Celestial optical theorem and its applications

刘晗（清华大学）

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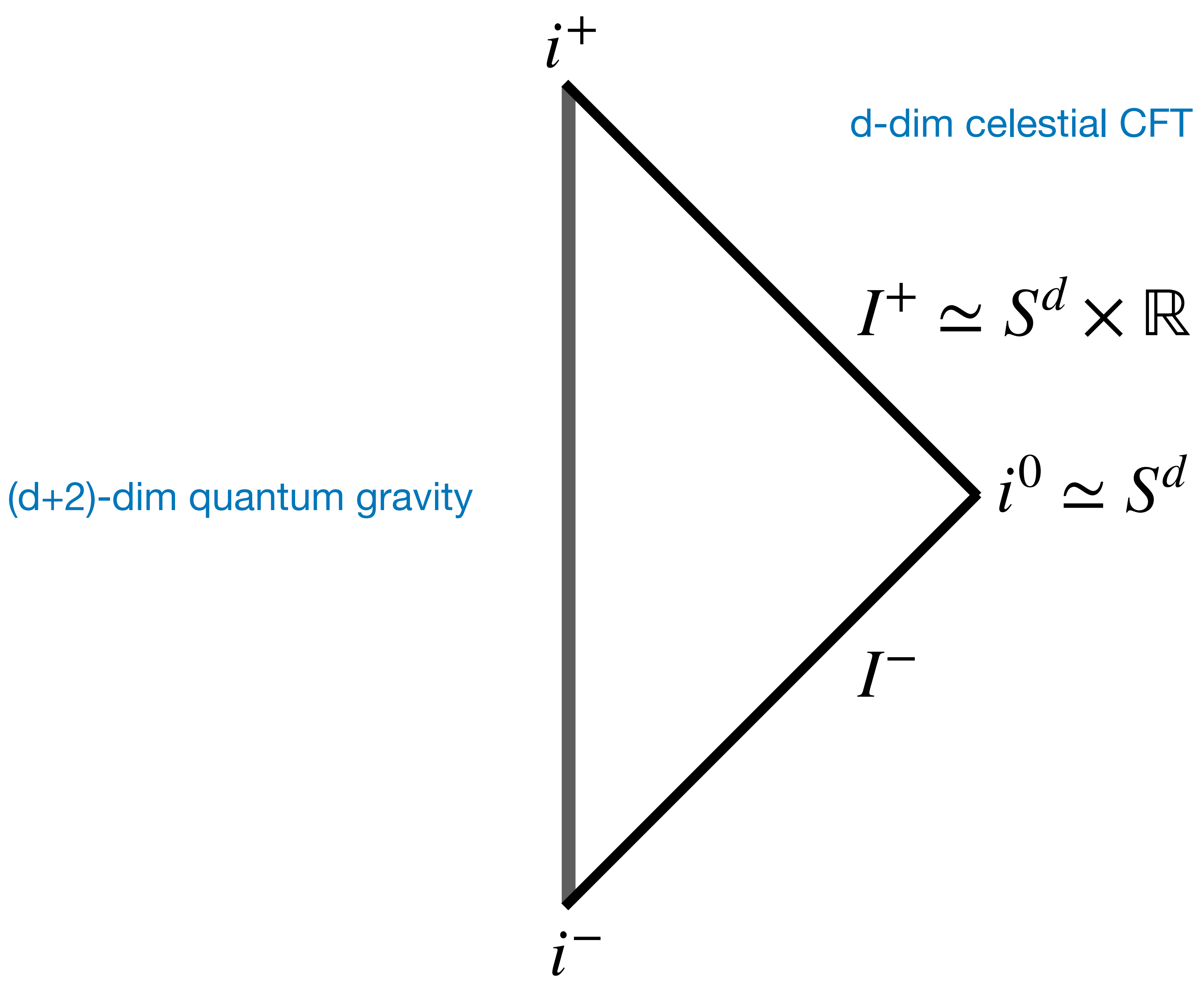
- Chi-Ming Chang, RL, and Wen-Jie Ma. Split representation in celestial holography. 11 2023. arXiv:2311.08736.
- RL and Wen-Jie Ma. Massive celestial amplitudes and celestial amplitudes beyond four points. 4 2024. arXiv:2404.01920.
- RL and Wen-Jie Ma. Celestial Optical Theorem. 4 2024. arXiv:2404.18898.

Introduction

Celestial holography

- Proposal:
  - bulk: (d+2)-dim quantum gravity on AF spacetime;
  - bdry: putative d-dim CCFT on the celestial sphere.
- Symmetry:
  - bulk: Lorentz symmetry SO(d+1,1);
  - bdry: Euclidean conformal symmetry SO(d+1,1);
  - here we do not discuss asymptotic symmetries/large gauge transformations.
- Observables:
  - bulk: scattering amplitudes $\mathscr{M}$;
  - bdry: celestial amplitudes $\mathscr{A}$ as conformal correlators.
- We focus on $d=2$, bosonic objects: $(\Delta, J) = (h+\bar{h}, h-\bar{h})$ and $J \in \mathbb{Z}$.
- We try to learn **universal properties** of CCFT from examples: **S-matrix unitarity**.



