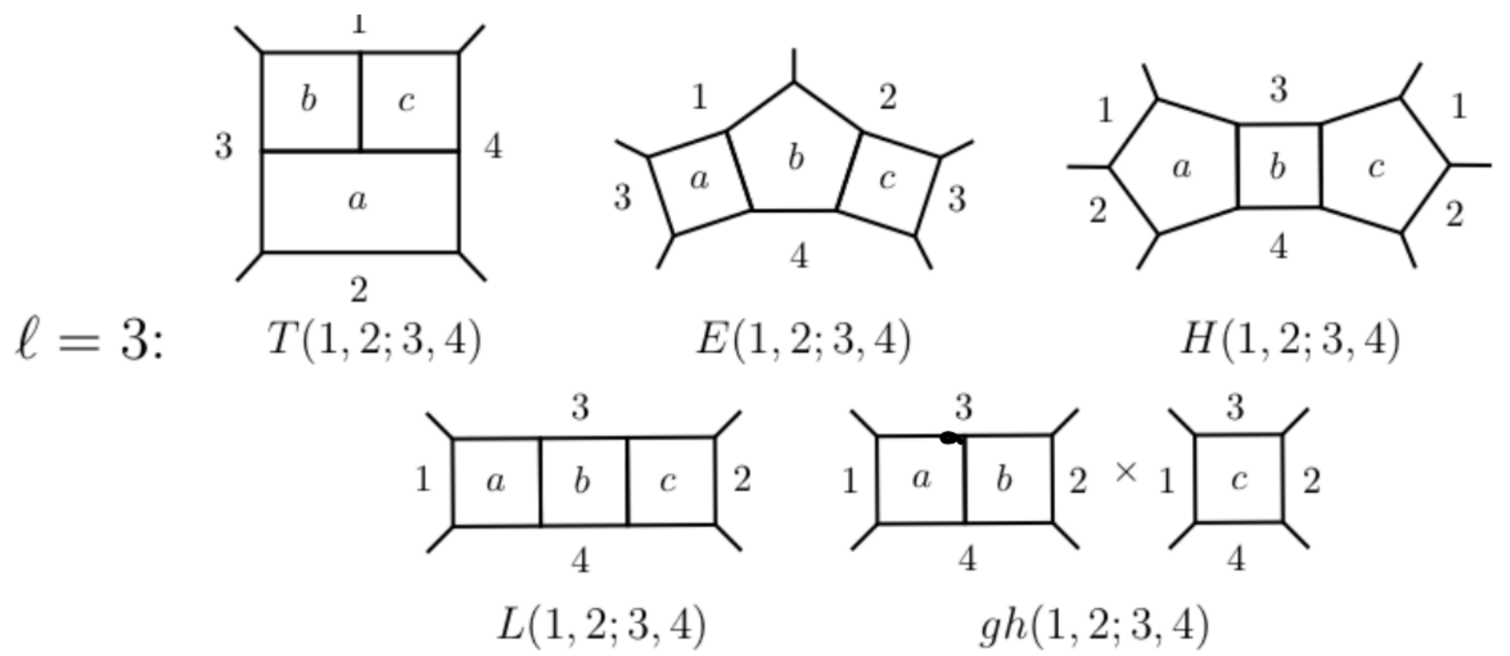


$h_{12,34}$

Loops as fibration over trees [w. Y. Huang, C. Kuo 2405.20292 + to appear]

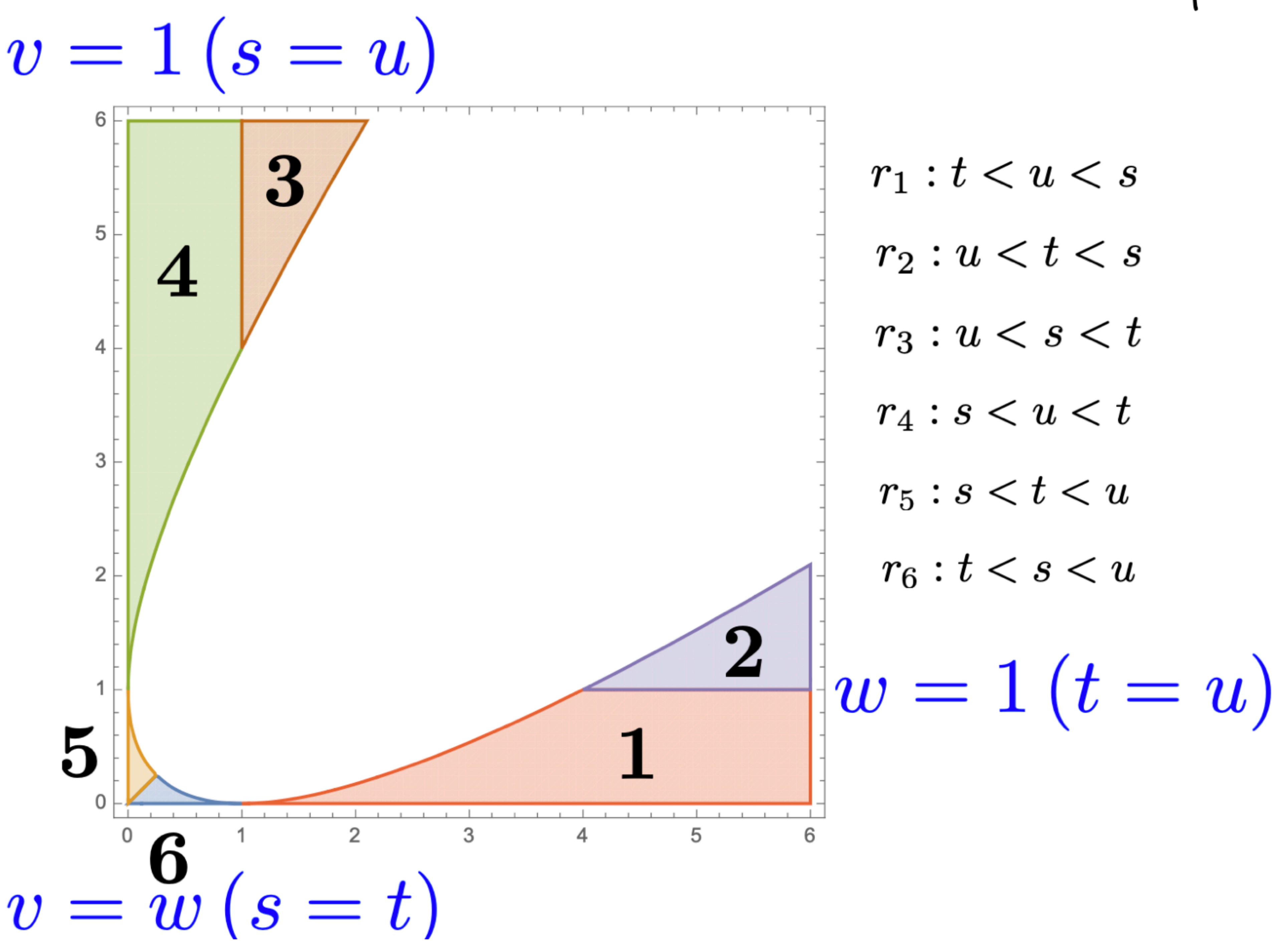
- In general, non-renormalizable theorem (prefactor for all loops)follows from tree geometry
- Starting L=3: distinct loop forms for different tree/kinematic regions! (new leading singularities etc)
- Computed loop forms for L=3,4 (sum over all 6 chambers/different LS)! 4-loop elliptic cut?





$$\Omega^{(3)\pm}_{r_i} = \Delta^2 A_{\sigma_3} \pm \Delta\big(B - \sigma_1(C_{\sigma_2} + C_{\sigma_3}) - \sigma_2 C_{\sigma_1}\big)\,,$$

$$\begin{aligned} A_s :=& \big[H(1,4;2,3) - E'(1,4;2,3) + (1,4)\leftrightarrow(2,3)\big] + (3\leftrightarrow 4) \\ &+ gh(1,2;3,4) + gh(3,4;1,2)\,, \\ B :=& T(1,2;3,4) + E(1,2;3,4) + 11\,\text{perms.} \\ &+ L(1,2;3,4) + (t+u)E'(1,2;3,4) + 5\,\text{perms.}\,, \\ C_s :=& 4(E'(1,2;3,4) + E'(3,4;1,2)) \end{aligned}$$



$r_1 : t < u < s$

$r_2 : u < t < s$

$r_3 : u < s < t$

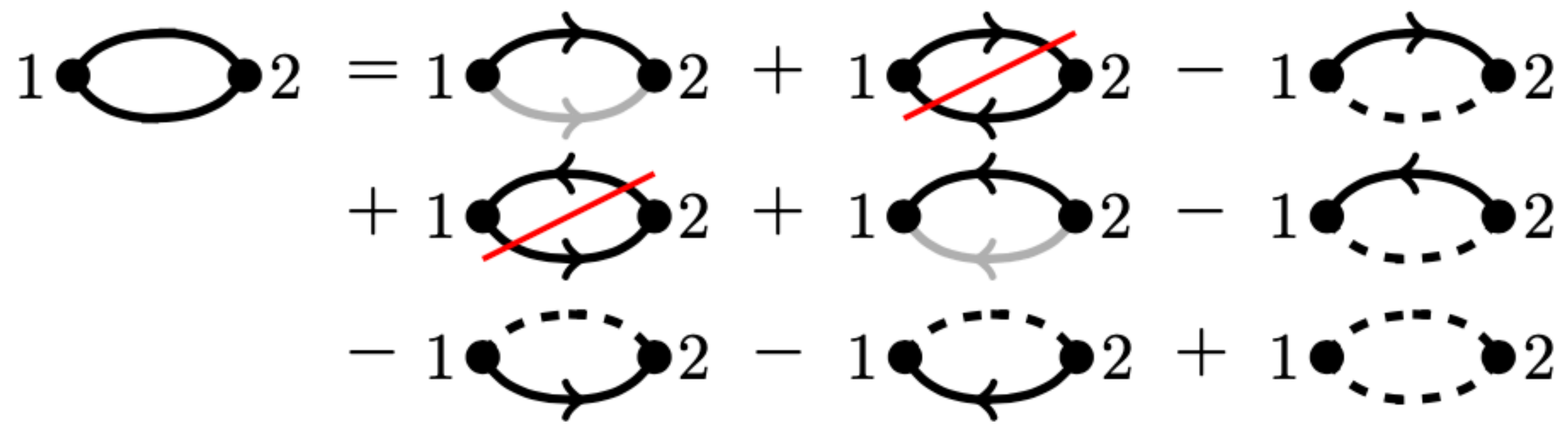
$r_4 : s < u < t$

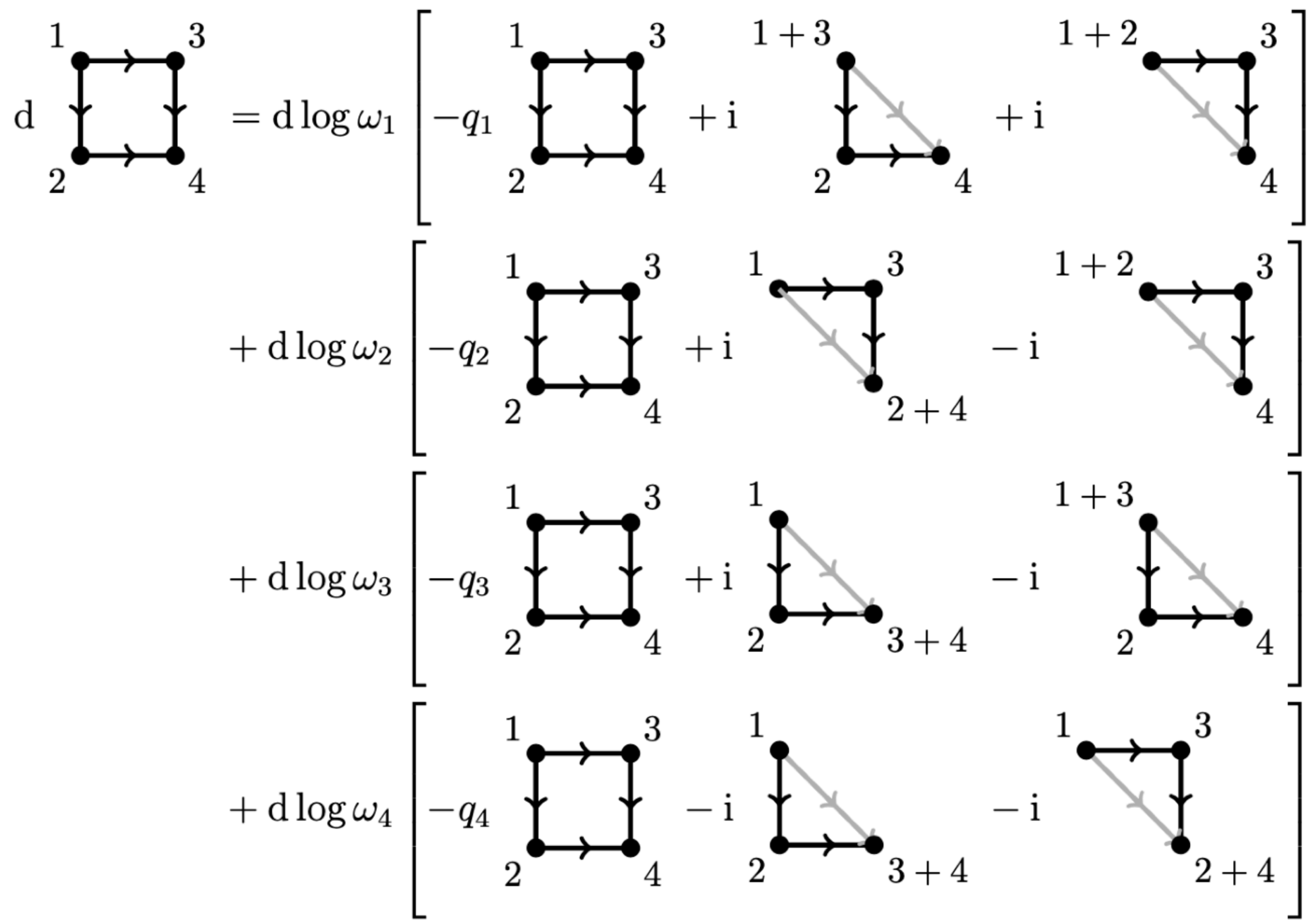
$r_5 : s < t < u$

$r_6 : t < s < u$

Cosmo. amplitudes: diff. eqs & recursion [w. 姜旭航, 刘家昊, 杨清霖, 张耀奇 2407.xxxxx]

- wave function coefficients & correlators for (conformal) scalars in FRW universe, e.g. q=0 for de Sitter
- Nested time integrals => naturally decompose into building blocks e.g. family trees, analytically solved in terms of gen. hypergeometric series [Fan, Zhong-zhi, 2024]; in general “loop integrands” —> all directed graphs



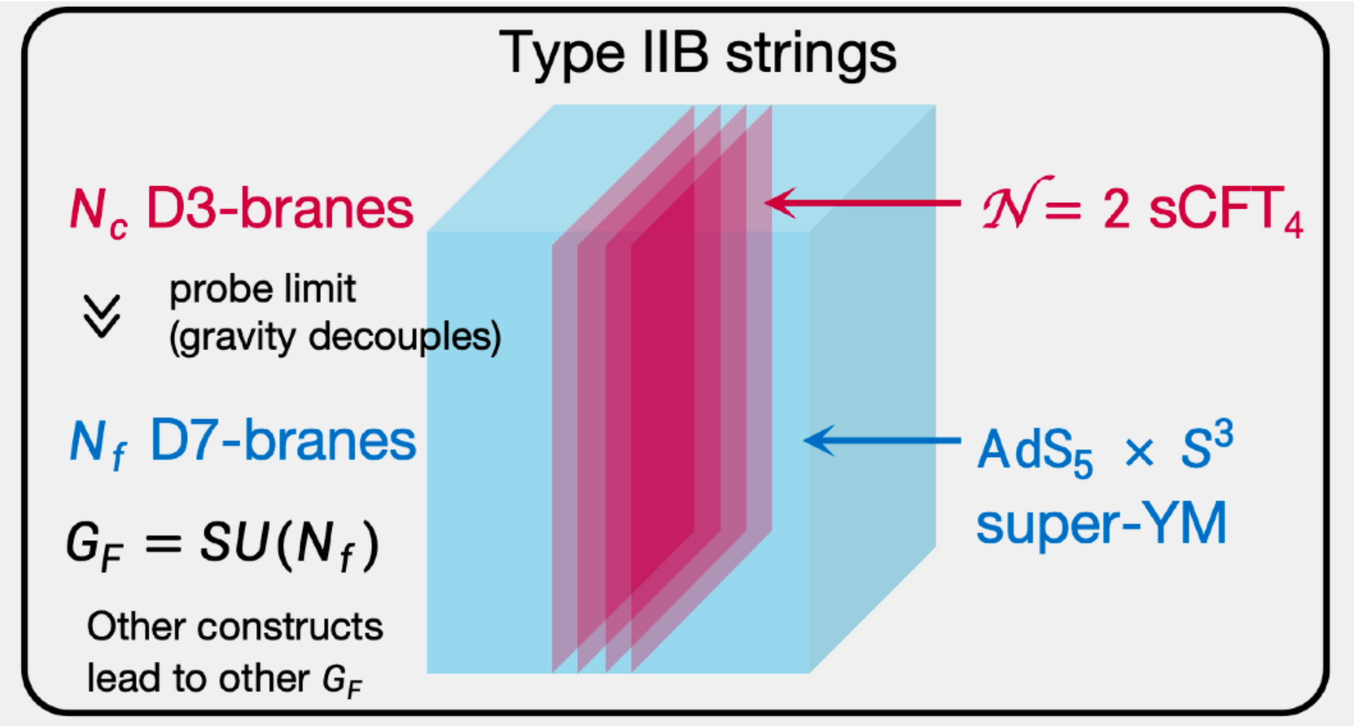


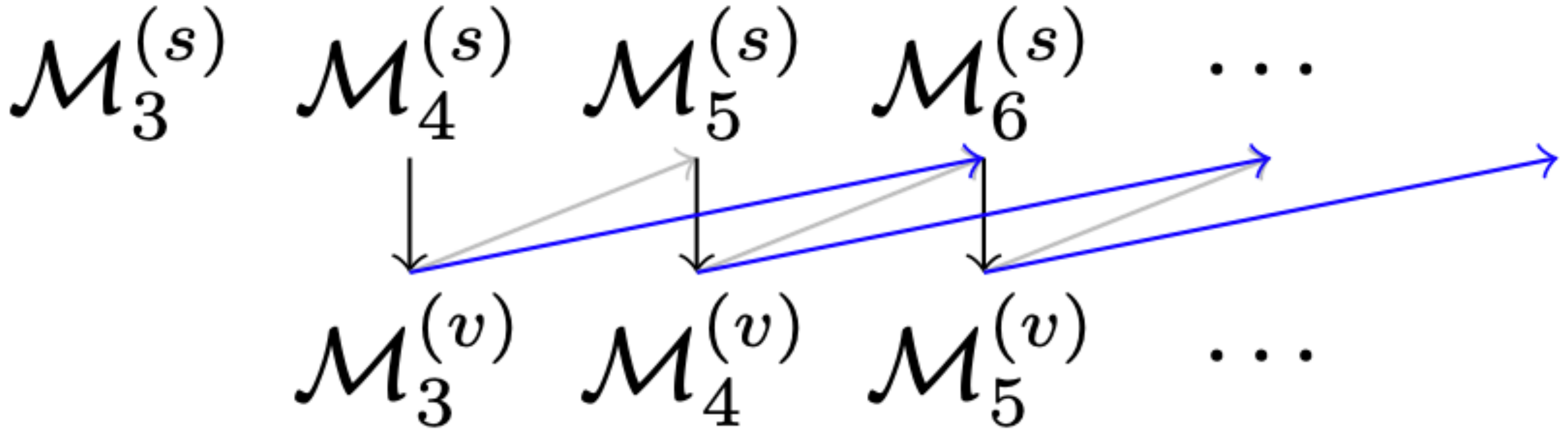
$$
\mathrm{d}\begin{pmatrix}\psi_{\text{2-chain}}\\ -\mathbf{B}_1/q_1\\ -\mathbf{B}_2/q_2\\ \mathbf{C}/(q_1q_2)\end{pmatrix}=\begin{pmatrix}-q_1\ell_1-q_2\ell_2 & q_1\ell_1-q_1\ell_3 & q_2\ell_2-q_2\ell_4 & 0\\ 0 & -q_1\ell_3-q_2\ell_2 & 0 & q_2\ell_2-q_2\ell_5\\ 0 & 0 & -q_1\ell_1-q_2\ell_4 & q_1\ell_1-q_1\ell_5\\ 0 & 0 & 0 & -(q_1+q_2)\ell_5\end{pmatrix}\begin{pmatrix}\psi_{\text{2-chain}}\\ -\mathbf{B}_1/q_1\\ -\mathbf{B}_2/q_2\\ \mathbf{C}/(q_1q_2)\end{pmatrix}
$$