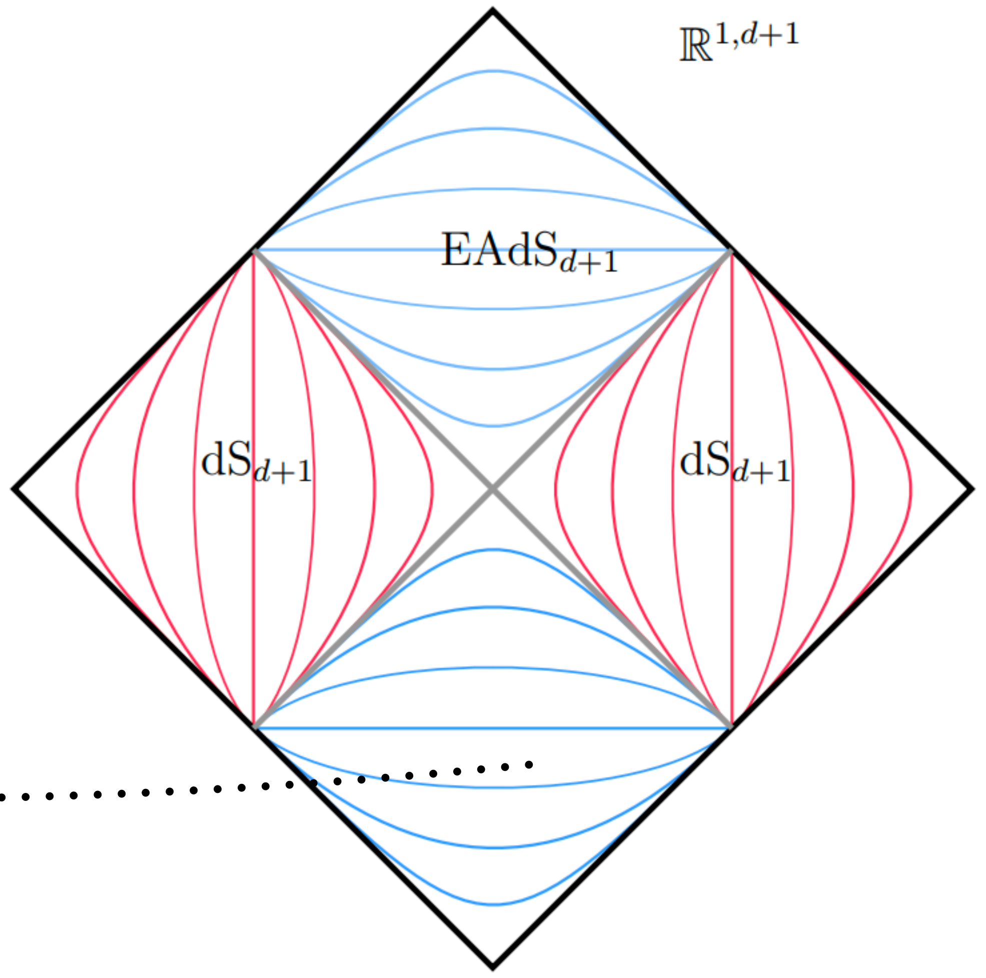
Dictionary

[Pasterski,Shao,Strominger]

mass-shell integral conformal basis

$$\mathcal{A}^{h_I|h_F}_{I\to F}(z_I|z_F) = \left(\prod_{i=1}^{n+m}\int \frac{d^3k_i}{k_i^0}\phi^{\{\mu\}_{\ell_i}}_{h_i,m_i}(z_i,k_i)\right)\mathcal{M}_{I\to F,\{\mu\}_{\ell_I}\{\mu\}_{\ell_F}}(k_I|k_F),$$

celestial amplitudes scattering amplitudes



Hyperbolic slicing of momentum space

- Continuous complex conformal dimension Δ:
  - $\Delta \in 1 + i\mathbb{R}$;
  - required by invertibility and completeness.
- Mass and conformal dimension are **not** related.
- Massless particle:
  - the integral transform is reduced to the Mellin transform;
  - collinear singularity of $\mathscr{M}$ $\Leftrightarrow$ OPE singularity of $\mathscr{A}$.

Conformal basis $\phi(z, k)$

- Defining property:
  - intertwine $\mathrm{SO}(3,1)$'s linear action on $\mathbb{R}^{3,1}$ and conformal action on $\mathbb{S}^2$;
  - solutions of EOM;
- Can be constructed from AdS bulk-to-bdry propagator;
- Quantum numbers:
  - mass m
  - conformal dimension Δ
  - conformal spin J
- bulk spin ℓ wrt $\mathrm{SO}(3,1) \to$ bdry spin J wrt $\mathrm{SO}(2)$
  - massive: $J \in L = \{-\ell, -\ell+1, \cdots, \ell-1, \ell\}$
  - massless: $J \in L = \{-\ell, \ell\}$

Particles (Wigner)

↓ conformal basis

Putative operators (PS)

↓ effectively as primary operators

SOC and inner product are modified. [Crawley,Miller,Narayanan,Strominger'21]

Primary operators (Verma)

Example: 3-pt scalar celestial amplitudes at tree-level

- 00-m

[Lam,Shao'17]

$$C^{\Delta_1,\Delta_2,\Delta_3}_{1^0_0+2^0_0\to 3^0_{m_3}} = 2^{-\Delta_{12}} m_3^{\Delta_{12}-d-2} \Gamma\left[\begin{matrix}\frac{\Delta_{13,2}}{2}, \frac{\Delta_{23,1}}{2} \\ \Delta_3\end{matrix}\right] .$$

- m0-m: hypergeometric fun.

[RL,Ma'24]

$$C^{\Delta_1,\Delta_2,\Delta_3}_{1^0_{m_1}+2^0_0\to 3^0_{m_3}} = d_{\Delta_1,\Delta_2,\Delta_3} 2^{1-\Delta_2} m_1^{-\Delta_{23}} m_3^{-d+\Delta_3} (m_3^2 - m_1^2)^{\Delta_2-1}\, {}_2F_1\left(\begin{matrix}\frac{\Delta_{23,1}}{2}, \frac{\Delta_{123}-d}{2} \\ \Delta_2\end{matrix} ; \frac{m_1^2 - m_3^2}{m_1^2}\right)$$

- mm-m: triple MB integral or double sum of Wilson poly.

[RL,Ma'24]

$$C^{\Delta_1,\Delta_2,\Delta_3}_{1^0_{m_1}+2^0_{m_2}\to 3^0_{m_3}} = d_{\Delta_1,\Delta_2,\Delta_3} \frac{(2\pi)^{\frac{d+1}{2}} (m_1 m_2)^{-\frac{d+1}{2}}}{(m_1+m_2)\Gamma[\frac{d+1}{2}]} \epsilon^{\frac{d-1}{2}} + O(\epsilon^{\frac{d+1}{2}}) .$$

- Proportional to the AdS three-point coefficient $d_{\Delta_1,\Delta_2,\Delta_3}$.

$$d_{\Delta_1,\Delta_2,\Delta_3} = \frac{1}{2}\pi^{d/2}\Gamma\left[\begin{matrix}\frac{\Delta_{12,3}}{2}, \frac{\Delta_{13,2}}{2}, \frac{\Delta_{23,1}}{2}, \frac{\Delta_{123}-d}{2} \\ \Delta_1, \Delta_2, \Delta_3\end{matrix}\right]$$

- In AdS/CFT, with $d_{\Delta_1,\Delta_2,\Delta_3}$ and split representation, double-trace operators appear in OPE.

- What about CCFT?

celestial amplitudes and OPE

Conformal partial wave expansion

Correlation functions → CPW coefficients/OPE

CPW expansion

inversion formula

$$\mathcal{A}^{h_i}(z_i) = \left(\prod_{j=1}^{n-3} \sum_{J'_j=-\infty}^{+\infty} \int_1^{1+i\infty} \frac{d\Delta'_j}{2\pi i \mu_{h'_j}}\right) \rho^{h_i}_{h'_i} \Psi^{h_i}_{h'_i}(z_i) \,,$$

celestial amplitude — integration over reps — CPW coefficient — CPW

$$\rho^{h_i}_{h'_i} = \frac{1}{\mathrm{Vol\,SO}(1,3)} \left(\prod_{j=1}^{n} \int d^2 z_j\right) \left(\Psi^{h_i}_{h'_i}(z_i)\right)^* \mathcal{A}^{h_i}(z_i) \,.$$

- Based on harmonic analysis of $\mathrm{SO}(3,1)$;
- Carrying exactly the same dynamical information as celestial amplitude;
  - pole of CPW coefficient ⇔ operator in OPE;
  - useful technique for deriving conformal block expansions and OPE.

Split representation of scalar Feynman propagator

[Chang,RL,Ma'23]

Feynman diagrams —(conformal basis)→ celestial amplitudes —(CPW expansion)→ CPW coefficients/OPE

Feynman diagrams —(split representation + 6j symbol)→ CPW coefficients/OPE

- An efficient tool for computing perturbative celestial amplitudes;
- Based on harmonic analysis of symmetric spaces.