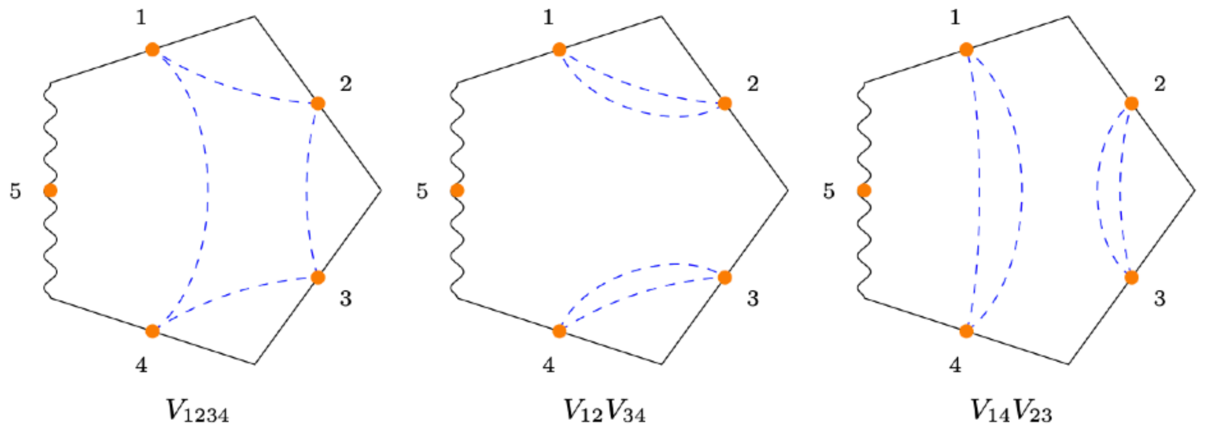
- Simplest DE for cosmo amps of any directed graph: contracting edge one at a time -> recursion relations
- For tree amps: combined to give canonical DE—> “kinematic flow"; more interesting CDE for loops
- byproduct: a compact, recursive formula for de Sitter (q=0) tree amps: multi-polylog & symbol structures

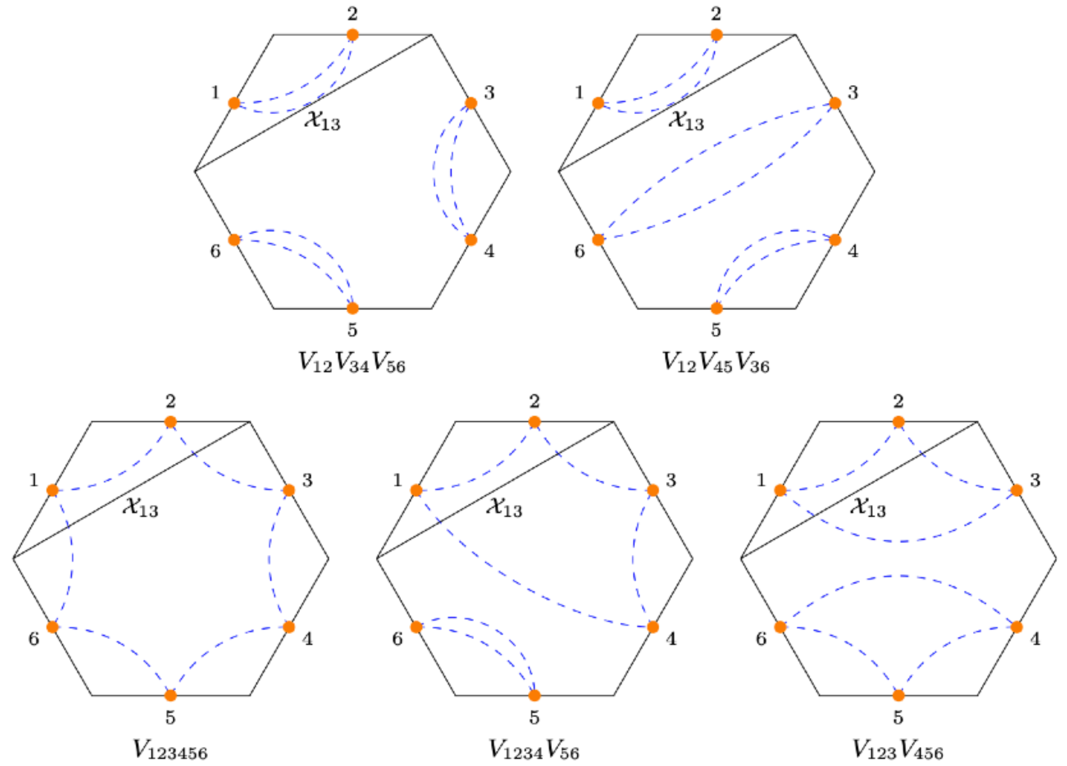
Supergluon amps in AdS to all n [w. 曹趣, 唐一朝, 2312.15484, + 李想, 2406.08538]

- Supergluon amplitudes in AdS_5 x S^3 (tree-level): rich data for CFT_4 & "scattering in AdS"; known up to n=6 based on factorizations (OPE) + flat-space limit [Alday, Goncalves, Nocchi, Zhou 2023]
- We find a recursive algorithm for supergluon & spinning amps to all n ("AdS constructibility")



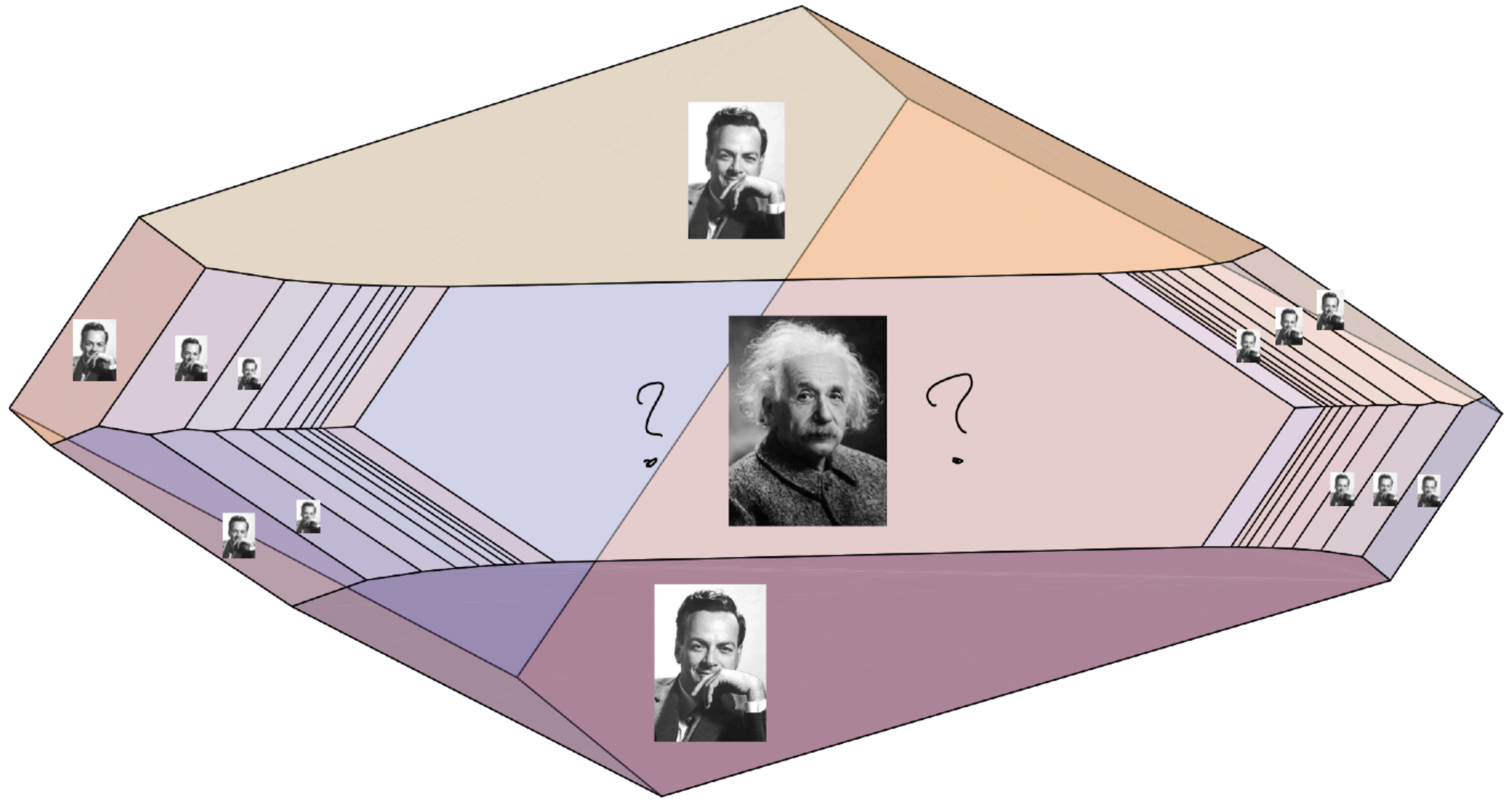






- Explicit, compact results up to n=8 (spinning for n=7), and the simplest R-symmetry case to all n
- New structures: general poles (truncation of descendents), nice Feynman rules, collinear/soft etc.
- They can be viewed as AdS generalizations of "scalar-scaffolded gluons" in flat-space!

Toy Models —> Real World via "numerators/zeros"



Combinatorial/geometries: e.g. SYM/ABJM, or Tr ϕ^3 (simplest colored scalars)
Amps uniquely determined by long-distance sing. or "denominators" !

More realistic theories: need "pole @ infinity" or numerators: N=4 (no pole @ infinity by DCI) vs. N<4;
Tr ϕ^3 (projective inv.) vs. other scalar theories e.g. ϕ^p , derivative coup. such as pions, even gluons?

What are "zeros" of (tree) amplitudes? already highly non-trivial for Tr ϕ^3: pattern of zeros (some $s_{i,j} = 0$) &
surprising factorizations near them; hidden in Feynman diagrams, manifest by geometries!

The same zeros+ factorizations are also present for non-linear sigma model & Yang-Mills:

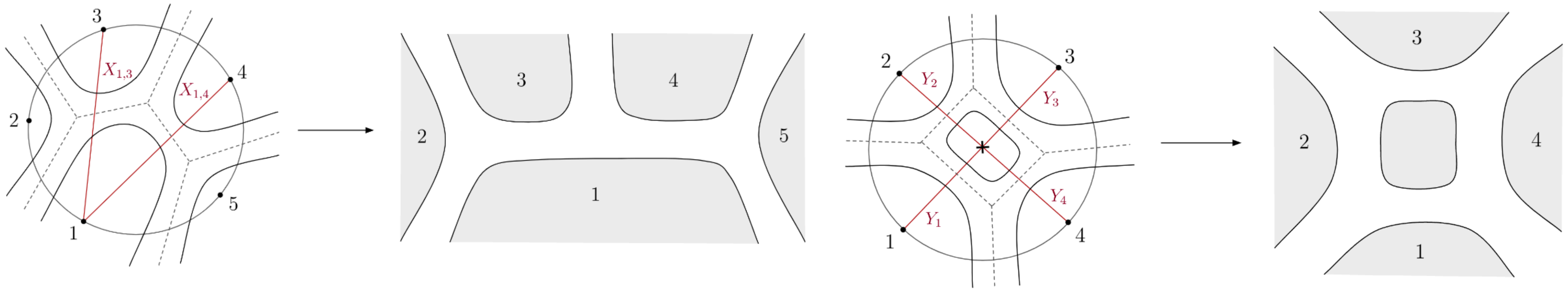
tree amps of Tr ϕ^3, pions & gluons given by one and same function at different kinematic points !

—> all-loop NLSM & (conjecturally) YM contained in all-loop (stringy) Tr ϕ^3

Tr ϕ^3 amplitudes [Arkani-Hamed, Bai, SH, Yan, '17; Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas, '23,…]

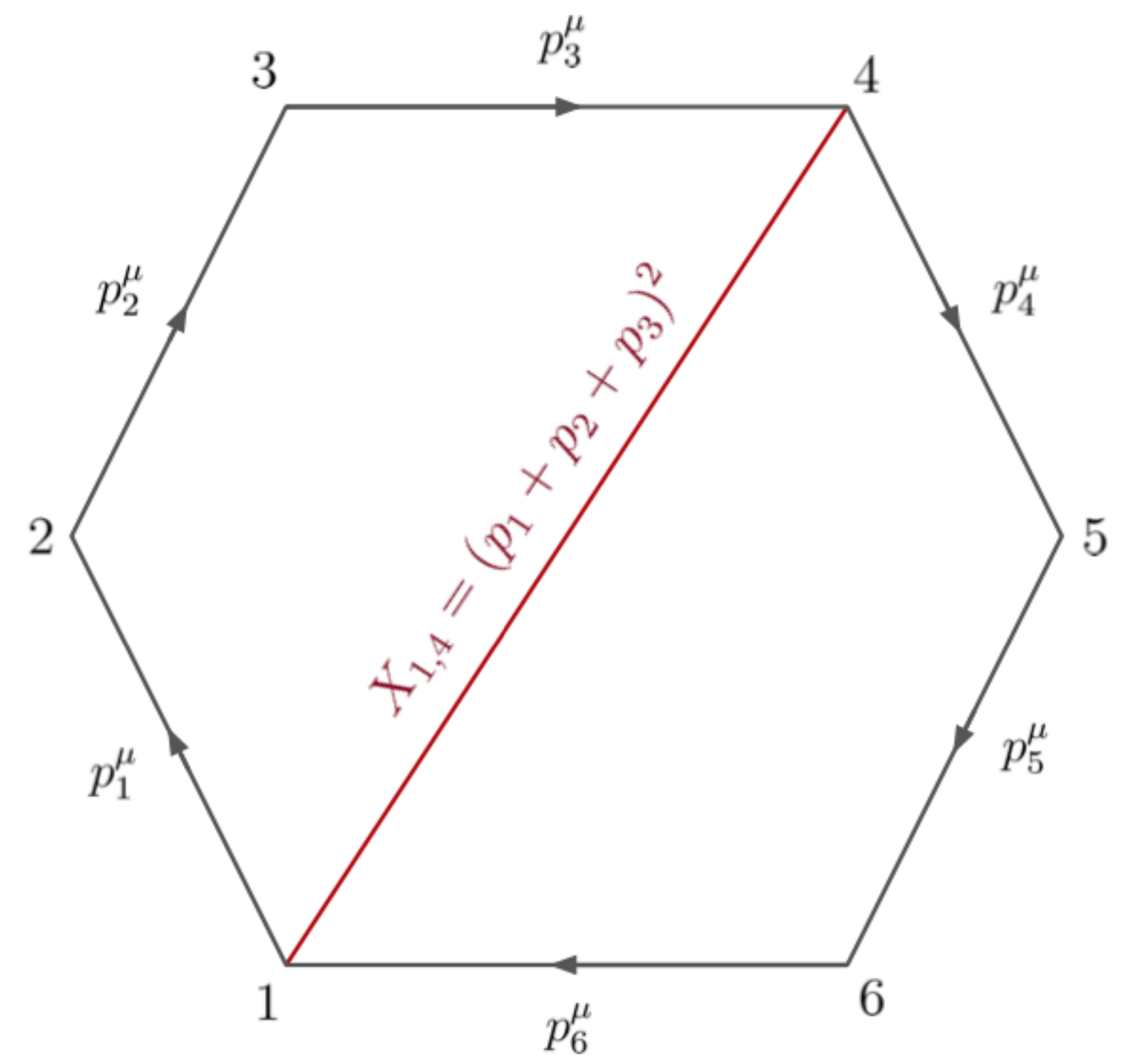
$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

ϕ : N by N matrix -> fat graphs, genus expansion (only planar graphs for $N \to \infty$)



planar variables: all poles of tree amps

$$X_{i,j} = (p_i + \cdots + p_{j-1})^2.$$



tree amp= sum over n-gon triangulations, e.g.

$$\mathcal{A}_4 = \frac{1}{X_{13}} + \frac{1}{X_{24}}\,,$$

$$\mathcal{A}_5 = \frac{1}{X_{1,3}X_{1,4}} + \frac{1}{X_{2,4}X_{2,5}} + \frac{1}{X_{1,3}X_{3,5}} + \frac{1}{X_{1,4}X_{2,4}} + \frac{1}{X_{2,5}X_{3,5}}.$$

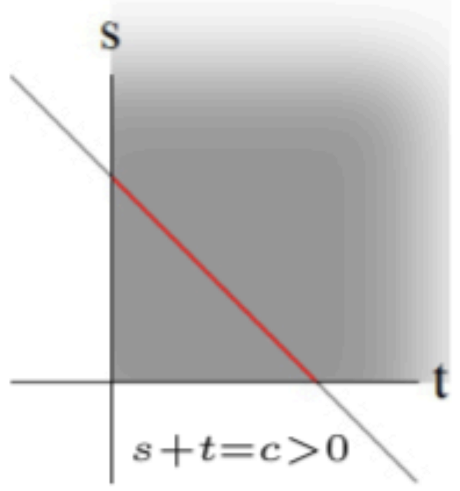
non-planar Mandelstam variables

$$c_{i,j} := -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1}\,,$$

Tree amplitude from associahedron [ABHY; Arkani-Hamed, SH, Salvatori, Thomas '19]

$\mathscr{A}_{n-3} : \{X_{i,j} \geq 0\} \cap$ (n-3)-dim subspace

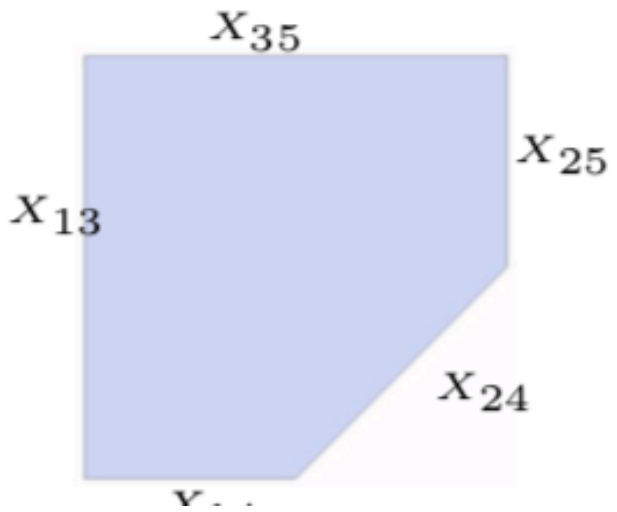
$c_{i,j} = \text{const.} > 0$ e.g. for $1 \leq i < j \leq n-1$

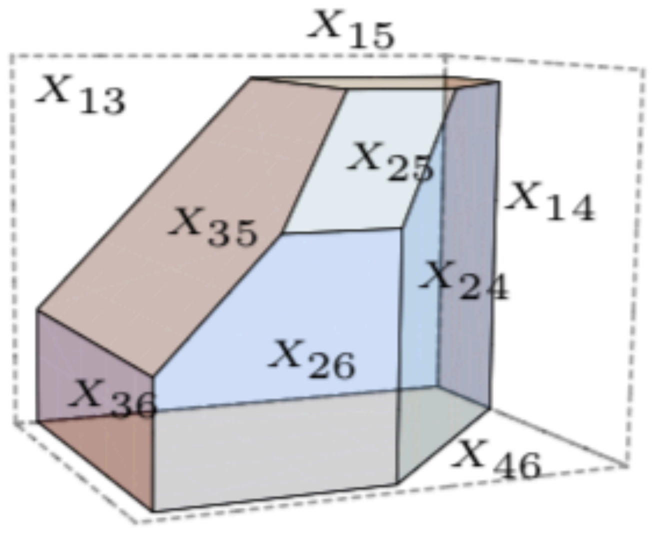


$$e.g.\ \mathcal{A}_1 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$$

$$\mathcal{A}_2 = \{X_{13}, \cdots, X_{25} > 0\} \cap \{-s_{13} = c_{13}, -s_{14} = c_{14}, -s_{24} = c_{24}\}$$

$$\mathscr{A}_2 : (n=5) \begin{cases} X_{1,3} > 0 \\ X_{1,4} > 0 \\ X_{2,4} > 0 \Leftrightarrow c_{1,3} - X_{1,3} + X_{1,4} > 0 \\ X_{2,5} > 0 \Leftrightarrow c_{1,3} + c_{1,4} - X_{1,3} > 0 \\ X_{3,5} > 0 \Leftrightarrow c_{1,4} + c_{2,4} - X_{1,4} > 0 \end{cases}$$