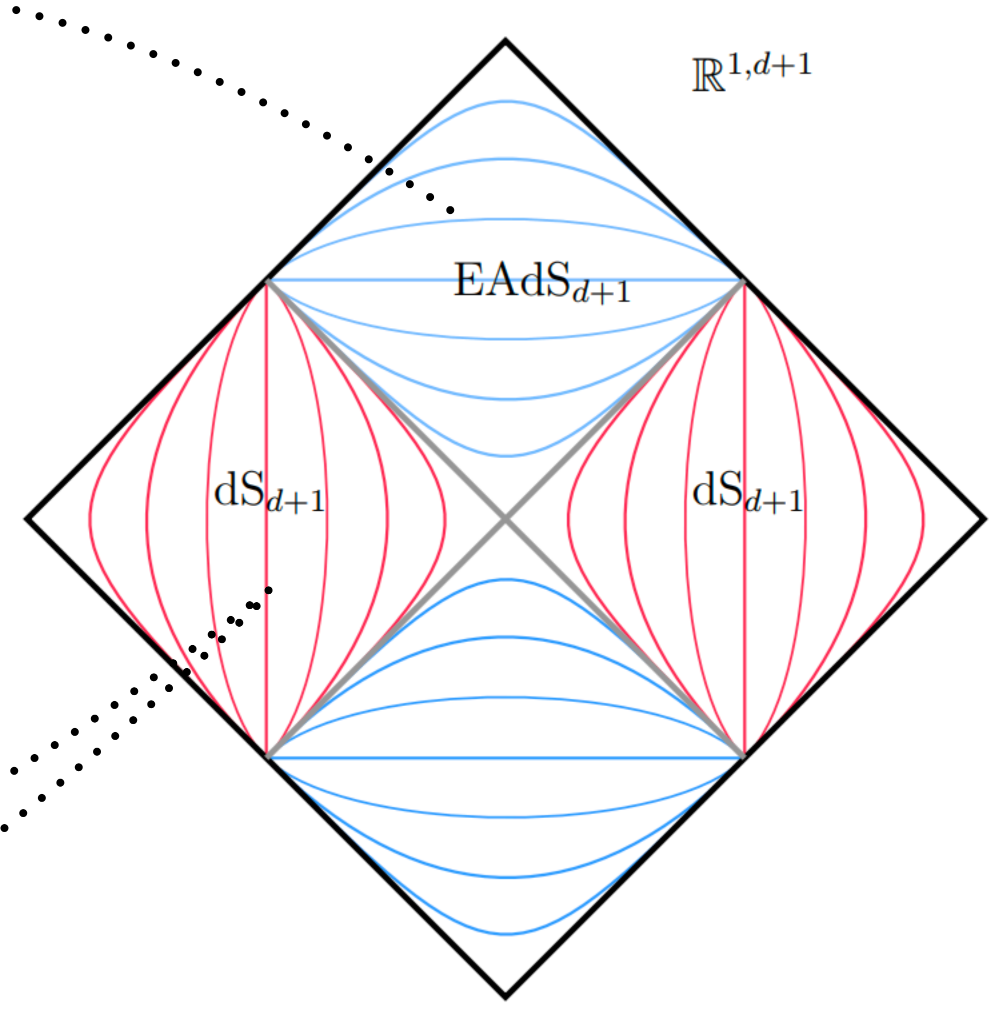
Feynman propagator — integration over mass leaves — integration over reps

$$
\begin{aligned}
\mathcal{K}_m(X_1,X_2) = & \int_0^{+\infty}\frac{dM}{(2\pi)^{d+2}}\frac{iM^{d+1}}{-M^2+m^2}\int_{\frac{d}{2}}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i\,\mu_\Delta}\int d^dx\left(\phi^-_{\Delta,M}\phi^+_{\tilde{\Delta},M}+\phi^+_{\Delta,M}\phi^-_{\tilde{\Delta},M}\right)\\
& +\frac{1}{2}\int_0^{\infty}\frac{dM}{(2\pi)^{d+2}}\frac{iM^{d+1}}{M^2+m^2}\left(\sum_{\epsilon=0,1}\int_{\frac{d}{2}}^{\frac{d}{2}+i\infty}\frac{(-1)^\epsilon d\Delta}{2\pi i\,\mu_\Delta}\int d^dx\,\phi_{\Delta,iM,\epsilon}\phi_{\tilde{\Delta},iM,\epsilon}\right.\\
& \left.+\alpha_d\sum_{(\Delta,\epsilon)\in D_{\rm dS}}\mathrm{Res}\,\frac{(-1)^\epsilon}{\mu_\Delta}\int d^dx\,\phi_{\Delta,iM,\epsilon}\phi_{d-\Delta,iM,\epsilon}\right).
\end{aligned}
$$

principal series of $L^2(\mathrm{EAdS})$

principal series of $L^2(\mathrm{dS})$

discrete series of $L^2(\mathrm{dS})$



Hyperbolic slicing of momentum space

Double-trace operators in celestial OPE

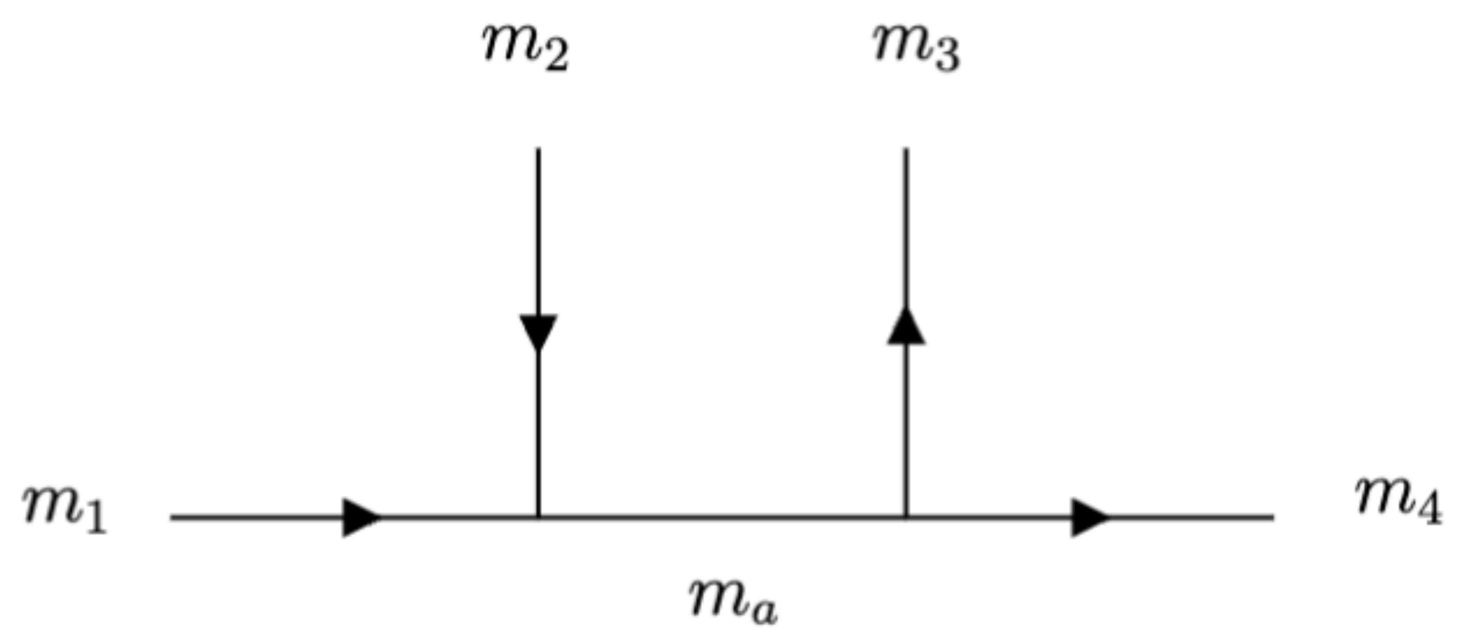
[RL,Ma'24]

- In AdS/CFT, there are double-trace operators in the OPE from tree-level exchange diagrams.

$$\mathcal{O}_{\Delta_1}\mathcal{O}_{\Delta_2} \sim \mathcal{C}_N \mathcal{O}_{\Delta_1+\Delta_2+2N} + \mathcal{C}'_N \mathcal{O}_{\Delta_3+\Delta_4+2N}$$

- In CCFT, we checked that if at most 2-of-5 masses are turned on, there are only double-trace exchanged.
  - 00000:

$$\sigma^{\Delta_i}_{\Delta_a} = -i\,\pi^{-\frac{3d}{2}-2} 2^{-\Delta_{1234}-d-2} 2\pi \delta_c\left(i(\Delta_{1234}-d-4)\right) \times \Gamma\begin{bmatrix} \Delta_3, \Delta_4, \frac{2d-\Delta_{34a}}{2}, \frac{\Delta_{1a,2}}{2}, \frac{\Delta_{2a,1}}{2}, \frac{\Delta_{a,34}+d}{2}, \frac{\Delta_{3a,4}}{2}, \frac{\Delta_{4a,3}}{2} \\ d-\Delta_3, d-\Delta_4, \frac{\Delta_{34,a}}{2}, \Delta_a, \Delta_a-\frac{d}{2}, \frac{\Delta_{34a}-d}{2} \end{bmatrix}.$$