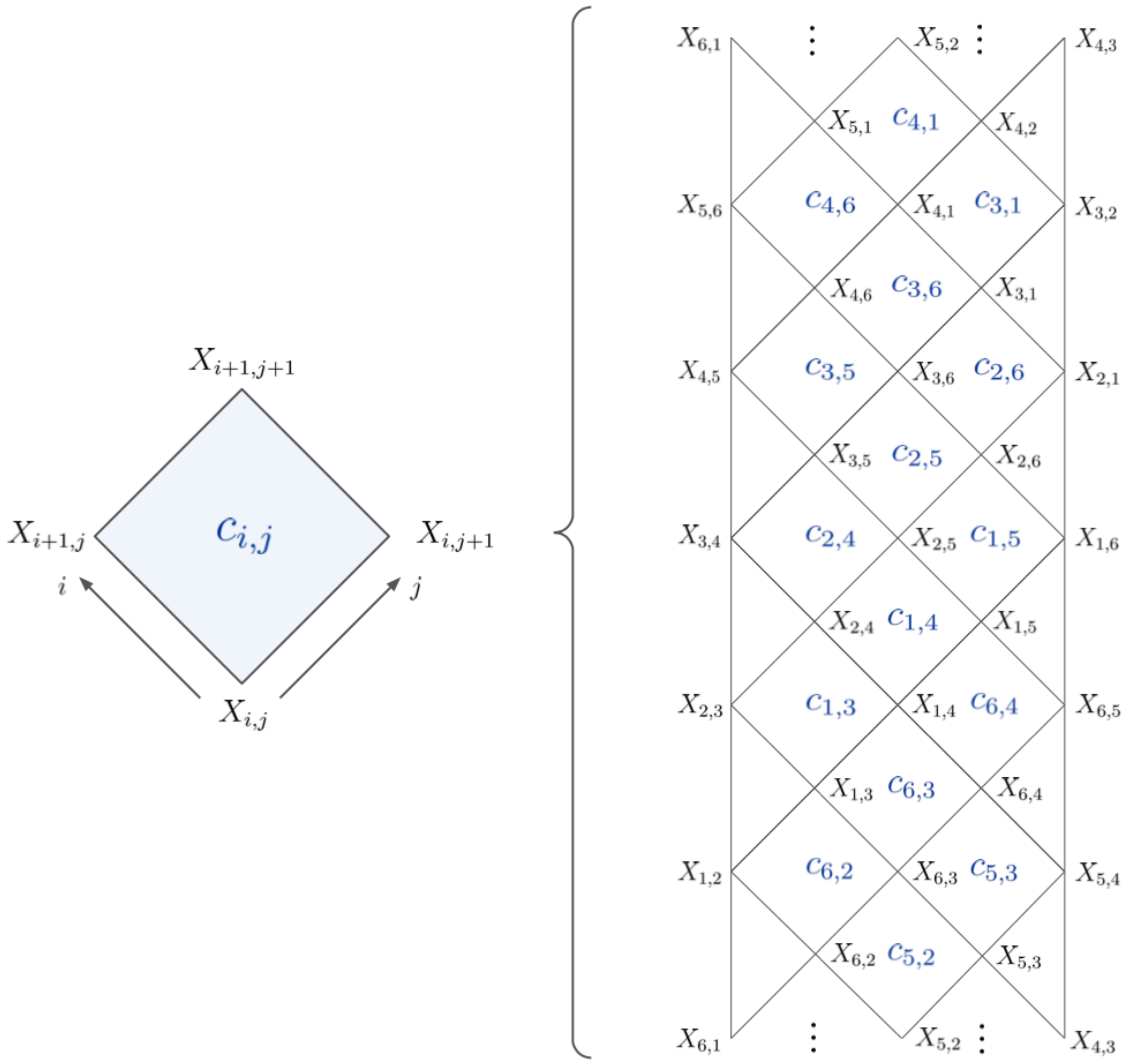
$$\Omega(\mathscr{A}_2) = d^2X \left(\frac{1}{X_{13}X_{14}} + \frac{1}{X_{13}X_{35}} + \frac{1}{X_{25}X_{35}} + \frac{1}{X_{25}X_{24}} + \frac{1}{X_{24}X_{14}} \right)$$

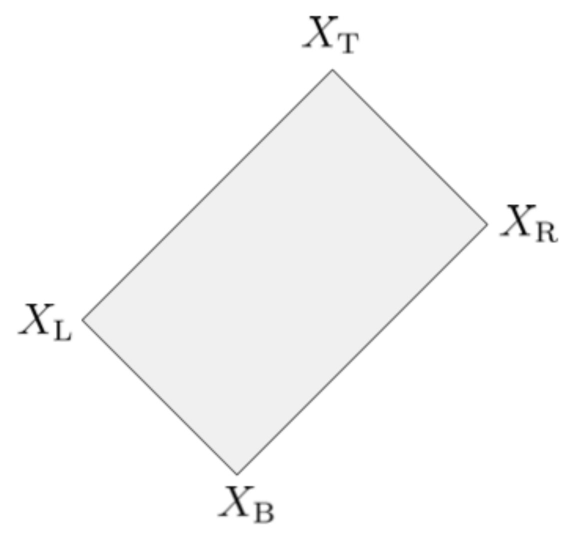
$$\Omega(\mathscr{A}_1) = \left(\frac{dX_{13}}{X_{13}} - \frac{dX_{24}}{X_{24}}\right)\Big|_{X_{13}+X_{24}=c_{13}} = dX_{13} \left(\frac{1}{X_{13}} + \frac{1}{c_{13} - X_{13}} \right)$$

geometric picture: FD expansion = a special triangulation, others → new formula & recursion for ϕ^3 amps

hidden symmetry of ϕ^3 amps (invisible in FD's), analog of dual conformal symmetry, manifest by geometry!

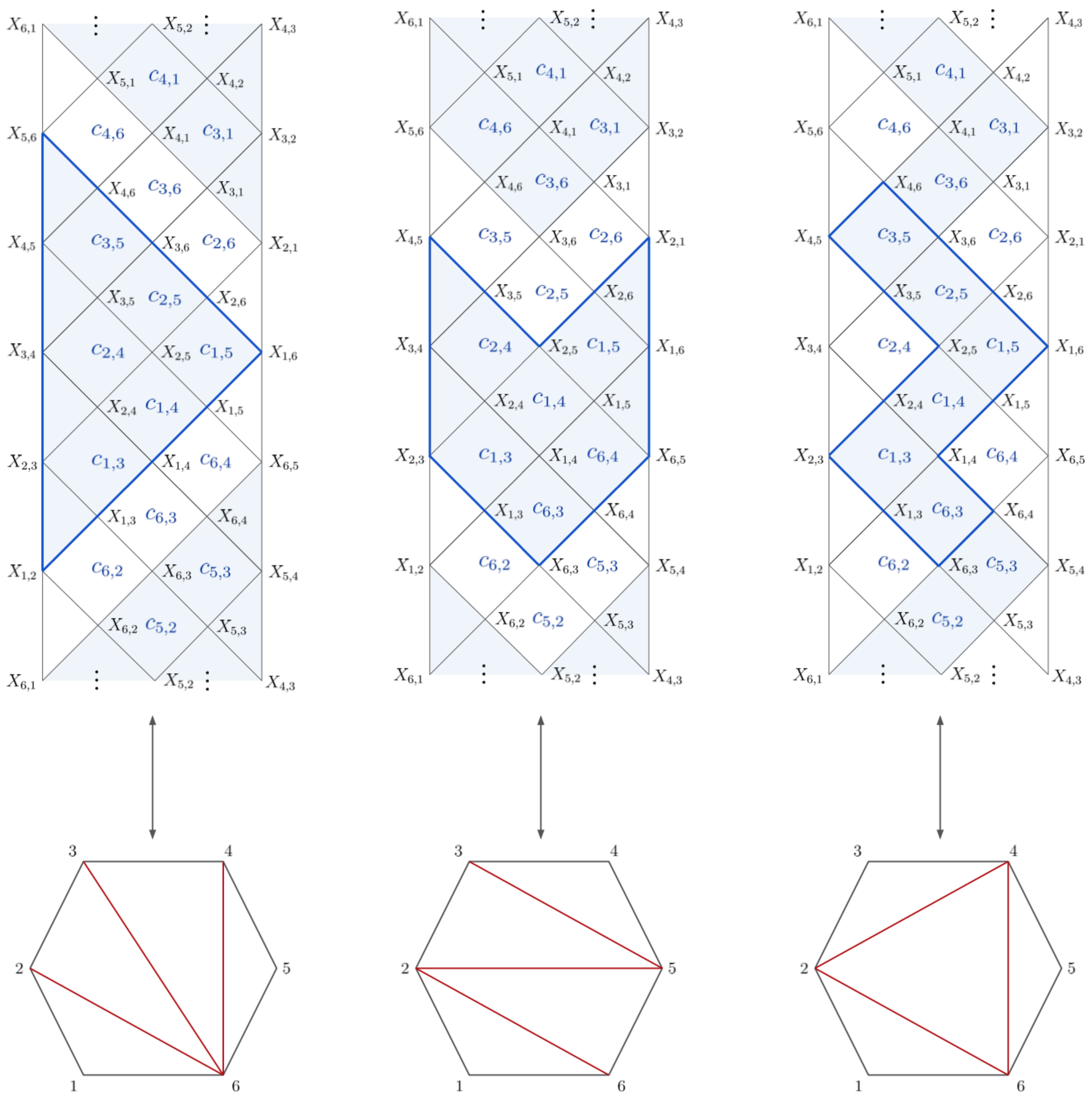
Kinematic mesh: 1+1 dim wave eqs & causal diamonds





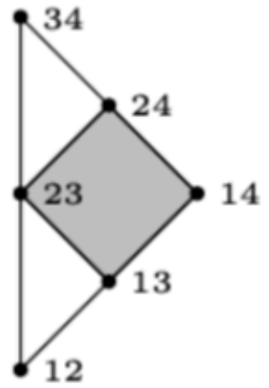
$$X_{\rm B} + X_{\rm T} - X_{\rm L} - X_{\rm R} = \sum_{c_{i,j} \in \diamond} c_{i,j}$$

mesh regions <=> initial triangulations (n=6)

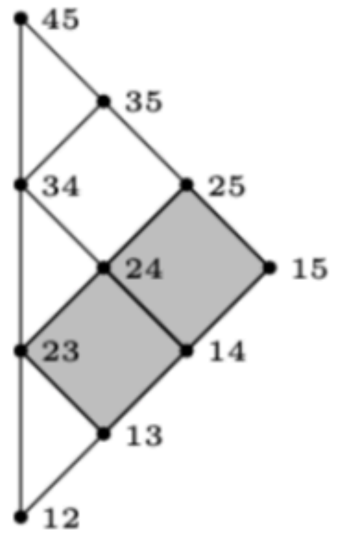


A surprise: zeros of Tr ϕ^3 on the mesh [Arkani-Hamed, Cao, Dong, Figuereido, SH, 2312.16282]

$$n = 4 : c_{13} = 0 \implies \frac{1}{X_{13}} + \frac{1}{X_{24}} = \frac{c_{13}}{X_{13}X_{24}} = 0$$



$$n = 5 : c_{13} = c_{14} = 0\,,\ \text{or}\ c_{14} = c_{24} = 0\,,\text{etc}\,.$$



the big cubic polynomial $N^{(3)} = 0$

Very difficult to see in Feynman diagrams: $$\frac{1}{X_{13}X_{14}} + \frac{1}{X_{13}X_{35}} + \frac{1}{X_{25}X_{35}} + \frac{1}{X_{25}X_{24}} + \frac{1}{X_{24}X_{14}} = \frac{N^{(3)}(\{X\})}{X_{13}X_{24}X_{35}X_{14}X_{25}}$$

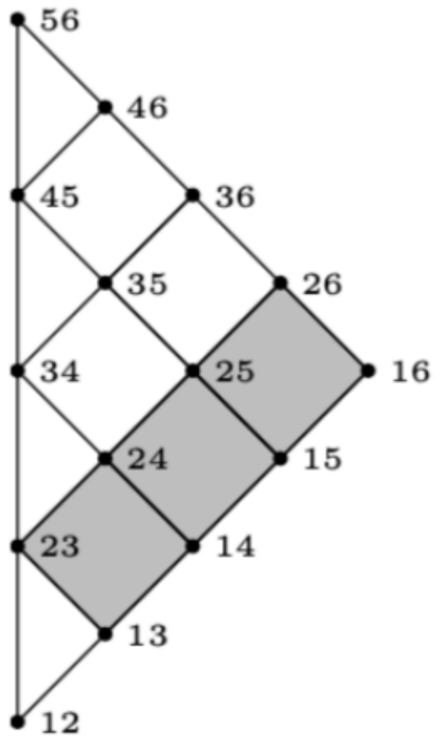
skinny rectangle ("soft limit")

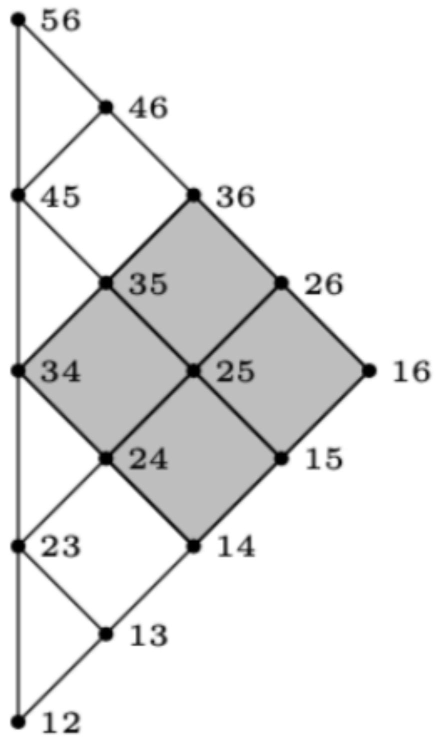
n-pt: highly non-trivial linear subspaces of the big numerator e.g. $c_{13} = c_{14} = \cdots = c_{1,n-1} = 0$

2 by 2 square

$$n = 6 : \text{also}\ c_{14} = c_{15} = c_{24} = c_{25} = 0\,;$$

generally any rectangle of the mesh



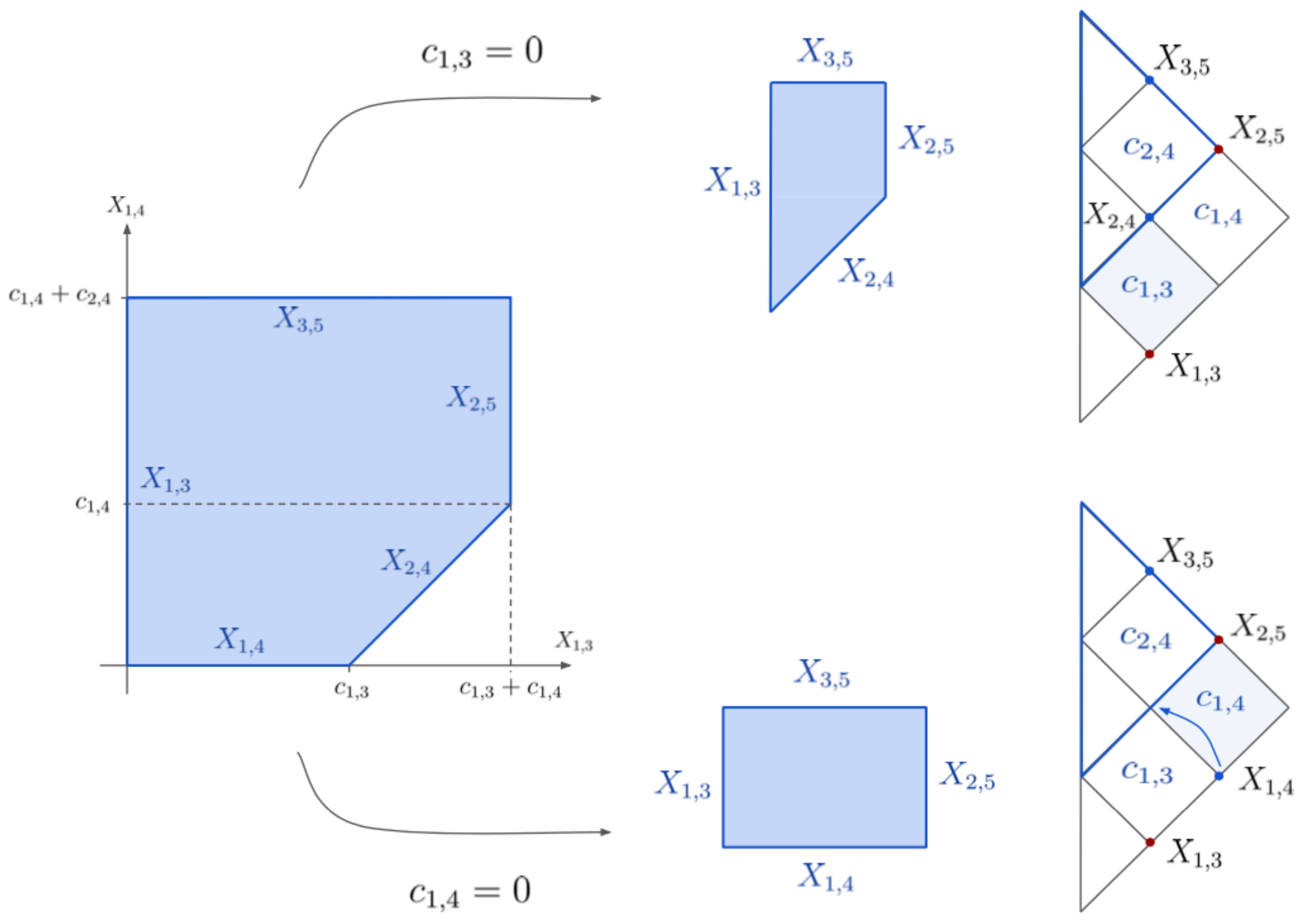


Factorization of 5-pt amplitude

the last step: turn on one $c \neq 0$

$$\mathcal{A}_5(X_{1,3}, X_{1,4}, X_{2,4}, X_{2,5}, X_{3,5}) \xrightarrow{c_{1,3}=0} \left(\frac{1}{X_{1,3}} + \frac{1}{X_{2,5}}\right) \times \left(\frac{1}{X_{2,4}} + \frac{1}{X_{3,5}}\right),$$

$c_{1,4} \neq 0$:
Mink. sum interval + triangle

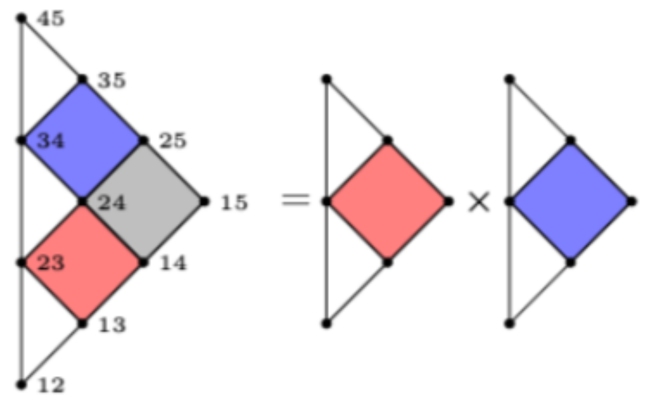


$$\left(\frac{1}{X_{1,3}} + \frac{1}{X_{2,5}}\right) \times \mathcal{A}_4(X_{2,4}, X_{3,5})$$

$$\left(\frac{1}{X_{1,3}} + \frac{1}{X_{2,5}}\right) \times \mathcal{A}_4(X_{1,4}, X_{3,5})$$

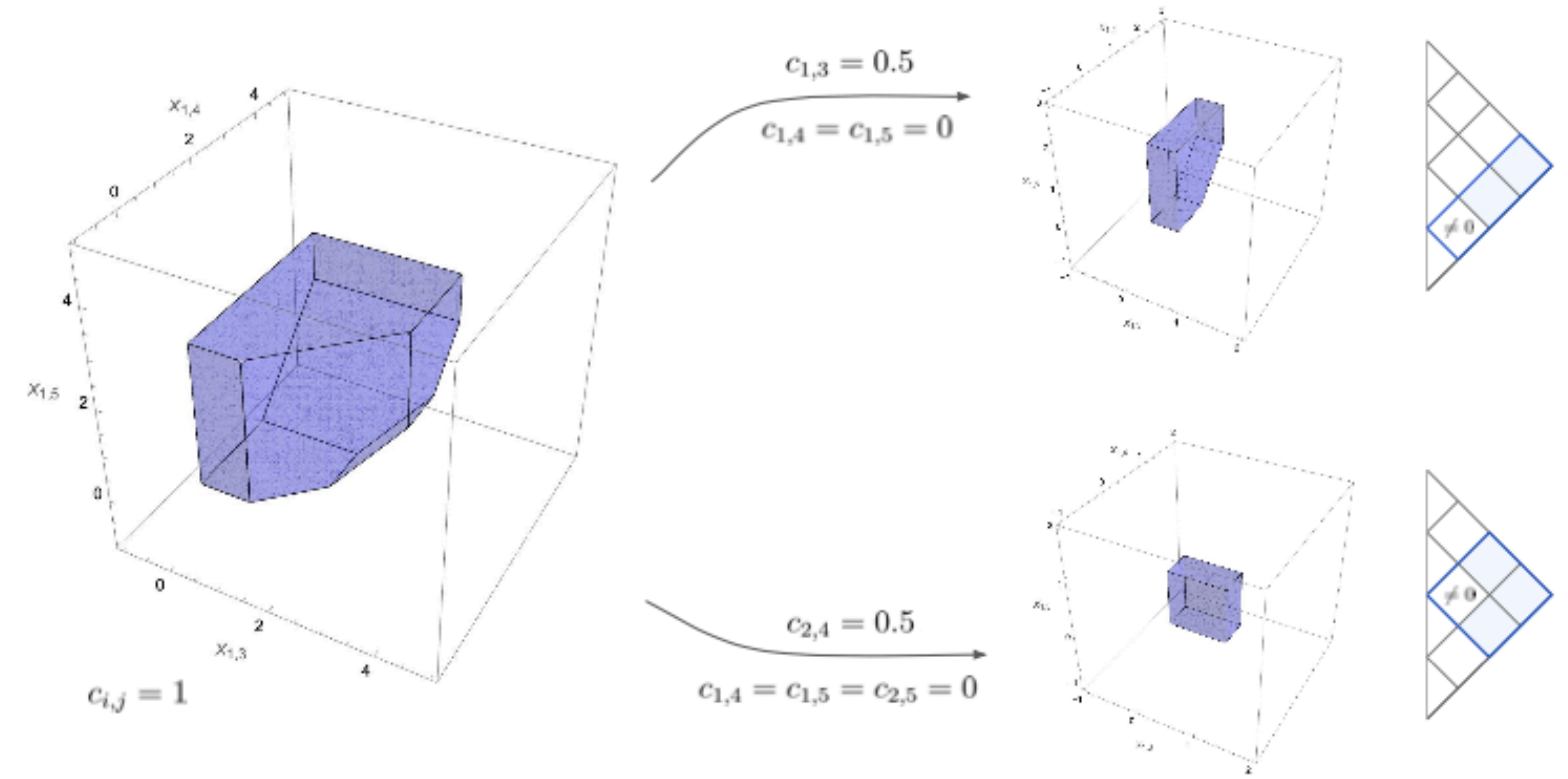
$c_{1,3} \neq 0$:
Mink. sum interval + interval

$$\mathcal{A}_5(X_{1,3}, X_{1,4}, X_{2,4}, X_{2,5}, X_{3,5}) \xrightarrow{c_{1,4}=0} \left(\frac{1}{X_{1,3}} + \frac{1}{X_{2,5}}\right) \times \left(\frac{1}{X_{1,4}} + \frac{1}{X_{3,5}}\right)$$

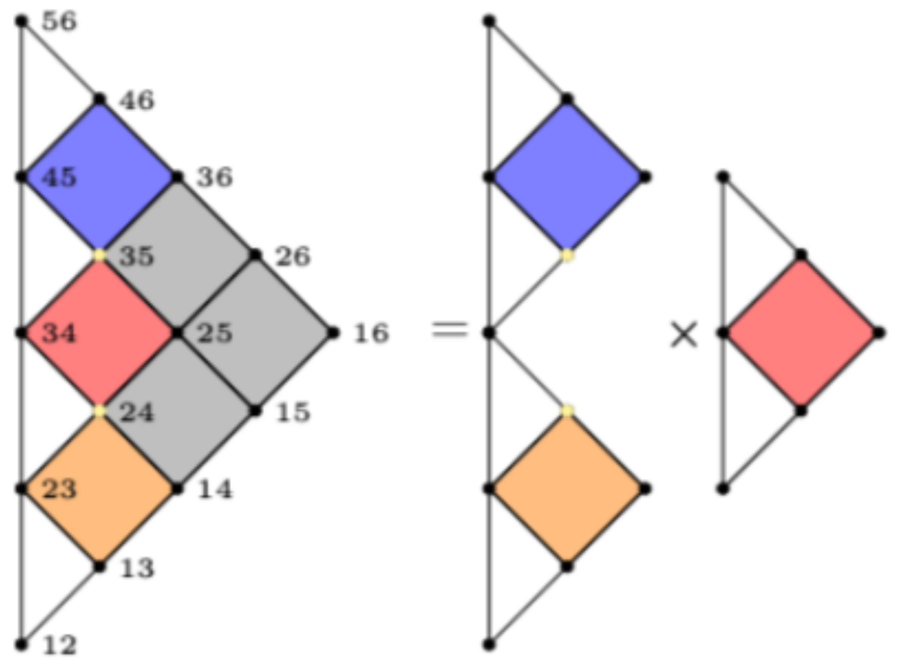


Factorization of 6-pt amplitude

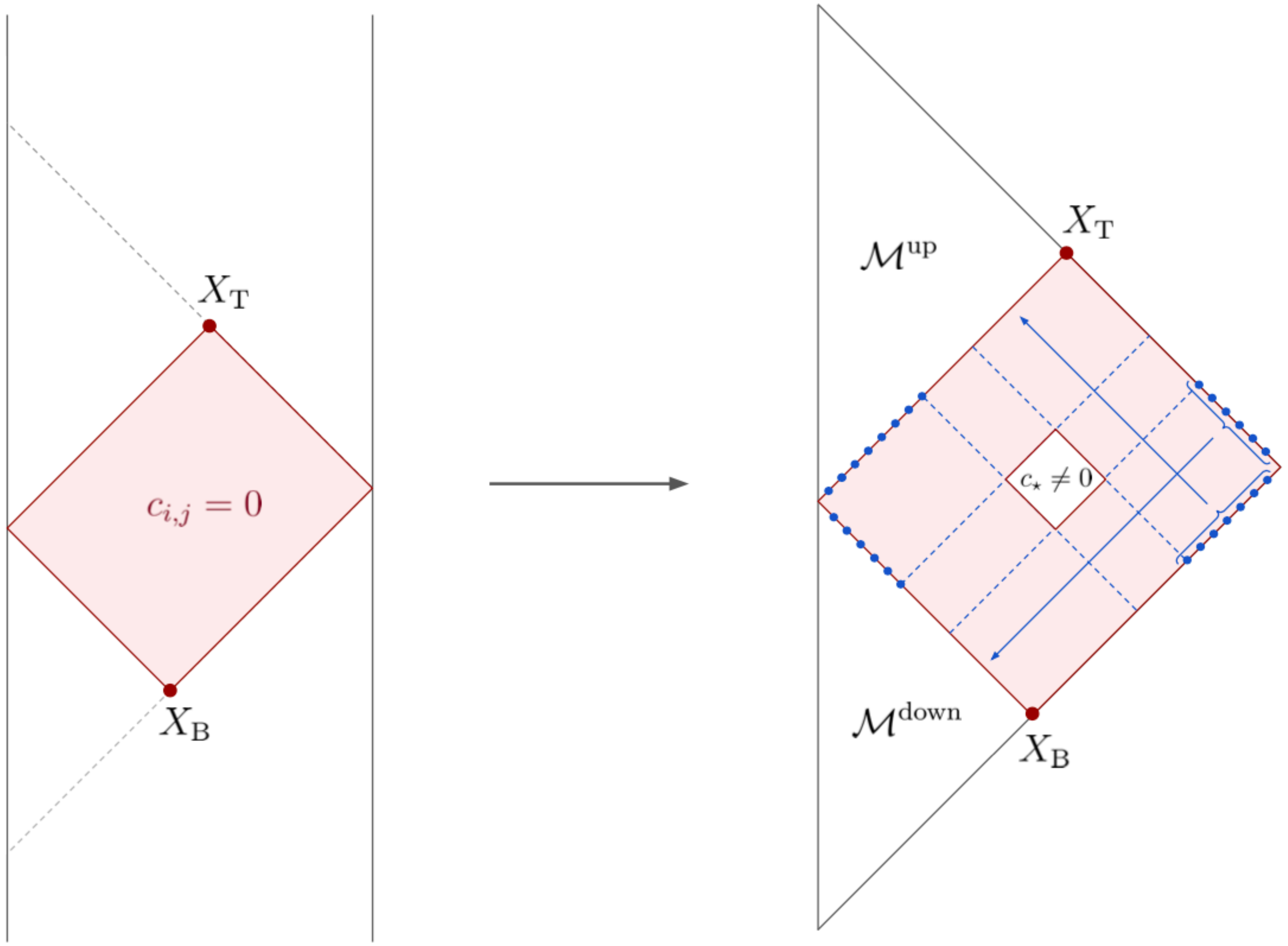
$$\mathcal{A}_6 \xrightarrow{c_{1,3}=0, c_{1,5}=0} \left(\frac{1}{X_{1,3}}+\frac{1}{X_{2,6}}\right)\times\left(\frac{1}{X_{2,4}X_{1,5}}+\frac{1}{X_{1,5}X_{3,5}}+\frac{1}{X_{3,5}X_{3,6}}+\frac{1}{X_{3,6}X_{4,6}}+\frac{1}{X_{4,6}X_{2,4}}\right)$$



$$\mathcal{A}_6 \xrightarrow{c_{1,4}=0, c_{1,5}=0, c_{2,5}=0} \left(\frac{1}{X_{1,4}}+\frac{1}{X_{3,6}}\right)\times\left(\frac{1}{X_{1,3}}+\frac{1}{X_{2,6}}\right)\times\left(\frac{1}{X_{1,5}}+\frac{1}{X_{4,6}}\right)$$



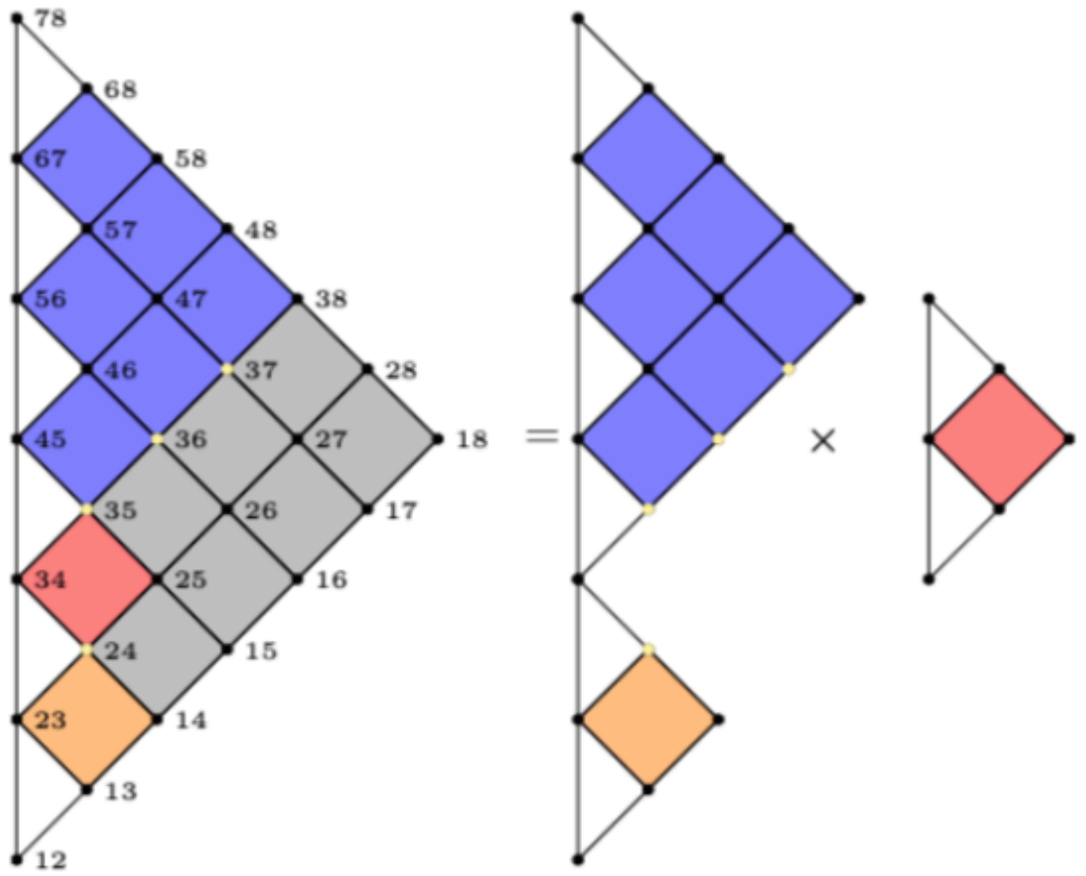
General zeros & factorizations



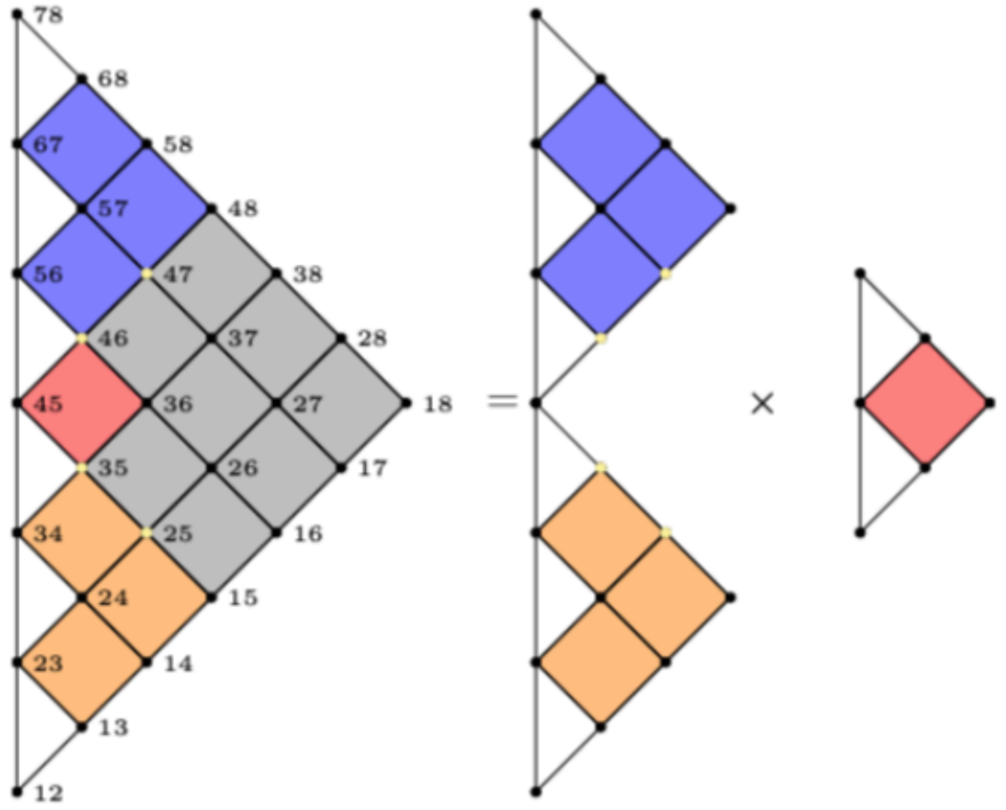
$$\mathcal{M}_n(c_\star \neq 0) = \left(\frac{1}{X_{\rm B}} + \frac{1}{X_{\rm T}}\right) \times \mathcal{M}^{\rm down} \times \mathcal{M}^{\rm up}.$$

$$\tilde{X} = X_{\rm T} + X, \quad X = X_B + \tilde{X}.$$

8-> 6*4*4



8-> 5*4*5



shifted kinematics: in terms of momenta, these are currents (with an off-shell leg)

Stringy Tr ϕ^3 amplitude [Arkani-Hamed, SH, Lam, 19']

Veneziano-Koba-Nielsen amplitudes (60's)

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3}(1,2,\ldots,n)=\int_{D(1\ldots n)}\frac{\mathrm{d}z_1\ldots\mathrm{d}z_n}{\mathrm{vol}\ \mathrm{SL}(2,\mathbb{R})}\underbrace{\frac{1}{z_{1,2}z_{2,3}\cdots z_{n,1}}}_{\mathrm{PT}(1,2,\ldots,n)}\times\underbrace{\prod_{i<j}z_{i,j}^{2\alpha' p_i\cdot p_j}}_{\text{Koba-Nielsen factor}}$$