  - m0000

$$\sigma^{\Delta_i}_{\Delta_a} = -i\,\pi^{-d-2} 2^{-\Delta_{234}-d-3} m^{\Delta_{234}-d-4} \times \Gamma\begin{bmatrix} \Delta_3, \frac{-\Delta_{1234}+d+4}{2}, \frac{\Delta_{1,234}+4}{2}, \Delta_4, \frac{\Delta_{12,a}}{2}, \frac{2d-\Delta_{34a}}{2}, \frac{\Delta_{1a,2}}{2}, \frac{\Delta_{2a,1}}{2}, \frac{\Delta_{12a}-d}{2}, \frac{\Delta_{a,34}+d}{2}, \frac{\Delta_{3a,4}}{2}, \frac{\Delta_{4a,3}}{2} \\ \Delta_1, d-\Delta_3, d-\Delta_4, \frac{-\Delta_{34a}+d+4}{2}, \frac{\Delta_{34,a}}{2}, \Delta_a, \Delta_a-\frac{d}{2}, \frac{\Delta_{a,34}+4}{2}, \frac{\Delta_{34a}-d}{2} \end{bmatrix}.$$

Multi-particle operators in celestial OPE

[Ball,Hu,Pasterski'23] [Guevara,Hu,Pasterski'24] [RL,Ma'24]

- New two-particle operator with $\Delta = \Delta_1 + \Delta_2 + 2\Delta_3 - 4$ were observed by OPE limit analysis. [Ball,Hu,Pasterski'23] [Guevara,Hu,Pasterski'24]

- Necessary for OPE associativity.

- Confirmed in 5,6-point tree diagrams, and there should be infinitely many new types of operators, even at tree-level.

[RL,Ma'24]

4-pt　　　　5-pt

$$\mathcal{O}_1^+\mathcal{O}_2^+ \sim \mathcal{C}_{\Delta_{12}-2+2N}\mathcal{O}^+_{\Delta_{12}-2+2N} + \mathcal{C}_{\Delta_{12}+2\Delta_3-4+2N}\mathcal{O}^+_{\Delta_{12}+2\Delta_3-4+2N} + \mathcal{C}_{\Delta_{12}+2\Delta_{34}-6+2N}\mathcal{O}^+_{\Delta_{12}+2\Delta_{34}-6+2N}\,.$$

6-pt

$$\sigma_{\Delta_a,\Delta_b,\Delta_c} = -i\,\pi^{-\frac{7d}{2}-6}2^{-\Delta_{123456}-3d-8}2\pi\delta_c\left(i(\Delta_{123456}-d-8)\right) \times\Gamma\begin{bmatrix}\Delta_5,\Delta_6,\frac{\Delta_{1a,2}}{2},\frac{\Delta_{2a,1}}{2},\frac{\Delta_{3a,b}}{2},\frac{\Delta_{3b,a}}{2},\frac{\Delta_{ab,3}}{2},\frac{\Delta_{4b,c}}{2},\frac{\Delta_{5c,6}}{2},\frac{\Delta_{6c,5}}{2},\frac{\Delta_{4c,b}}{2},\frac{\Delta_{bc,4}}{2}\\ d-\Delta_5,d-\Delta_6,\Delta_a,\Delta_a-\frac{d}{2},\Delta_b,\Delta_b-\frac{d}{2},\frac{\Delta_{56,c}}{2},\Delta_c,\Delta_c-\frac{d}{2}\end{bmatrix} \times\Gamma\begin{bmatrix}\frac{-\Delta_{3456a}+d+6}{2},\frac{\Delta_{a,3456}+6}{2},\frac{-\Delta_{456b}+d+4}{2},\frac{\Delta_{b,456}+4}{2},\frac{\Delta_{3ab}-d}{2},\frac{2d-\Delta_{56c}}{2},\frac{\Delta_{c,56}+d}{2},\frac{\Delta_{4bc}-d}{2}\\ \frac{-\Delta_{456b}+d+6}{2},\frac{\Delta_{b,456}+6}{2},\frac{-\Delta_{56c}+d+4}{2},\frac{\Delta_{c,56}+4}{2},\frac{\Delta_{56c}-d}{2}\end{bmatrix}.$$

Staggered operators in celestial OPE

[Chang,RL,Ma'23] [Chang,RL,Ma,to appear]

- Former examples are s-channel diagram expanded in s-channel OPE; only EAdS part contributes.