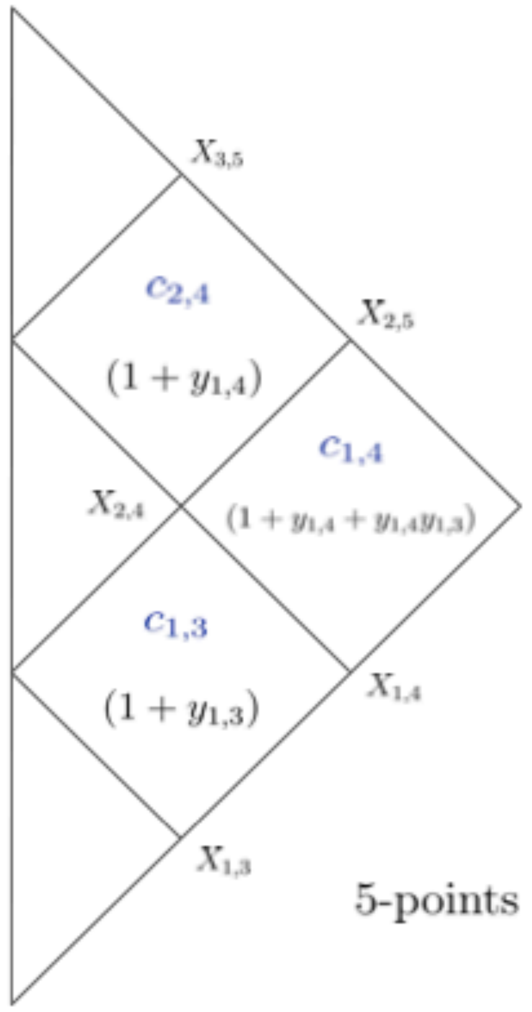
u variables & positive para. y

$$u_{i,j}=\frac{z_{i-1,j}z_{i,j-1}}{z_{i,j}z_{i-1,j-1}}.$$

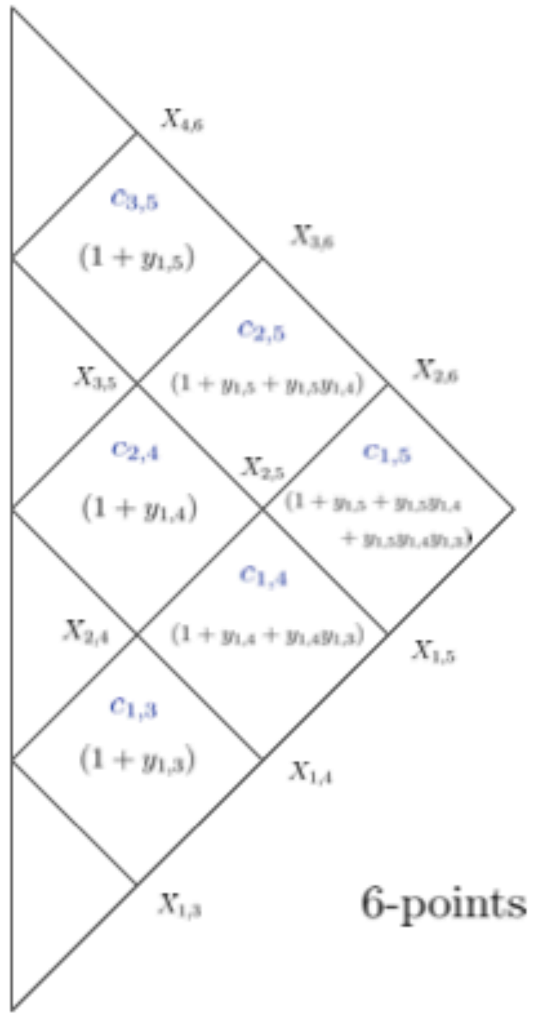
$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3}=\int_{\mathbb{R}^{n-3}_{>0}}\prod_{I=1}^{n-3}\frac{\mathrm{d}y_I}{y_I}\prod_{(a,b)}u_{a,b}^{\alpha' X_{a,b}}(y)=\int_{\mathbb{R}^{n-3}_{>0}}\prod_{I=1}^{n-3}\frac{\mathrm{d}y_I}{y_I}y_I^{\alpha' X_I}\prod_{i,j}F_{i,j}(y)^{-\alpha' c_{i,j}}$$

ray-like triangulation =>

$$F_{i,j}=1+y_{1,j}+y_{1,j}y_{1,j-1}+\cdots+y_{1,j}\cdots y_{1,i+2}.$$



5-points



6-points

$$\mathcal{I}_4^{\mathrm{Tr}(\phi^3)}=\int_{\mathbb{R}_{>0}}\frac{\mathrm{d}y_{1,3}}{y_{1,3}}y_{1,3}^{\alpha' X_{1,3}}(1+y_{1,3})^{-\alpha' c_{13}}=\frac{\Gamma[\alpha' X_{1,3}]\Gamma[\alpha'(c_{1,3}-X_{1,3})]}{\Gamma[\alpha' c_{1,3}]}$$

$$\mathcal{I}_5^{\mathrm{Tr}\,\phi^3}=\int_0^\infty\prod_{i=3}^{4}\frac{\mathrm{d}y_{1,i}}{y_{1,i}}y_{1,i}^{\alpha' X_{1,i}}\prod_{i<j}F_{i,j}(\mathbf{y})^{-\alpha' c_{i,j}}$$

$$=\int_0^\infty\frac{\mathrm{d}y_{1,3}}{y_{1,3}}\frac{\mathrm{d}y_{1,4}}{y_{1,4}}y_{1,3}^{\alpha' X_{1,3}}y_{1,4}^{\alpha' X_{1,4}}\,(1+y_{1,3})^{-\alpha' c_{1,3}}\,(1+y_{1,4})^{-\alpha' c_{2,4}}\,(1+y_{1,4}+y_{1,3}y_{1,4})^{-\alpha' c_{1,4}}\,.$$

Zeros of string amplitude [see also D'Adda, Sciuto, D'Auria, Gliozzi, 71']

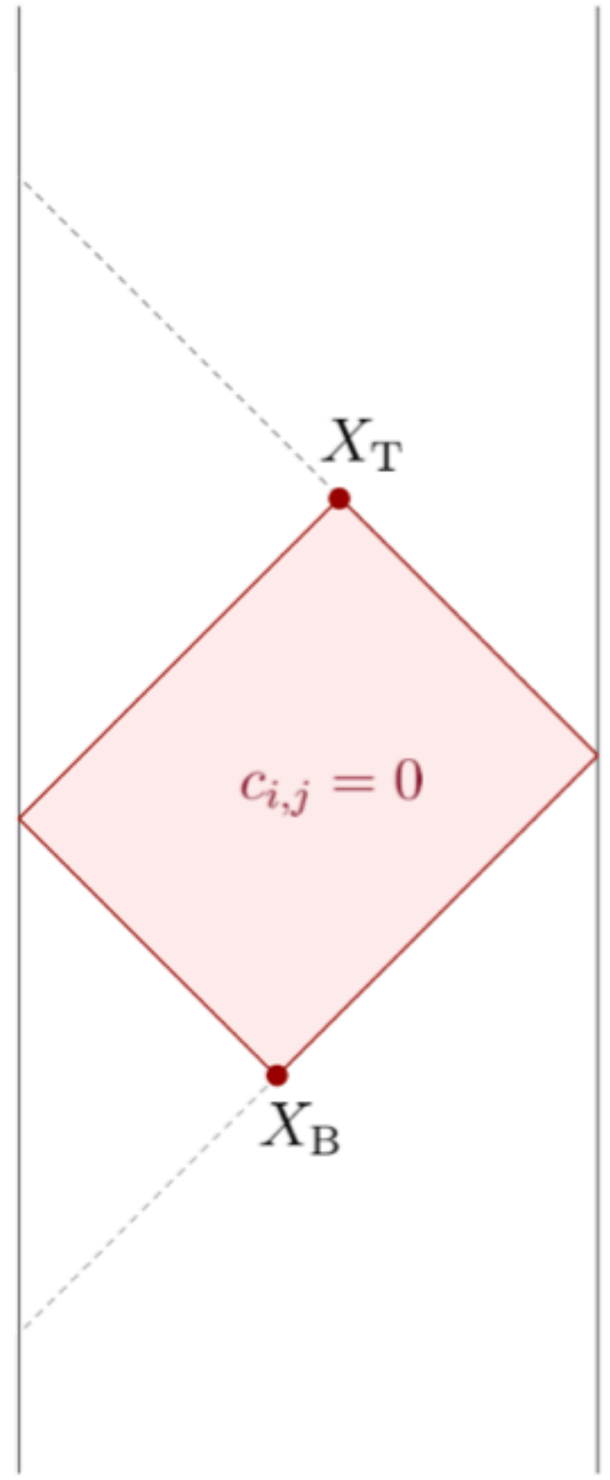
$$\mathcal{I}_4^{\mathrm{Tr}(\phi^3)} = \int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}} (1+y_{1,3})^{-\alpha' c_{13}} = \frac{\Gamma[\alpha' X_{1,3}]\Gamma[\alpha'(c_{1,3} - X_{1,3})]}{\Gamma[\alpha' c_{1,3}]}$$

any non-positive integer works: e.g.

by setting $\alpha' c_{1,3} = -n$,

$$\mathcal{I}_4^{\mathrm{Tr}(\phi^3)} \to \sum_{k=0}^{n} \underbrace{\int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}+k}}_{=0} = 0.$$

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \sum_{k_{a_1,b_1},\ldots,k_{a_N,b_N}=0}^{n_{a_1,b_1},\ldots,n_{a_N,b_N}} (\text{remaining integrals}) \times \underbrace{\int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}+k_{a_1,b_1}+\cdots+k_{a_N,b_N}}}_{=0} = 0$$



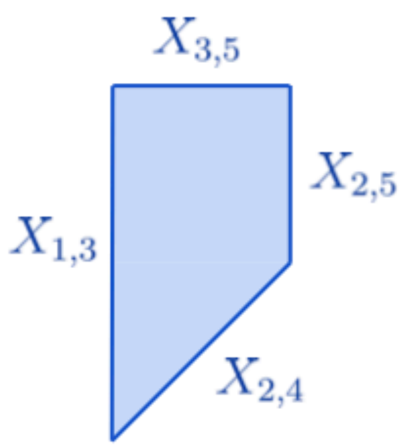
$c_{i,j} = -n_{ij}\,, \quad 1 \leq i < a-1,\ a \leq j < n$

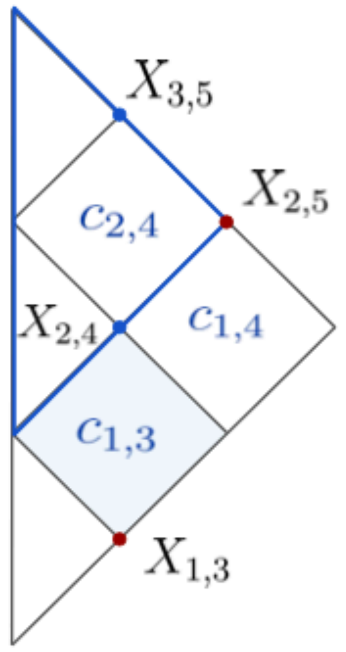
n(n-3)/2 infinite families of zeros

Factorizations

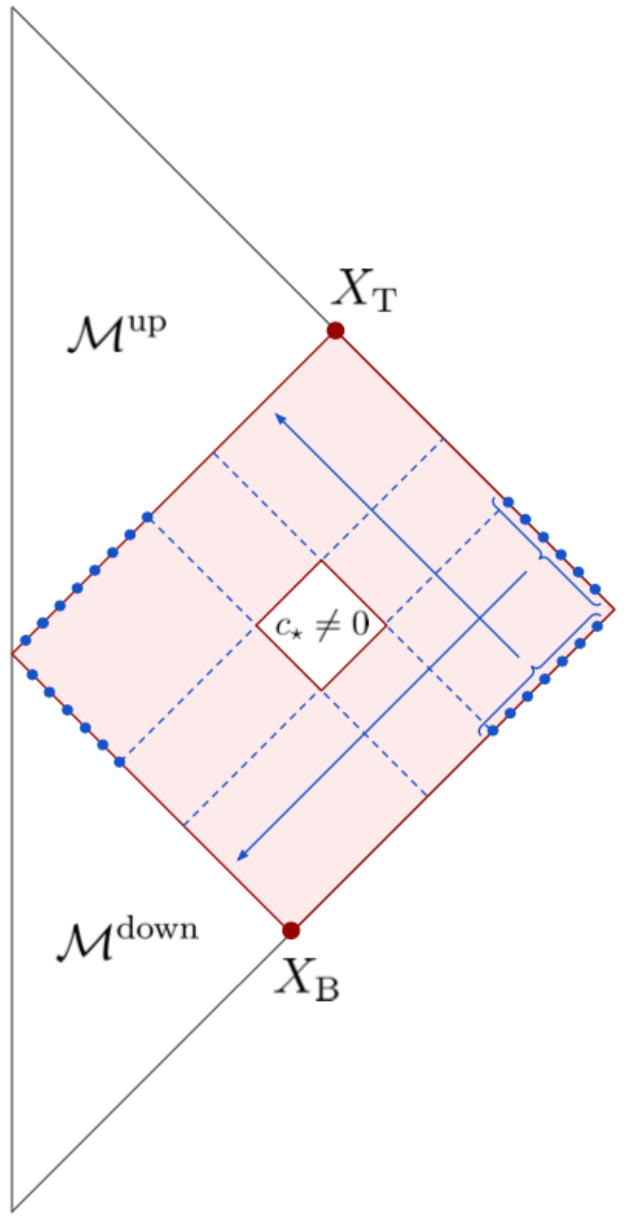
$$c_{1,3}=0\,,\quad \text{but}\quad c_{1,4}\neq 0\,,\qquad \mathcal{I}_5^{\mathrm{Tr}\,\phi^3}\to\int_0^\infty \frac{\mathrm{d}y_{1,3}}{y_{1,3}}y_{1,3}^{\alpha' X_{1,3}}\int_0^\infty \frac{\mathrm{d}y_{1,4}}{y_{1,4}}y_{1,4}^{\alpha' X_{1,4}}\,(1+y_{1,4})^{-\alpha' c_{2,4}}\,(1+y_{1,4}+y_{1,3}y_{1,4})^{-\alpha' c_{1,4}}$$

$$\begin{aligned}\mathcal{I}_5^{\mathrm{Tr}\,\phi^3}&\to\int_0^\infty \frac{\mathrm{d}\tilde{y}_{1,3}}{\tilde{y}_{1,3}}\tilde{y}_{1,3}^{\alpha' X_{1,3}}\,(1+\tilde{y}_{1,3})^{-\alpha' c_{1,4}}\int_0^\infty \frac{\mathrm{d}y_{1,4}}{y_{1,4}}y_{1,4}^{\alpha'(X_{1,4}-X_{1,3})}\,(1+y_{1,4})^{-\alpha'(c_{2,4}+c_{1,4}-X_{1,3})}\\&=\mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,3},\alpha'(c_{1,3}-X_{1,3}))\times\mathcal{I}_4^{\mathrm{up},\mathrm{Tr}\,\phi^3}(\alpha'(X_{1,4}-X_{1,3}),\alpha'(c_{2,4}+c_{1,4}-X_{1,4}))\\&=\mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,3},\alpha' X_{2,5})\times\mathcal{I}_4^{\mathrm{up},\mathrm{Tr}\,\phi^3}(\alpha' X_{2,4},\alpha' X_{3,5}).\end{aligned}$$





$$\left(\frac{1}{X_{1,3}}+\frac{1}{X_{2,5}}\right)\times\mathcal{A}_4(X_{2,4},X_{3,5})$$



$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3}\to\mathcal{I}_i^{\mathrm{down},\mathrm{Tr}\,\phi^3}\times\mathcal{I}_{n-i+2}^{\mathrm{up},\mathrm{Tr}\,\phi^3}\times\mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,i},\alpha'(c_{km}-X_{1,i})).$$

$$\begin{aligned}&X_{l,i}\to X_{l,i}+X_{1,i}=X_{l,n},\text{for}\quad l=2,\ldots,k.\\&X_{i-1,j}\to X_{i-1,j}-X_{i-1,n}=X_{1,j},\text{for}\quad j=m,\ldots,n-1.\end{aligned}$$

non-positive integers: sum of such factorizations

Deformed to the real world [ACDFH 23]

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left(\frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha'\delta}, \qquad \mathcal{I}_{2n}^{\delta} = \mathcal{I}_{2n}^{\mathrm{Tr}\,\phi^3}[\alpha' X_{e,e} \to \alpha'(X_{e,e}+\delta), \alpha' X_{o,o} \to \alpha'(X_{o,o}-\delta)]\,.$$

key: all $c_{i,j} = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$ are preserved, thus all zero + fact. unchanged!

$\alpha'\delta = 0$

$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

$0 < \alpha'\delta < 1 \quad (\text{or } \mathbb{R}/\mathbb{Z})$

$\alpha' \to 0$

$$\mathcal{L}_{\mathrm{NLSM}} = \frac{1}{8\lambda^2} \mathrm{Tr}\left(\partial_\mu \mathrm{U}^\dagger \partial^\mu \mathrm{U}\right), \quad \text{with} \quad \mathrm{U} = (\mathbb{I} + \lambda\Phi)(\mathbb{I} - \lambda\Phi)^{-1}$$

$\alpha'\delta = \pm 1$

$$\mathcal{L}_{\mathrm{YMS}} = -\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\phi^I D_\mu\phi^I - \frac{g_{\mathrm{YM}}^2}{4}\sum_{I\neq J}[\phi^I,\phi^J]^2\right)$$

2n-pt Tr ϕ^3 string amps => 2n-pion in NLSM or 2n-scalar (n-gluon) in YMS: same function @ different pts!

NLSM directly from Tr ϕ^3

$$\mathcal{I}^{\delta}_{2n} = \int_{\mathbb{R}^{2n-3}_{>0}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(e,e)} u_{e,e}^{\alpha'(X_{e,e}+\delta)} \times \prod_{(o,o)} u_{o,o}^{\alpha'(X_{o,o}-\delta)} \times \prod_{(o,e)} u_{o,e}^{\alpha' X_{o,e}}$$

$$\to \mathcal{A}_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

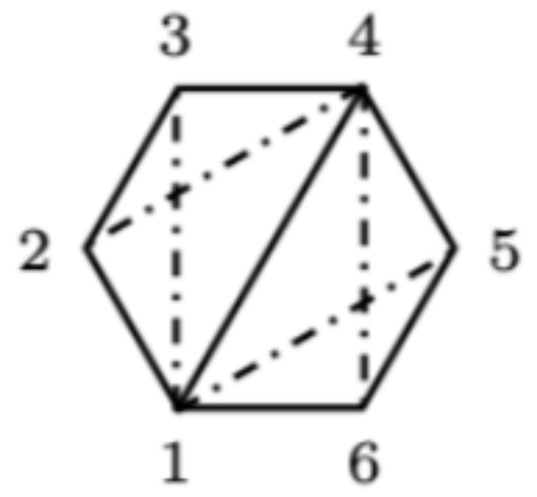
(note: if both $+\delta$ => tr ϕ^4 amplitude)

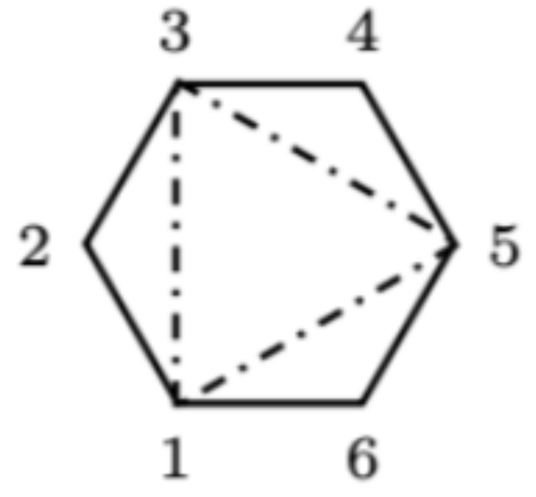
Field-theory directly take $\delta \to \infty$:

$$\mathcal{A}_{2n}^{\mathrm{NLSM}} = \lim_{\delta \to \infty} \delta^{n-1} \mathcal{A}_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

$$\mathcal{A}_4^{\mathrm{Tr}\,\phi^3}(X_{1,3}-\delta, X_{2,4}+\delta) \xrightarrow{\delta\to\infty} \frac{1}{\delta}(1-1) - \frac{1}{\delta^2}\underbrace{(X_{1,3}+X_{2,4})}_{\mathcal{A}_4^{\mathrm{NLSM}}} + \mathcal{O}(1/\delta^3),$$

$$\mathcal{A}_6^{\mathrm{Tr}\,\phi^3}(X \to X \pm \delta) \xrightarrow{\delta\to\infty} -\frac{1}{\delta^4}\Big(-\frac{(X_{1,3}+X_{2,4})(X_{1,5}+X_{4,6})}{X_{1,4}} + (\text{cyclic}, i \to i+2)$$
$$+X_{1,3}+X_{3,5}+X_{1,5}+X_{2,4}+X_{4,6}+X_{2,6}\Big) + \mathcal{O}(1/\delta^5),$$





Proof: 1. correct pole structure (only $X_{o,e}=0$) 2. correct factorization 3. Adler zero from “skinny zero”

NLSM zeros and factorizations [ACDFH 23]

All zeros directly follow; factorizations?

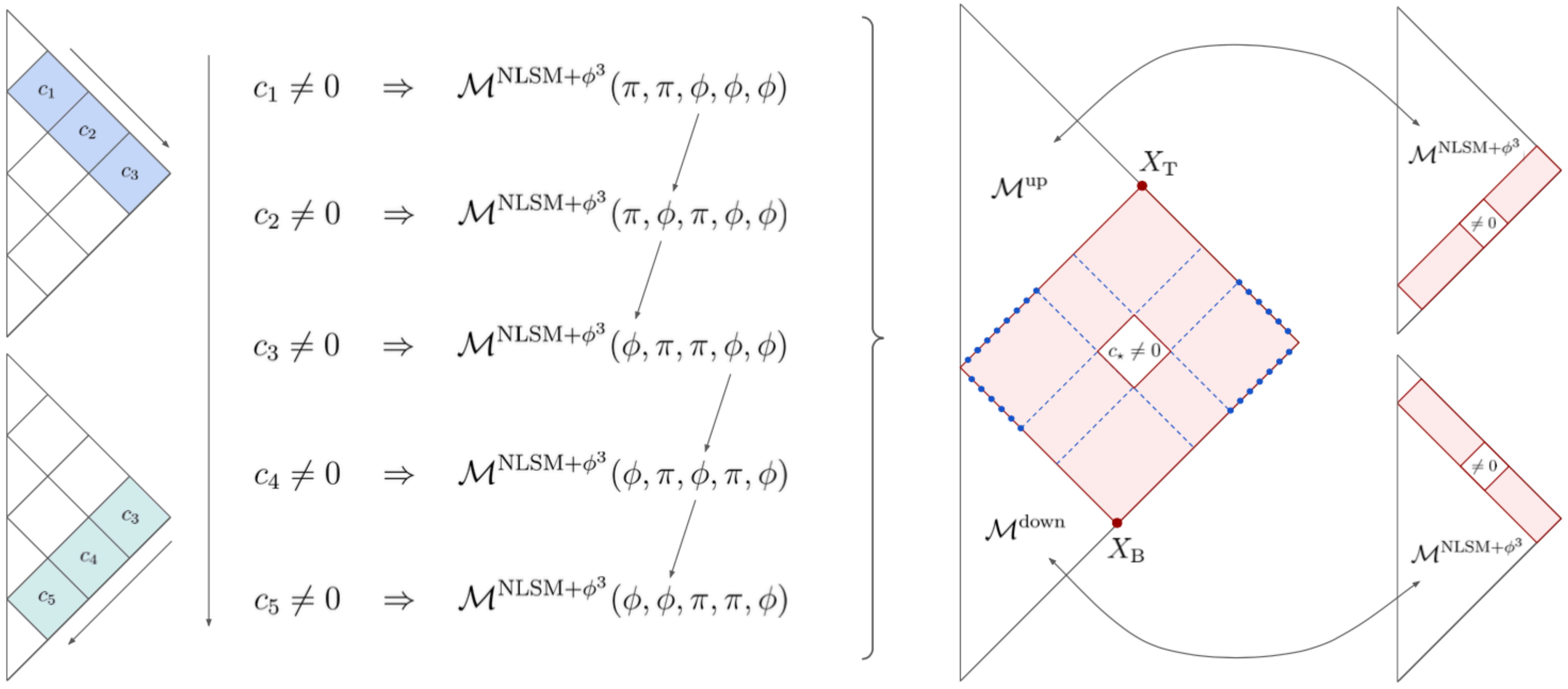
1. even-even-> NLSM* 4pt ϕ^3 *NLSM:
2. odd-odd-> mixed* 4pt NLSM* mixed:

$$\mathcal{A}_6(c_{1,4}=c_{1,5}=c_{2,5}=0)=\frac{c_{1,3}c_{2,4}c_{3,5}}{X_{1,4}(X_{1,4}-c_{2,4})}=\left(\frac{1}{X_{1,4}}+\frac{1}{X_{3,6}}\right)\cdot c_{1,3}\cdot c_{3,5}$$
$$=\left(\frac{1}{X_{1,4}}+\frac{1}{X_{3,6}}\right)\cdot\underbrace{(X_{1,3}+X_{2,6})}_{\mathcal{M}_4^{\mathrm{down,NLSM}}}\cdot\underbrace{(X_{1,5}+X_{4,6})}_{\mathcal{M}_4^{\mathrm{up,NLSM}}},$$

$$\mathcal{A}_6(c_{1,3}=c_{1,4}=0)=-c_{1,5}\cdot\left(\frac{c_{3,5}}{X_{3,6}}+\frac{c_{2,4}}{X_{2,5}}+1\right)$$
$$=-(X_{1,3}+X_{2,6})\cdot\underbrace{\left(\frac{X_{3,5}+X_{4,6}}{X_{3,6}}+\frac{X_{2,4}+X_{3,5}}{X_{2,5}}-1\right)}_{\mathcal{A}_5^{\mathrm{NLSM}+\phi^3}(\phi,\pi,\pi,\phi,\phi)},$$

$$\mathcal{A}_6(c_{1,3}=c_{1,5}=0)=c_{1,4}\cdot\frac{c_{3,5}}{X_{3,6}}$$
$$=(X_{1,3}+X_{2,6})\cdot\underbrace{\left(\frac{X_{3,5}+X_{4,6}}{X_{3,6}}-1\right)}_{\mathcal{A}_5^{\mathrm{NLSM}+\phi^3}(\phi,\pi,\phi,\pi,\phi)},$$

$$\mathcal{A}_6(c_{1,4}=c_{1,5}=0)=c_{1,3}\cdot\left(\frac{c_{3,5}}{X_{3,6}}+\frac{c_{3,5}+c_{2,5}}{X_{1,4}}\right)$$
$$=(X_{1,3}+X_{2,6})\cdot\underbrace{\left(\frac{X_{3,5}+X_{4,6}}{X_{3,6}}+\frac{X_{4,6}+X_{1,5}}{X_{1,4}}-1\right)}_{\mathcal{A}_5^{\mathrm{NLSM}+\phi^3}(\phi,\phi,\pi,\pi,\phi)},$$

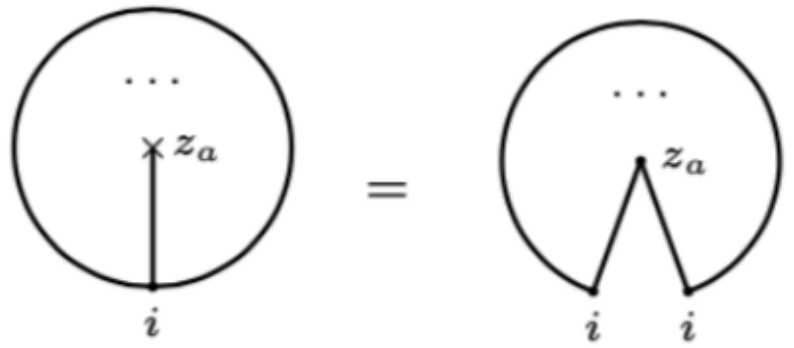


All-loop NLSM contained in Tr ϕ^3 [ACDFH]

Same shift works for planar integrand of NLSM: $X_{e,e} \to X_{e,e} + \delta\,, \quad X_{o,o} \to X_{o,o} - \delta$ (inc. loop punctures)

$$\lim_{\delta\to\infty} \sum_{z_{a=1,\cdots,L}\ \text{even/odd}}^{2^L} (\delta)^{n+2L-2} A^{\delta}_{n,L} = A^{\text{NLSM}}_{n,L}\,.$$

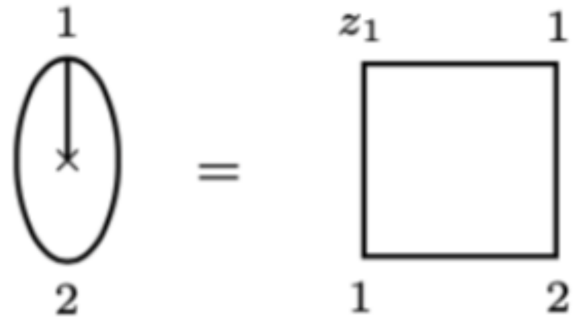
Proof: satisfies all-loop single-cuts + fact. (follow from cuts/fact. of ϕ^3)



e.g. 1-loop 2-pt:

$$A^{\delta}_{2,1} = \frac{1}{(X_{1,z_1}-\delta)(X_{2,z_1})} + \frac{1}{(X_{1,z_1}-\delta)(X_{1,1}-\delta)} + \frac{1}{(X_{2,z_1})(X_{2,2}-\delta)} + (1 \leftrightarrow 2, \delta \to -\delta)$$
$$\to \delta^{-2}\left(2 - \frac{X_{1,1}+X_{2,z_1}}{X_{1,z_1}} - \frac{X_{2,2}+X_{1,z_1}}{X_{2,z_1}}\right) = \delta^{-2} A^{\text{NLSM}}_{2,1},$$

single-cut=forward-limit of tree 4-pt



$$\operatorname*{Res}_{X_{1,z_1}=0} A^{\text{NLSM}}_{2,1} = -(X_{1,1}+X_{2,z_1}) = A^{\text{NLSM}}_{4,0}(1,z_1,1,2).$$

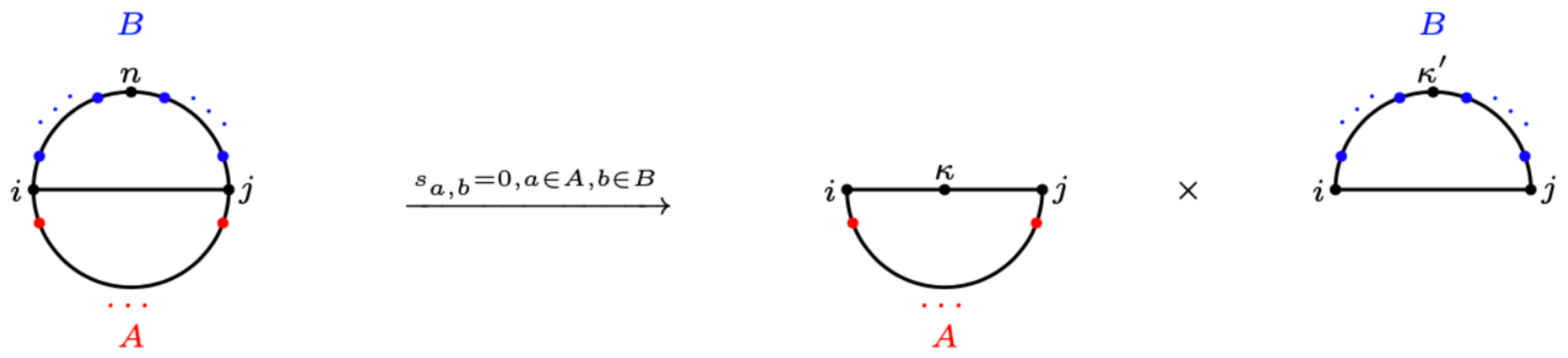
"Adler zero": soft limit -> scaleless integrals! Very practical, e.g. 3-loop 4-pt NLSM integrand

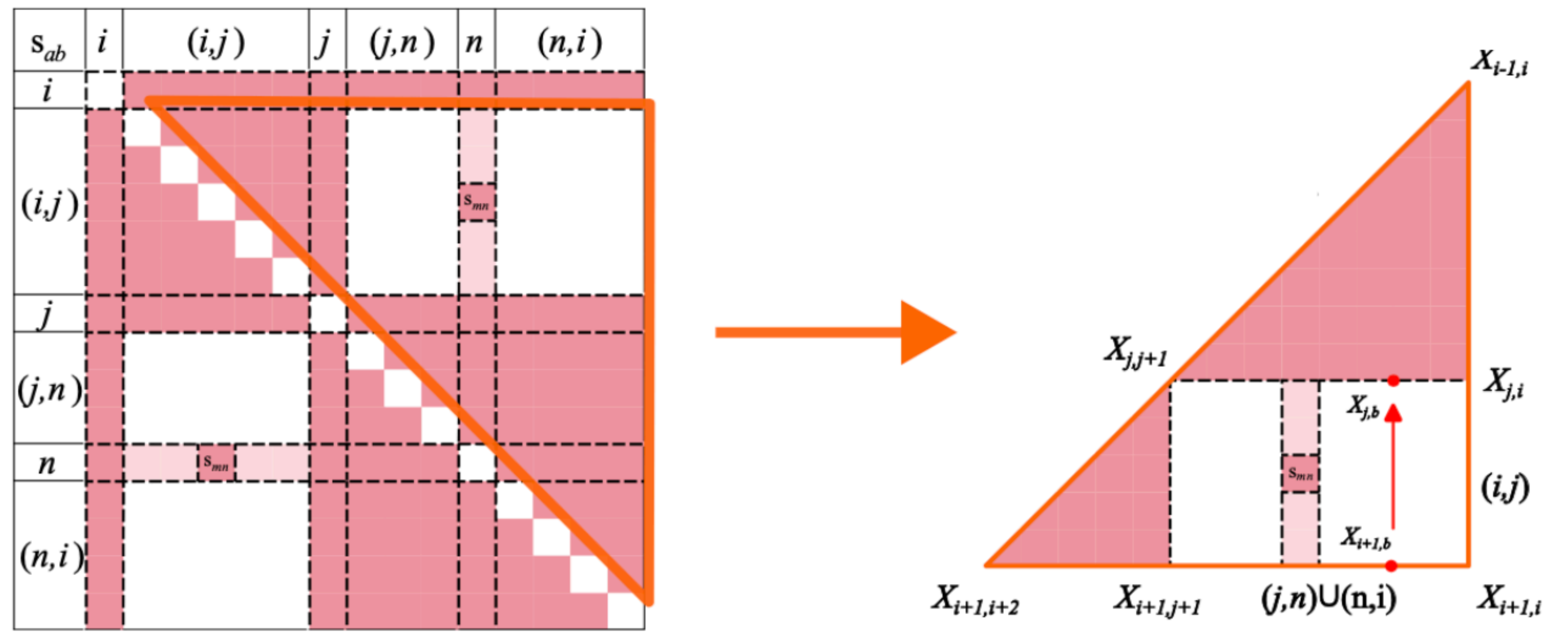
Interlude: universal splittings for string/particle amps [w. Cao, Dong, Shi, Zhu, 2024]

A remarkable new behavior for a wide class of string/particle tree amps: on a subspace, string/particle amp factorizes into two off-shell currents (total #=n+3)

Universally hold for scalars (ϕ^3,NLSM, sGal, YMS/DBI/EMS) + gluons, gravitons in (bosonic/super) string theories implies & extends factorizations near zeros & multi splitting to all these theories see also [J. Trnka et al] [L. Rodina] [Y. Zhang 2024]

Very nicely, extends to all loops for stringy ϕ^3 (NLSM etc.) from "gluing surfaces" [N. Arkani-Hamed, C. Figueiredo, 2024]





Origin: splittings of string/CHY integrals

$$\mathcal{S}_n = \sum_{a<b} s_{a,b} \log z_{a,b} = \sum_{a<b\neq k,(a,b)\neq(i,j)} s_{a,b} \log |ab| \qquad |ab| = z_{a,b} \quad \text{with } z_k \to \infty \text{ and } z_i = 0, z_j = 1,$$

$$s_{a,b} = 0, \qquad \forall a \in A, b \in B\,,$$

$$\mathcal{S}_n \to \underbrace{(\mathcal{S}_A + \mathcal{S}_{i,A} + \mathcal{S}_{j,A})}_{\mathcal{S}_L(i,A,j;\kappa)} + \underbrace{(\mathcal{S}_B + \mathcal{S}_{i,B} + \mathcal{S}_{j,B})}_{\mathcal{S}_R(j,B,i;\kappa')}\,,$$

where $\mathcal{S}_A = \sum_{a<b,a,b\in A} s_{a,b} \log |ab|$, $\mathcal{S}_{i,A} = \sum_{a\in A} s_{i,a} \log |ia|$, $\mathcal{S}_{j,A} = \sum_{a\in A} s_{a,j} \log |aj|$ (in the gauge fixing above, $|ia| = z_a - z_i = z_a$, $|aj| = z_j - z_a = 1 - z_a$)

$$d\mu_n^{\mathbb{R}} := (\alpha')^{n-3} \prod_{a\neq i,j,k} dz_a \exp(\alpha' \mathcal{S}_n)$$
$$d\mu_n^{\mathbb{C}} = d\mu_n^{\mathbb{R}}(z) d\mu_n^{\mathbb{R}}(\bar{z})$$

$$d\mu_n^{\mathbb{R}} \to d\mu_L^{\mathbb{R}}(i,A,j;\kappa) d\mu_R^{\mathbb{R}}(j,B,i;\kappa')\,.$$

similarly for the closed-string case

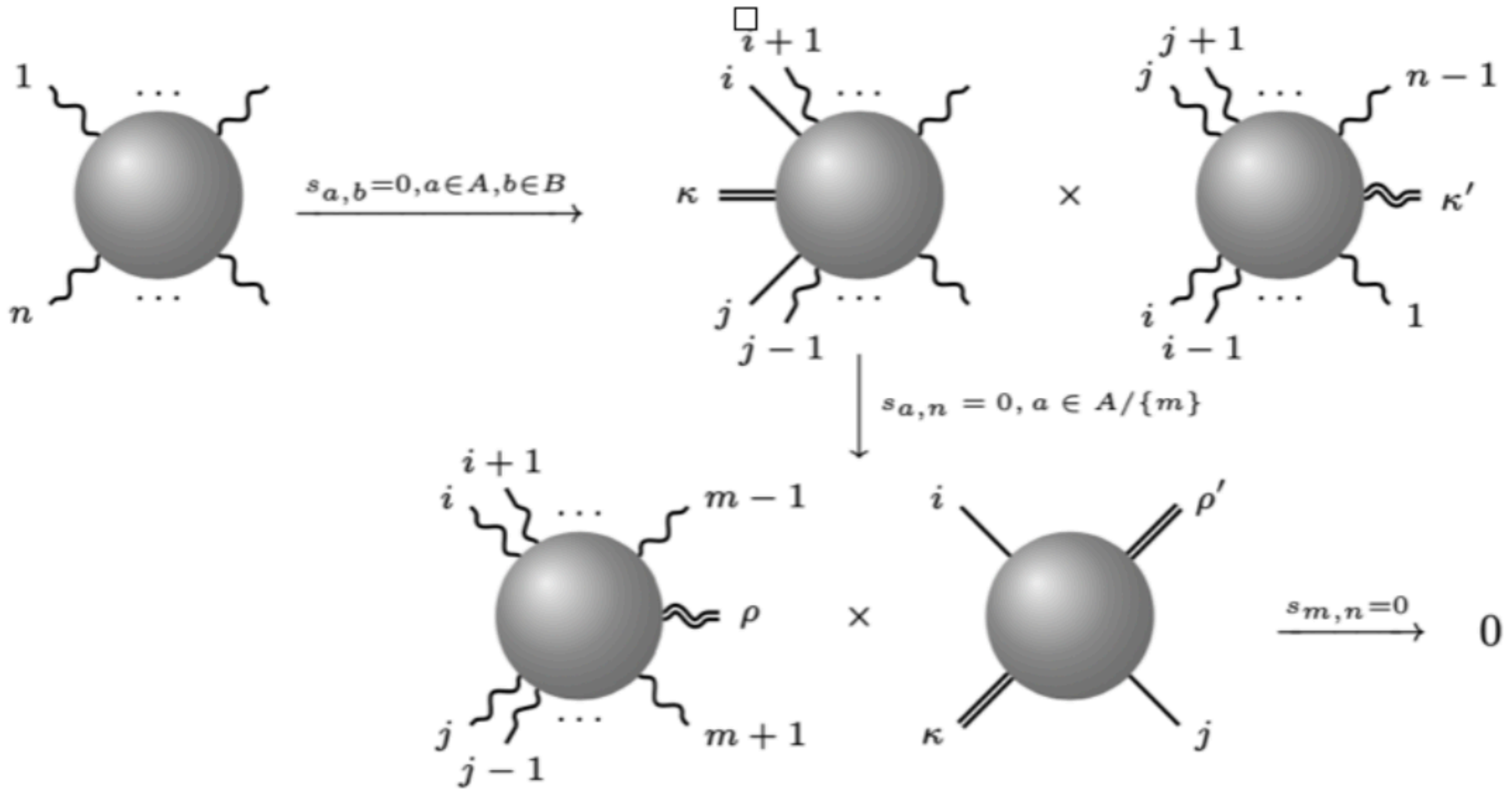
split it again -> "smooth splitting" [Cachazo et al]

$$\mathcal{S}_n \to \mathcal{S}(i,A,j;\kappa_A) + \mathcal{S}(j,B,k;\kappa_B) + \mathcal{S}(k,C,i;\kappa_C)$$

further set $s_{a,k} = 0$ for all $a \in A$

except for $a = m$, and the left-potential further splits

$$\mathcal{S}_L(i,A,j;\kappa) \to \mathcal{S}_L(i, A/\{m\}, j; \rho) + \mathcal{S}(i, \rho', j, \kappa),$$