- t-channel diagram in t-channel OPE; dS (continuous and discrete) parts contribute.

$$
{}_t\mathcal{A}^{\Delta_i}_{1^0_0+2^0_0\to 3^0_0+4^0_0}(x_i) = \frac{im^{\beta-2}}{2^{5+\beta}(2\pi)^4}\csc[\frac{\pi\beta}{2}]\left[2a_0 G^{\Delta_{13},\Delta_{24}}_{\frac{1}{2},\frac{1}{2}} + \sum_{n=0}^{+\infty}\left(b_n G^{\Delta_{13},\Delta_{24}}_{\frac{n+2}{2},\frac{n+2}{2}} + G^{\Delta_{13},\Delta_{24}}_{\text{sta},-\frac{n}{2},\frac{n}{2}+1} + G^{\Delta_{13},\Delta_{24}}_{\text{sta},\frac{n}{2}+1,-\frac{n}{2}}\right)\right],
$$

primary operators — staggered operators

$$
G^{\Delta_{13},\Delta_{24}}_{\Delta}(\chi) = \chi^{\Delta}\sum_{k=0}^{\infty}\frac{(\Delta-\Delta_{13})_k(\Delta+\Delta_{24})_k}{\Gamma[k+1](2\Delta)_k}\chi^k
$$

$$
\begin{aligned}
G^{\Delta_{13},\Delta_{24}}_{\text{sta},-\frac{n}{2}}(\chi) = &\sum_{k=0}^{n}\frac{(-1)^k(-\frac{n}{2}-\Delta_{13})_k(-\frac{n}{2}+\Delta_{24})_k(n-k)!}{n!\Gamma[k+1]}\chi^{k-\frac{n}{2}}\\
&+\sum_{s=0}^{n}\sum_{k=0}^{\infty}\frac{(-1)^n(-\frac{n}{2}-\Delta_{13})_{k+n+1}(-\frac{n}{2}+\Delta_{24})_{k+n+1}}{n!\Gamma[k+n+2]\Gamma[k+1](k+1+s)}\chi^{k+\frac{n+2}{2}}.
\end{aligned}
$$

- Why they appear? Relation to celestial diamonds?

$$
\begin{aligned}
L_0|\mathcal{O}^1\rangle &= \Delta|\mathcal{O}^1\rangle, & L_1|\mathcal{O}^1\rangle &= 0,\\
L_0|\mathcal{O}^2\rangle &= (1-\Delta)|\mathcal{O}^2\rangle, & L_1|\mathcal{O}^2\rangle &= aL_{-1}^{-2\Delta}|\mathcal{O}^1\rangle.
\end{aligned}
$$

Celestial optical theorem

Celestial optical theorem

[RL,Ma'24]

S-matrix unitarity —(completeness)→ optical thm

S-matrix unitarity —(completeness + conformal basis + on-shell split-rep + CPW expansion)→ celestial optical thm

$$i\mathcal{M}^*_{F\to I} - i\mathcal{M}_{I\to F} = \sum_{n=1}^{+\infty}\sum_{\alpha_n}\int d\Pi_{\alpha_n}\mathcal{M}^*_{F\to\alpha_n}\mathcal{M}_{I\to\alpha_n}.$$

$$S^\dagger S = 1$$

$$i\left(\mathcal{A}^{\overline{h}_3^*\overline{h}_4^*|\overline{h}_1^*\overline{h}_2^*}_{3+4\to1+2}\right)^* - i\mathcal{A}^{h_1h_2|h_3h_4}_{1+2\to3+4} = \sum_{n=1}^{\infty}\sum_{\alpha_n}\frac{1}{S_{\alpha_n}}\left(\prod_{j_{\alpha_n}=1}^{n}\sum_{J_{j_{\alpha_n}}\in L_{j_{\alpha_n}}}\int_{1-i\infty}^{1+i\infty}\frac{d\Delta_{j_{\alpha_n}}}{2\pi i N^{\ell_{j_{\alpha_n}}}_{h_{j_{\alpha_n}},m_{j_{\alpha_n}}}}\int d^2w_{j_{\alpha_n}}\right)$$
$$\times\left(\mathcal{A}^{\overline{h}_3^*\overline{h}_4^*|h_{\alpha_n}}_{3+4\to\alpha_n}(z_3,z_4|w_{\alpha_n})\right)^*\mathcal{A}^{h_1h_2|h_{\alpha_n}}_{1+2\to\alpha_n}(z_1,z_2|w_{\alpha_n}).$$

2-2 CPW coefficients — 3-pt coefficients — 3-pt coefficients in shadow basis

$$i\left(\rho^{\overline{h}_3^*\overline{h}_4^*|\overline{h}_1^*\overline{h}_2^*}_{3+4\to1+2,h}\right)^* - i\rho^{h_1h_2|h_3h_4}_{1+2\to3+4,h} = \sum_{\alpha}\frac{\mu_h\sigma_{J,L_\alpha}}{N^{\ell_\alpha}_{h,m_\alpha}}\left(\left(C^{\overline{h}_3^*\overline{h}_4^*|h}_{3+4\to\alpha}\right)^*C^{h_1h_2|h}_{1+2\to\alpha} + \left(C^{\overline{h}_3^*\overline{h}_4^*|h}_{3+4\to\mathcal{S}[\alpha]}\right)^*C^{h_1h_2|h}_{1+2\to\mathcal{S}[\alpha]}\right)$$
$$+\sum_{n=2}^{\infty}\sum_{\alpha_n}\frac{1}{S_{\alpha_n}}\left(\prod_{j_{\alpha_n}=1}^{n}\sum_{J_{j_{\alpha_n}}}\int_{1-i\infty}^{1+i\infty}\frac{d\Delta_{j_{\alpha_n}}}{2\pi i N^{\ell_{j_{\alpha_n}}}_{h_{j_{\alpha_n}},m_{j_{\alpha_n}}}}\right)\left(\prod_{j=2}^{n-1}\sum_{J'_j=-\infty}^{+\infty}\int_{1-i\infty}^{1+i\infty}\frac{d\Delta'_j}{4\pi i\mu_{h'_j}}\right)\left(\rho^{\overline{h}_3^*\overline{h}_4^*|h_{\alpha_n}}_{3+4\to\alpha_n,h'_i}\right)^*\rho^{h_1h_2|h_{\alpha_n}}_{1+2\to\alpha_n,h'_i}$$