Splitting string amps of gluons and gravitons

$$\epsilon_a \cdot \epsilon_b = \epsilon_a \cdot \epsilon_{i,j,k} = 0$$
$$p_a \cdot \epsilon_b = p_a \cdot \epsilon_{i,j,k} = 0$$
$$\epsilon_a \cdot p_b = p_a \cdot p_b = 0$$

$$\mathcal{M}_n^{\text{open}} \to \mathcal{J}^{\text{mixed}}(i^\phi, A, j^\phi; \kappa^\phi) \times \mathcal{J}(j, B, i; \kappa')_\mu \epsilon_n^\mu \,,$$

e.g. $$\mathcal{M}^{\text{YM}}(1,2,3,4,5,6,7) \xrightarrow{i=1,j=4,k=7} \mathcal{J}^{\text{YM}+\phi^3}(1^\phi,2,3,4^\phi;\kappa^\phi)\,\mathcal{J}^{\text{YM}}(1,4,5,6;\kappa')_\mu \epsilon_7^\mu,$$

same for $\tilde{\epsilon}$: $$\mathcal{M}_n^{\text{closed}} \to \mathcal{J}(i^\phi, A, j^\phi; \kappa^\phi) \times \mathcal{J}(j, B, i; \kappa')_{\mu\nu} \epsilon_n^\mu \tilde{\epsilon}_n^\nu$$

swap a, b for $\tilde{\epsilon}$: $$\mathcal{M}_n^{\text{closed}} \to \mathcal{J}(i^g, A, j^g; \kappa^g)_\mu \epsilon_n^\mu \times \mathcal{J}(j^g, B, i^g; \kappa'^g)_\nu \tilde{\epsilon}_n^\nu \,.$$

e.g. $$\mathcal{M}^{\text{GR}}(1,2,3,4,5,6,7) \xrightarrow{i=1,j=4,k=7} \mathcal{J}^{\text{GR}+\phi^3}(1^\phi,2,3,4^\phi;\kappa^\phi)\,\mathcal{J}^{\text{GR}}(1,4,5,6;\kappa')_{\mu\nu} \epsilon_7^\mu \tilde{\epsilon}_7^\nu,$$

$$\mathcal{M}^{\text{GR}}(1,2,3,4,5,6,7) \xrightarrow{i=1,j=4,k=7} \epsilon_7^\mu \mathcal{J}^{\text{EYM}}(1^g,2,3,4^g;\kappa^g)_\mu\,\mathcal{J}^{\text{EYM}}(1^g,4^g,5,6;\kappa'^g)_\nu \tilde{\epsilon}_7^\nu.$$

Soft theorems from "skinny" splitting

$$\epsilon_a \cdot \epsilon_b = \epsilon_a \cdot \epsilon_{i,j,k} = 0$$
$$p_a \cdot \epsilon_b = p_a \cdot \epsilon_{i,j,k} = 0$$
$$\epsilon_a \cdot p_b = p_a \cdot p_b = 0$$

$$\mathcal{M}_n^{\text{open}} \to \mathcal{J}^{\text{mixed}}(i^\phi, A, j^\phi; \kappa^\phi) \times \mathcal{J}(j, B, i; \kappa')_\mu \epsilon_n^\mu\,,$$

the "skinny" case: $A = \{a\}$:

$$\mathcal{J}^{\text{mixed}}(i^\phi, a, j^\phi; \kappa^\phi) = \epsilon_a \cdot p_i B(s_{i,a}, s_{j,a}+1) - \epsilon_a \cdot p_j B(s_{i,a}+1, s_{j,a})$$

soft gluon limit: (n-1)-pt current-> amplitude

sum over choices of k (i,j fixed to be adjacent to a):

$$\sum_{k\neq i,j,a} \mathcal{J}^{\text{mixed}} \times \mathcal{J}_{n-1} \to \left(\frac{\epsilon_a \cdot p_i}{p_a \cdot p_i} - \frac{\epsilon_a \cdot p_j}{p_a \cdot p_j}\right) \times \mathcal{M}_{n-1}^{\text{YM}},$$

similarly for gravity: sum over i, j ,k

$$\sum_{k,i,j\neq a} \mathcal{J}^{\text{mixed}} \times \mathcal{J}_{n-1} \to \left(\sum_{b\neq a} \frac{\epsilon_a \cdot p_b \tilde{\epsilon}_a \cdot p_b}{p_a \cdot p_b}\right) \mathcal{M}_{n-1}^{\text{GR}}$$

soft limit of NLSM, DBI, sGal: (enhanced) Adler zero with t^1, t^2, t^3 behavior from 4-pt (pure) current

extract "sub-leading" soft theorems? multi-soft limits from more general splitting!

Scaffolded gluons: combinatorial origin of YM [ACDFH, 2024]

$\alpha'\delta = 1$ gives 2n-scalar stringy amplitude = 2n-scalar in bosonic string!

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,2n) = \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \left(\prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}\right) \exp\left(\sum_{i\neq j} 2\frac{\epsilon_i\cdot\epsilon_j}{z_{i,j}^2} - \frac{\sqrt{\alpha'}\epsilon_i\cdot p_j}{z_{i,j}}\right)\Bigg|_{\text{multi-linear in } \epsilon_i},$$

$$p_i\cdot\epsilon_j = 0, \quad \forall\ (i,j)\in(1,\ldots,2n),$$

special component $\epsilon_1\cdot\epsilon_2 \ \ldots\ \epsilon_{2n-1}\cdot\epsilon_{2n}$

$$\epsilon_i\cdot\epsilon_j = \begin{cases} 1 & \text{if } (i,j)\in\{(1,2);(3,4);(5,6);\ldots;(2n-1,2n)\}, \\ 0 & \text{otherwise.}\end{cases}$$

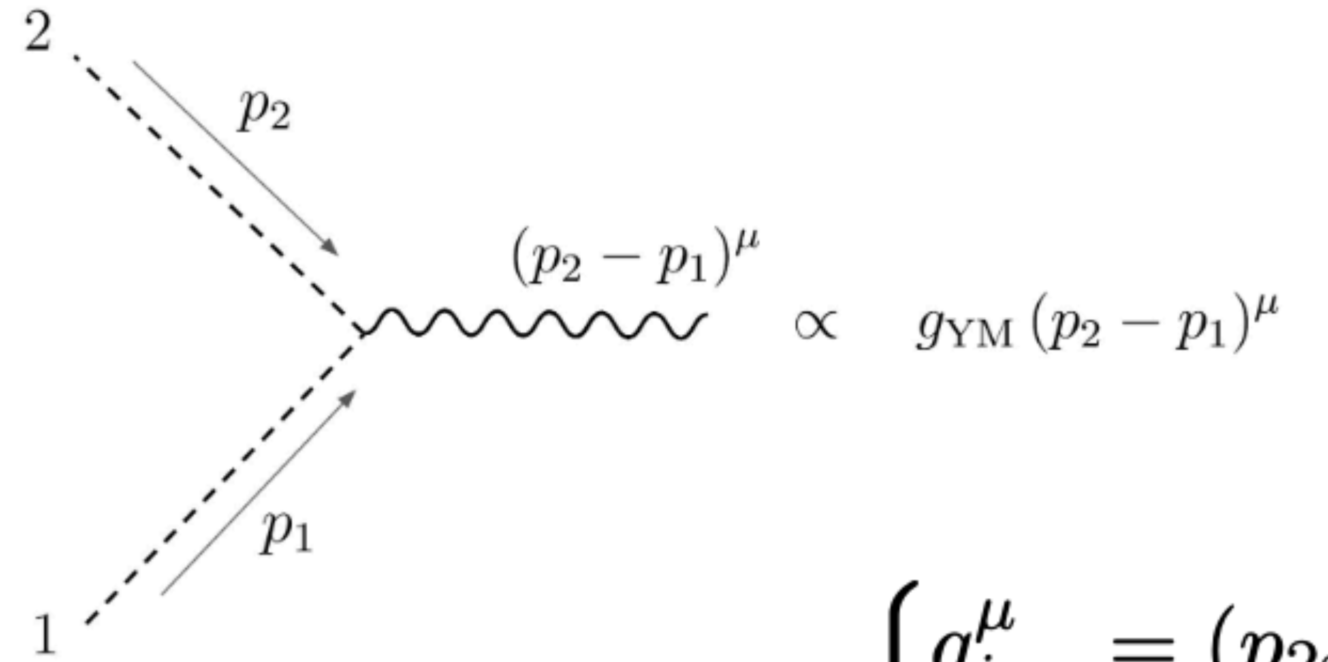
$$\mathcal{A}_{2n}(1,2,\ldots,2n) \xrightarrow{\text{special kinematics}} \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j} \frac{1}{z_{1,2}^2 z_{3,4}^2 z_{5,6}^2 \ldots z_{2n-1,2n}^2}$$

$$= \underbrace{\int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \frac{1}{z_{1,2}z_{2,3}z_{3,4}\ldots z_{2n,1}} \prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}}_{\text{Stringy Tr}\,\phi^3} \frac{z_{2,3}z_{4,5}z_{6,7}\ldots z_{2n,1}}{z_{1,2}z_{3,4}z_{5,6}\ldots z_{2n-1,2n}} \qquad \left(\prod u_{e,e}/\prod u_{o,o}\right)$$

exactly corresponds to $\alpha'\delta = 1$: n pairs of scalars in bosonic string, $(1,2)(3,4)\cdots(2n-1,2n)$

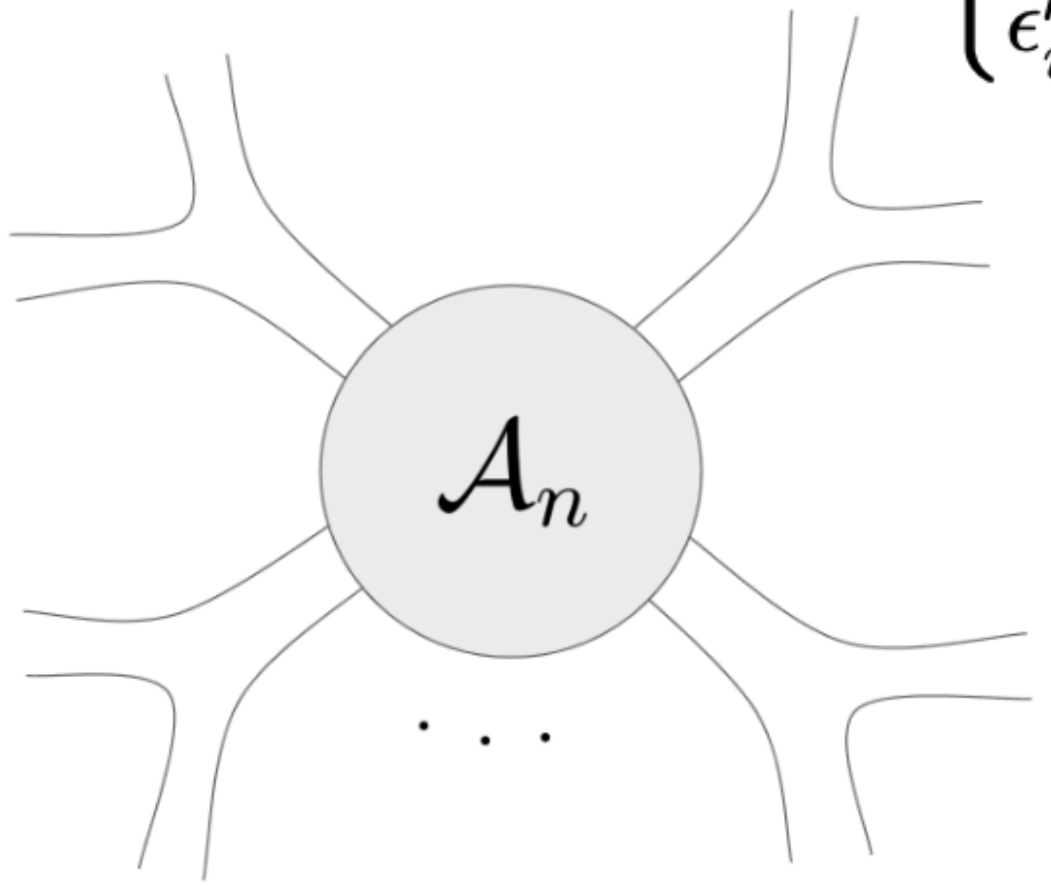
note $\alpha'\delta = -1$ => $(2,3)(4,5)\cdots(2n,1)$

by taking n ``scaffolding residues” -> n-gluon bosonic string amp in X (scalar) language!

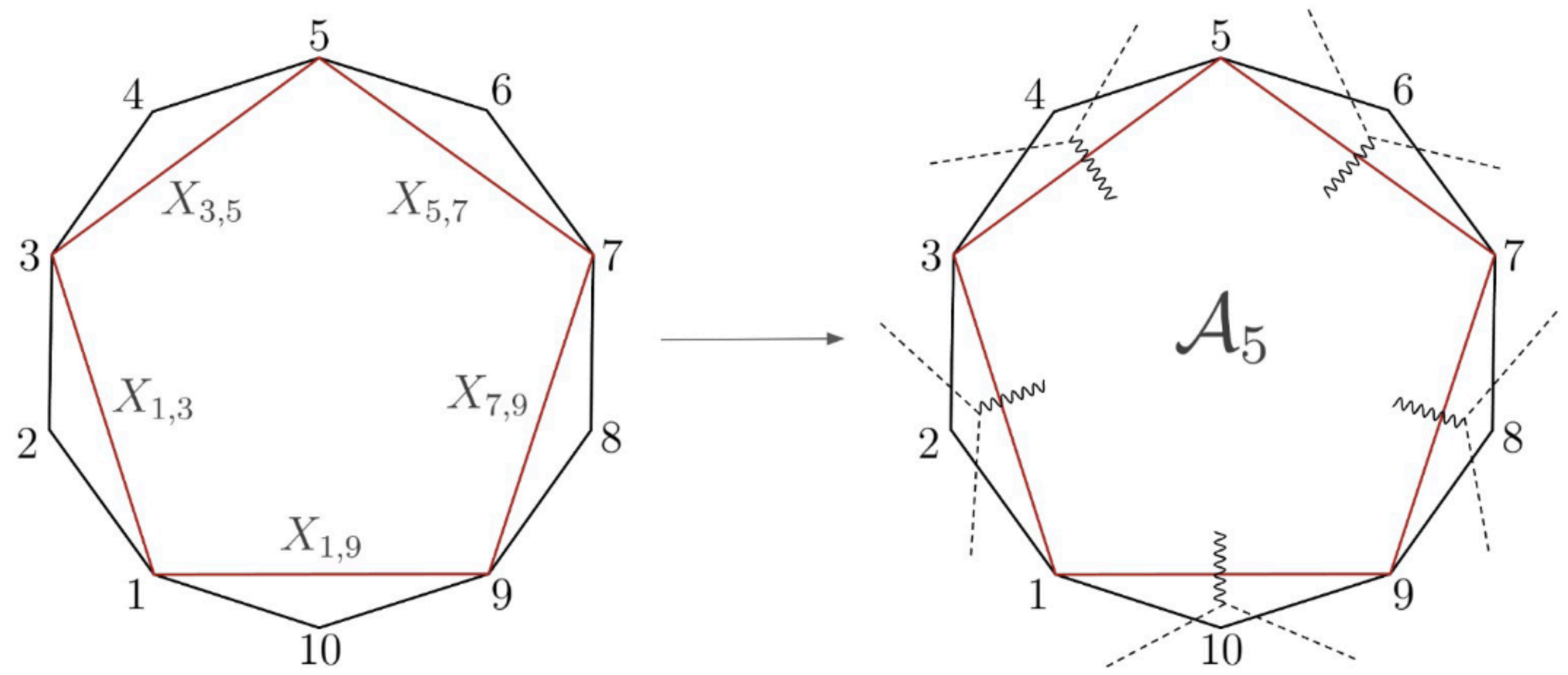


$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}^{2n-3}_{>0}} \underbrace{\prod_{i=1}^{n} \frac{dy_{2i-1,2i+1}}{y^2_{2i-1,2i+1}} \prod_{I\in\mathcal{T}'} \frac{dy_I}{y_I^2} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}}}_{\Omega_{2n}},$$

$$\begin{cases} q_i^\mu & = (p_{2i} + p_{2i-1})^\mu \\ \epsilon_i^\mu & \propto (p_{2i} - p_{2i-1})^\mu \end{cases},$$



$$X_{1,3} = X_{3,5} = \cdots = X_{1,2n-1} = 0.$$



$$\mathcal{I}_n^{\text{gluon}} = \int_{\mathbb{R}^{n-3}_{>0}} \text{Res}_{y_{1,3}=0} \left(\text{Res}_{y_{3,5}=0} \left(\ldots \left(\text{Res}_{y_{1,2n-1}=0} \left(\Omega_{2n}\right)\right) \ldots\right)\right) \Big|_{X_{2i-1,2i+1}=0}$$

$$A_3^{\text{gluon}} = \alpha'^2 \left(c_{1,3}c_{1,5} + c_{1,3}c_{2,5} + c_{1,3}c_{3,5} + c_{1,4}c_{3,5} + c_{1,5}c_{3,5} + c_{1,5}c_{3,6}\right) - \alpha'^3 \left(X_{1,4}X_{2,5}X_{3,6}\right)$$

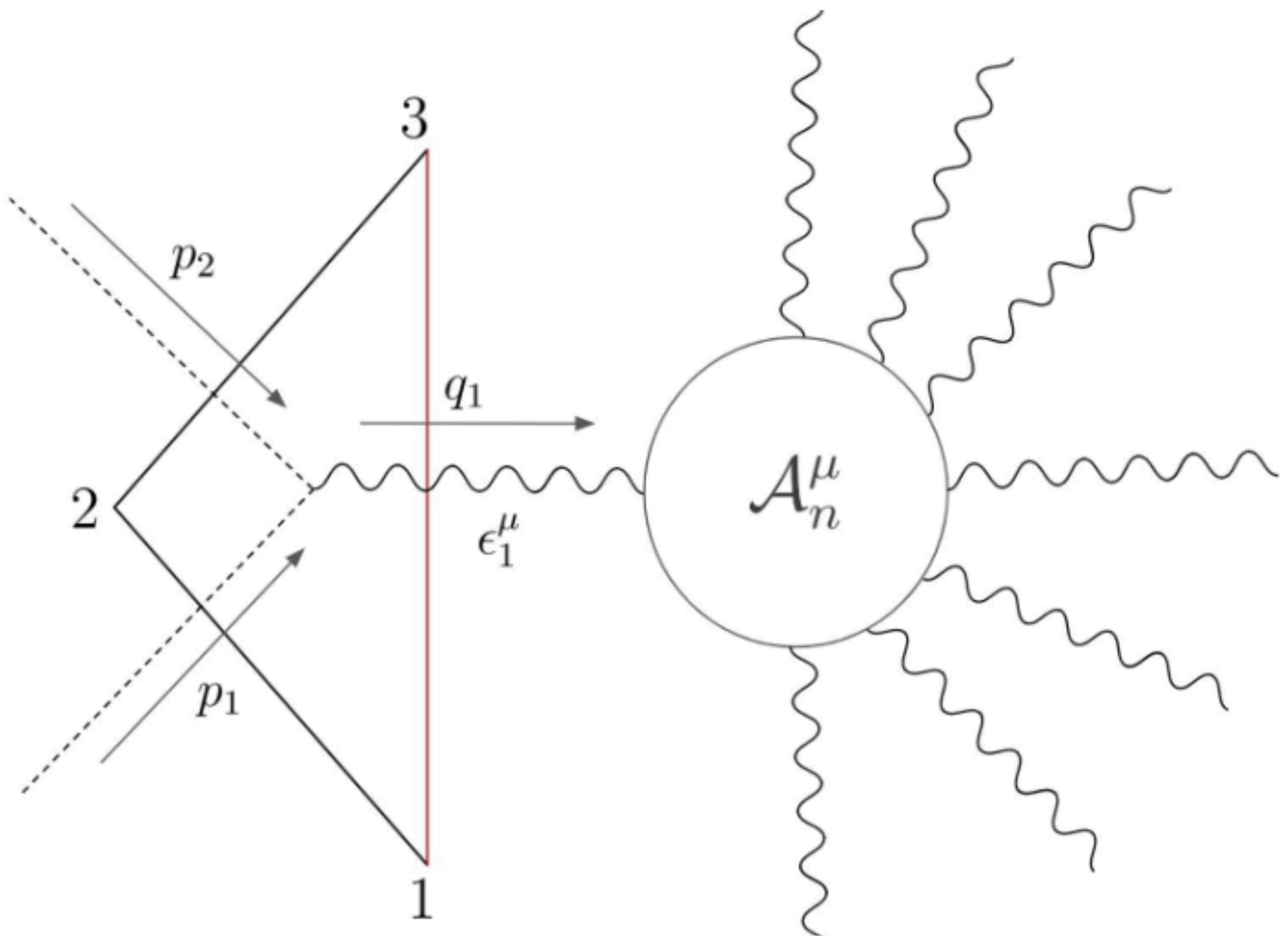
$$A_3^{\text{YM}}(1,2,3) = \frac{1}{2}(\epsilon_1\cdot\epsilon_3\ \epsilon_2\cdot q_1 - \epsilon_1\cdot q_2\ \epsilon_2\cdot\epsilon_3 + \epsilon_1\cdot\epsilon_2\ q_2\cdot\epsilon_3).$$

$$A_3^{F^3}(1,2,3) = \epsilon_1\cdot q_3\ \epsilon_2\cdot q_1\ \epsilon_3\cdot q_2,$$

Conjecture: all-loop YM in stringy Tr ϕ^3 [ACDFH, 2024]

Generalize stringy tree amp (disk) to loops (higher-genus surfaces):

$$A_n^{\text{gluon}} = \int_0^\infty \prod_i \frac{\mathrm{d}y_i}{y_i^2} \operatorname{Res}_{y_{s_1}=0}\left(\operatorname{Res}_{y_{s_2}=0}\left(\ldots\left(\operatorname{Res}_{y_{s_n}=0}\Omega_{2n}\right)\ldots\right)\right).$$

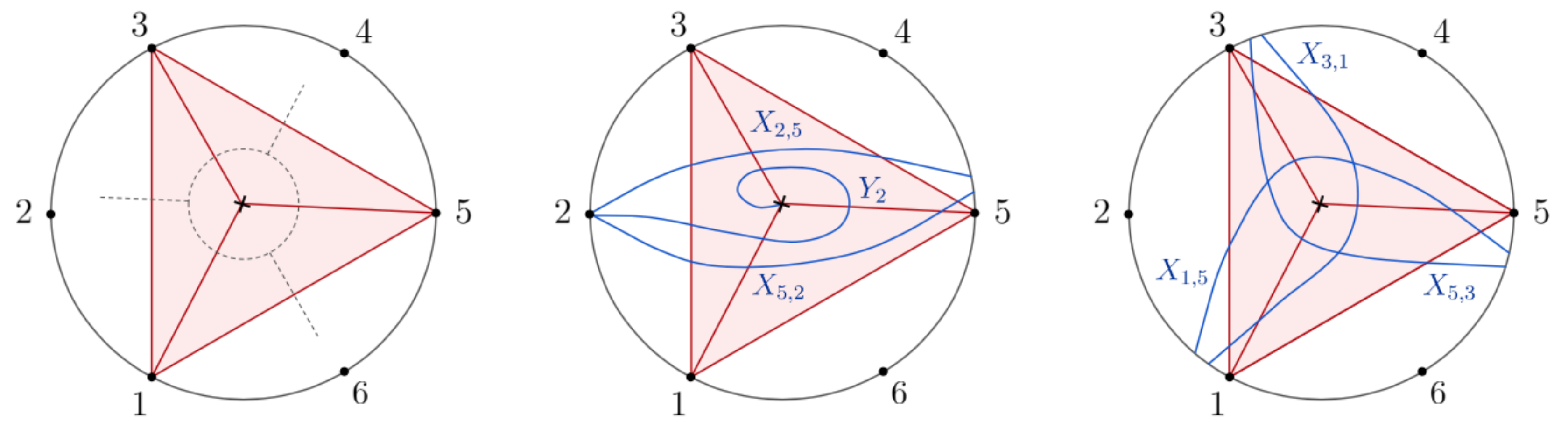


e.g. 1-loop w. self-intersecting curves & closed curve Δ (absent for scalars)

$$\mathcal{I}_{2n}^{\text{1-loop}}(1,2,\ldots,2n) = \int_0^\infty \prod_i \frac{\mathrm{d}y_i}{y_i^2} \prod_C u_C^{\alpha' X_C} \times \prod_{C' \in \text{ s.i.}} u_{C'}^{\alpha' X_{C'}} \times u_\Delta^\Delta$$

$$\epsilon_1^\mu \propto (X_3 - X_2)^\mu - (X_2 - X_1)^\mu \propto X_2^\mu - \frac{(X_3 + X_1)^\mu}{2}$$

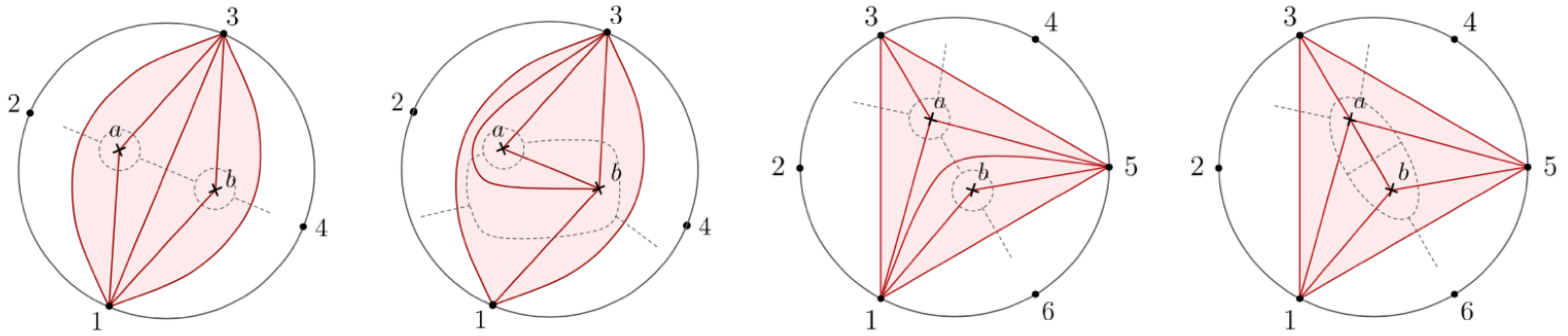
not bosonic string beyond tree, but conjecturally gives all-loop integrands!



extend the notion of (loop) gauge invariance + factorization from surfaceology!

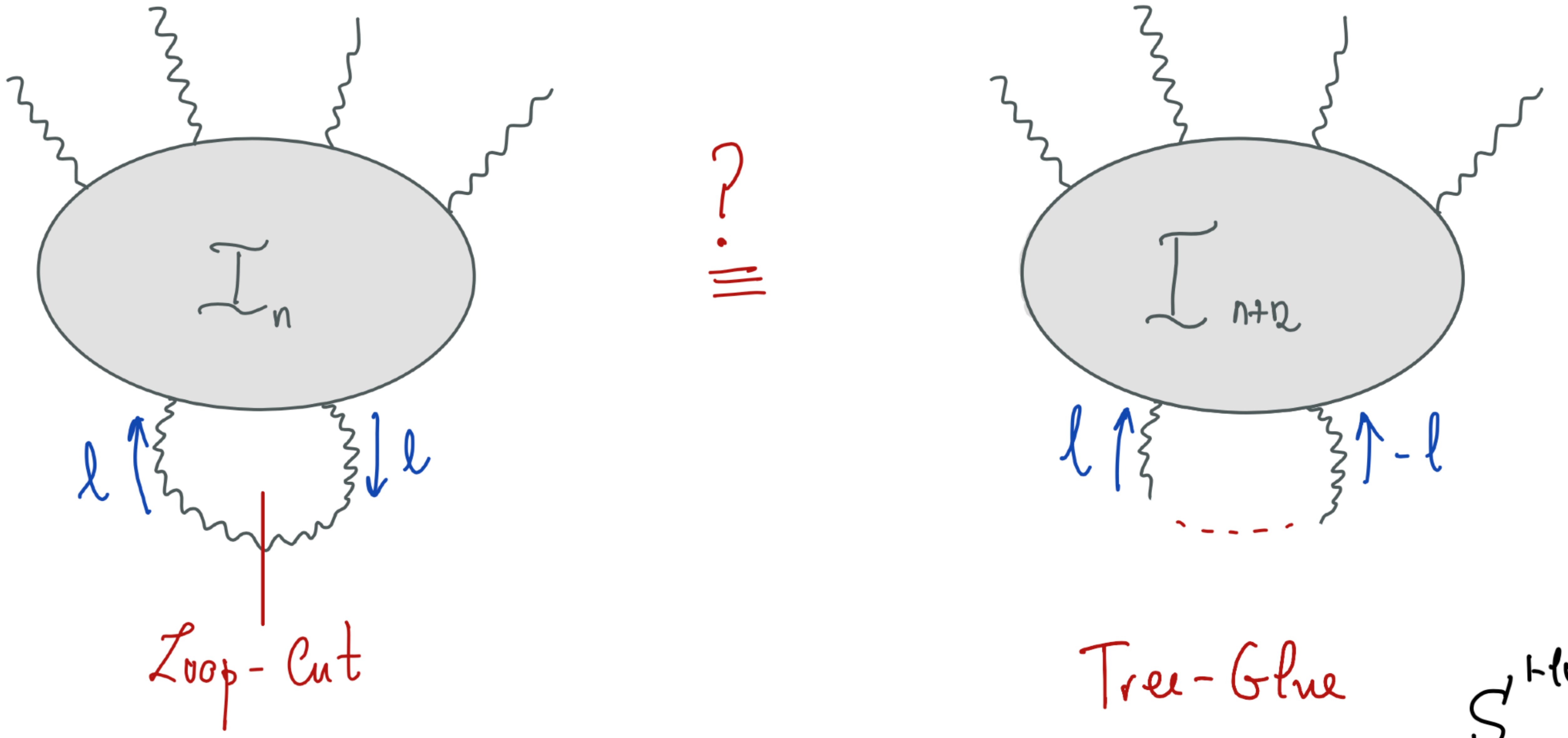
strong evidence from leading singularities (residues only): checked up to 2 loops;

LS = residue of $\int \prod \frac{dy}{y^2} \prod u^X$ = gluing of 3pt (in X space) iff $\Delta = 1 - D$.



How to determine “perfect” YM loop integrands?

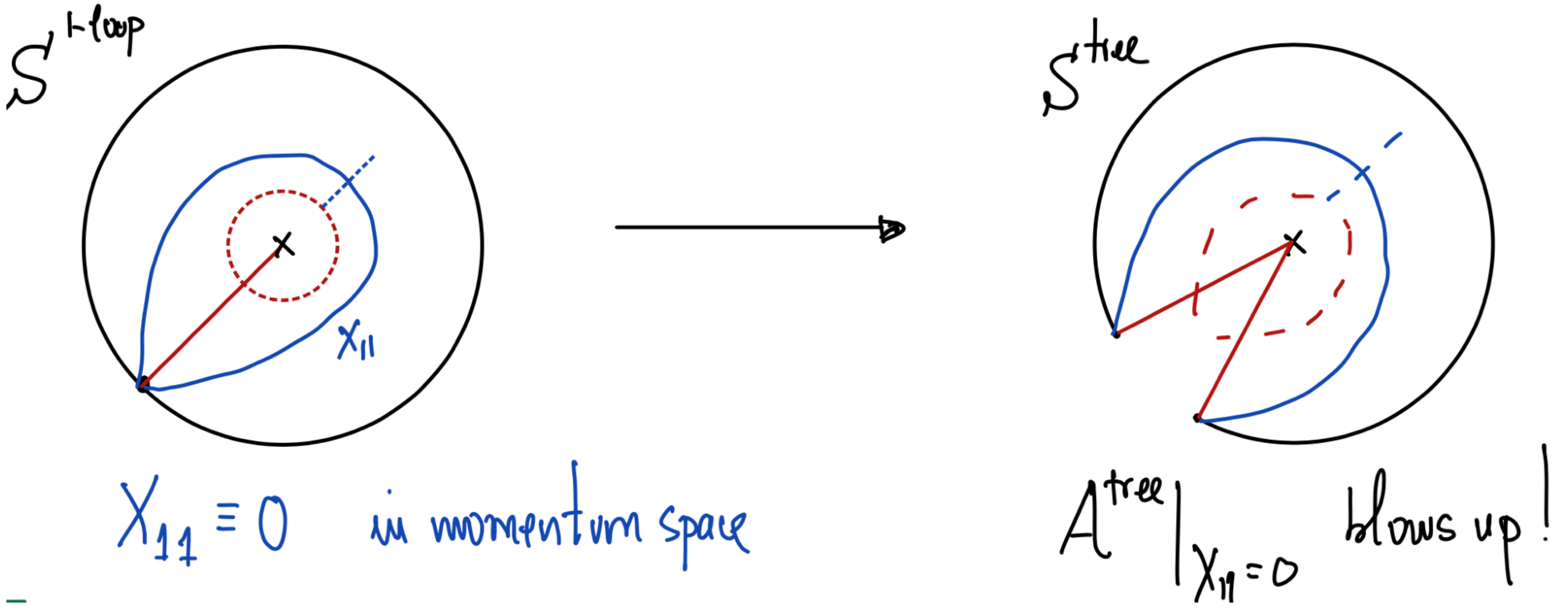
Similar to tree factorization on poles, just need loop cuts: e.g. 1-loop single-cut = forward limit (gluing tree)



naively divergent => “the” integrand
(e.g. Adler zero, gauge inv.) ill defined!

no issues for scalars, but for gluons 1/0 !
(cancels in super-Yang-Mills)

pic from Carolina’s talk @ Strings



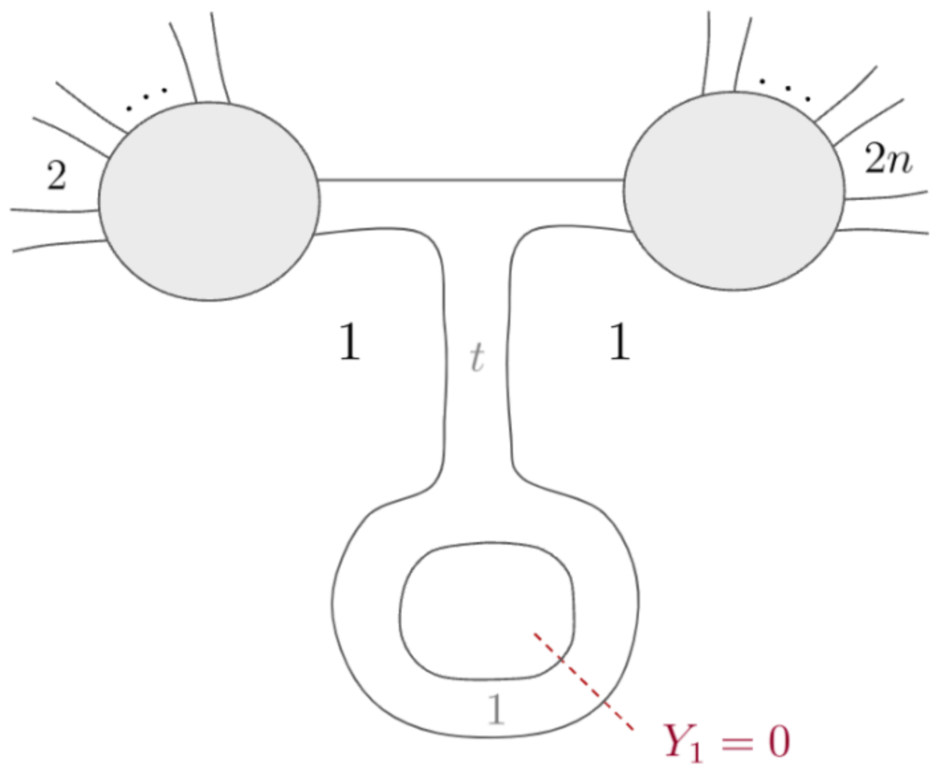
surface provides a natural way out: curves without standard momentum (e.g. tadpoles) => “perfect” integrand

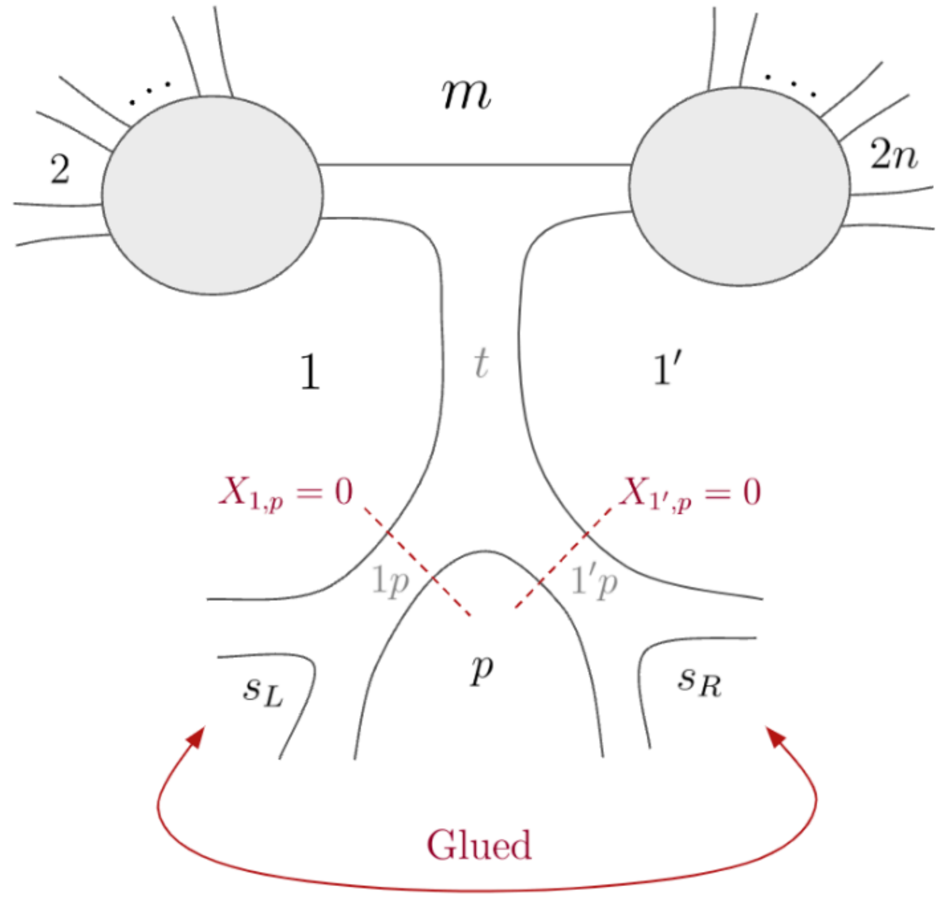
“doubling” variables: similar to Lorentzian -> complex in 4d tree kinematics

Recursion relations for YM loop integrands [ACDFH, to appear]

Surface makes it clear that all-loop "perfect" integrands exist (also beyond planar limit); can be reconstructed from these "residues" => recursions for perfect 1-loop integrand & all-loop integrand up to scaleless terms!

$$\tilde{\mathcal{A}}^{\text{YM}}_{n,L} = \int_0^1 \frac{dt}{t} \sum_{i=1}^{n} \sum_{a=1}^{L} \tilde{X}_{2i-1,z_a} \left(\sum_{j,k} (X_{z_a,j} + X_{z_a,k} - X_{j,k}) \frac{\partial^2 \tilde{\mathcal{A}}^{\text{YM}}_{n+2,L-1}(1,\ldots,i',z_a,i,\ldots,n)}{\partial X_{2i',j}\,\partial X_{2i'+2,k}} - D \frac{\partial \tilde{\mathcal{A}}^{\text{YM}}_{n+2,L-1}(1,\ldots,i',z_a,i,\ldots,n)}{\partial X_{2i',2i'+2}} \right) \Bigg|_{i=i',\ \tilde{X}_{2i-1,z_a} \to t\tilde{X}_{2i-1,z_a}} .$$





"perfect": a notion of surface gauge invariance+ factorization

in practice, e.g. explicit results up to 1-loop 6pt, 2-loop 4pt (D-dim) integrands -> reproduce correct amps after loop integrations!

huge simplifications when going back to 4d spinor-helicity!

Summary

Combinatorial geometries: "polytopes" in kinematic space (@ infinity) encode QM + spacetime:
combinatorics → geometries → (stringy) forms → integrated (non-perturbative) results

amplituhedra (SYM/ABJM), correlahedron/squared-amplituhedron, AdS/dS + cosmological amps
associahedra + surfacehedra (Tr ϕ^3), binary geometries ("strings") => real world: pions, gluons etc.

- Stringy Tr ϕ^3, NLSM & YM (2n-scalar) has (infinite) hidden zeros & factorizations near them
- String δeformation: same function describe them all; universal splitting (extend to all loops from surfaces)
- Generalize to all loops: NLSM in field-theory Tr ϕ^3; YM in stringy Tr ϕ^3 (binary integrals on surfaces)

Thank you!