internal conformal dimensions — external conformal dimensions — 2-n CPW coefficients

Celestial optical theorem

- Optical thm:
  - unitarity of S-matrix + completeness relation;
  - nonlinear relations between scattering amplitudes;
  - on-shell, nonperturbative.
- Celestial optical thm:
  - unitarity of S-matrix + completeness relation;
  - nonlinear relations between CPW coefficients;
  - on-shell, nonperturbative.

- Optical thm:
  - phase space integrals are hard to manipulate;
  - no natural partial wave expansion for higher-points.
- Celestial optical thm:
  - scalar integral equations of CPW coefficients;
  - natural basis for all points.

Positivity

- We focus on elastic scattering $1+2 \to 1+2$.
- Notice: $\rho^{h_1 h_2|h_3 h_4}_{1+2\to 1+2,h} \equiv \rho^{J_i}_J(\Delta;\Delta_i)$ is a function of Δ and Δ_i for each fixed spin J and J_i.
- Positivity:
  - nonperturbative criterion of CCFT;
  - similar to that in cosmological bootstrap from dS unitarity; [Hogervorst,Penedones,Vaziri'21] [Di Pietro,Gorbenko,Komatsu'21]

$$\mathrm{Im}(\rho^{h_1 h_2|\overline{h}_1^* \overline{h}_2^*}_{1+2\to 1+2,h}) \geqslant 0\,, \quad \text{if } \Delta \in 1 + i\,\mathbb{R}.$$

Analyticity assumptions

- Analyticity assumptions:
  - $\rho_J^{J_i}(\Delta;\Delta_i)$ is meromorphic with respect to $\Delta \in \mathbb{C}$ and decays to zero as $\Delta \to \infty$;
    - natural assumption for CFT;
    - can be loosened to include branch points.
  - $\rho_J^{J_i}(\Delta;\Delta_i)$ is is also meromorphic with respect to the external conformal dimensions $\Delta_i \in \mathbb{C}^4$;
    - technical assumption, inspired by perturbative results;
    - can be loosened to smaller domain of Δ_i, to include branch points.

Applications

- Exactly one of the following statements is true:
  - for any J and J_i, $\rho_J^{J_i}(\Delta;\Delta_i)=0$;
  - for $J_1=-J_3$ and $J_2=-J_4$, $\rho_J^{J_i}(\Delta;\Delta_i)$ contains at least one Δ-pole.
- Given any celestial three-point coefficient $C_{1+2\to\alpha}^{h_1h_2|h}$, if $\Delta=f(\Delta_1,\Delta_2)$ is a pole of $C_{1+2\to\alpha}^{h_1h_2|h}$ for some meromorphic f, and the intersection between the hypersurface $\Delta=f(\Delta_1,\Delta_2)$ and the principal series $\Re\Delta=1$ is nonempty, then $\Delta=f(\Delta_1,\Delta_2)$ is a pole of $\rho_J^{J_i}(\Delta;\Delta_i)$ when $J_1=-J_3$ and $J_2=-J_4$.
  - Hence double-trace poles $\Delta=\Delta_1+\Delta_2+\cdots$ do not have anomalous dimensions, in contrast to AdS/CFT!
- If the two incoming particles are both massless, exactly one of the statements is true:
  - for any J and J_i, $\rho_J^{J_i}(\Delta;\Delta_i)=0$;
  - for $J_1=-J_3$ and $J_2=-J_4$, $\rho_J^{J_i}(\Delta;\Delta_i)$ contains only simple Δ-poles in a definite set;
  - in shadow basis, **only** double-trace operators appear in the OPE.
- All the properties are consistent with known perturbative examples.

Future directions

Future directions

- (Off-shell) Split representation for spinning particles are still incomplete.
- S-matrix crossing symmetry should lead further constraints on CPW coefficients.
- Solving the bootstrap equations from the celestial optical theorem.
- Better understanding of staggered modules (work in progress).

Thanks for attention!