Schur index and modularity

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work in progress

Outline

- Background and goal
- Modularity of FMLDE
- Modularity of index
- High temperature behavior and 3d mirror

Background

- SCFT/VOA correspondence
- Goal

SCFT/VOA: 4d $\mathcal{N} = 2$ SCFT $\mathcal{T} \to$ 2d chiral algebra $\mathbb{V}[\mathcal{T}]$

> [Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees, '13], [Lemos, Peelaers, '14], [Fredrickson, Pei, Yan, Ye, '17], [Cordova, Gaiotto, Shao, '17] [Song, Xie, Yan, '17] [Buican, Nishinaka, '20] [Xie, Yan, '19] ....

- Symmetries of $\mathcal{T}$: $P_{\alpha,\dot\alpha}, K^{\alpha,\dot\alpha}, H, M_\alpha{}^\beta, M^{\dot\beta}{}_{\dot\alpha}, r, R, R^\pm, Q^I_\alpha, \tilde Q_{I\dot\alpha}, S_I^\alpha, \tilde S^{I\dot\alpha}$
- **Schur operators**: local, quarter-BPS $(Q^1_-, \tilde Q_{2\dot -}, S_1^-, \tilde S^{2\dot -})$

$$E - 2R - j_1 - j_2 = r + j_2 - j_1 = 0\ .$$

- Schur operators of $\mathcal{T}$ form a **2d chiral algebra** $\mathbb{V}[\mathcal{T}]$

  > a.k.a. vertex operator algebra (VOA)

SCFT/VOA: 4d $\mathcal{N} = 2$ SCFT $\mathcal{T}$

- **Non-trivial**: contains a **Virasoro** subalgebra
- **Non-unitary**: $c = -12c_{\text{4d}} < 0$
- **Flavor symmetry** $\to$ **Kac-Moody** subalgebra
- Crucial superconformal invariant
- **Non-rational, non-unitary** (but quasi-lisse): non-trivial Higgs branch

> `[Arakawa, Kawasetsu]` `[Arakawa]` `[Beem, Rastelli]`, `...`

Schur index

- **Schur index**: special limit of 4d $\mathcal{N}=2$ superconformal index

$$\mathcal{I}(b,q) = q^{\frac{c_{4d}}{2}}\,\mathrm{str}\, q^{E-R} b^{f}\ , \quad q = e^{2\pi i\tau},\ b = e^{2\pi i\mathfrak{b}}$$

- Counts Schur operators with sign
- Identified with the **vacuum character**,

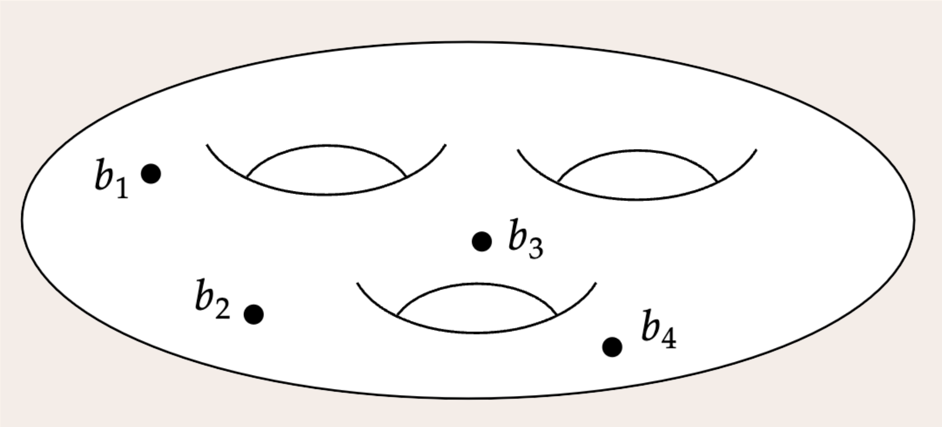
$$\mathcal{I}(\mathcal{T}) = \mathrm{ch}_0(\mathbb{V}[\mathcal{T}])\ .$$

- Schur index of **Lagrangian** theories are computed in **closed forms**

> [Bourdier, Drukker, Felix] [YP, Peelaers]
> [Hatsuda, Okazaki] [Huang] [Beem, Singh, Razamat]
> [Du, Huang, Wang] ...

A_1 class-$\mathcal{S}$ theories

- 4d $\mathcal{N} = 2$ superconformal
- Specified by a punctured Riemann surface $\Sigma_{g,n}$

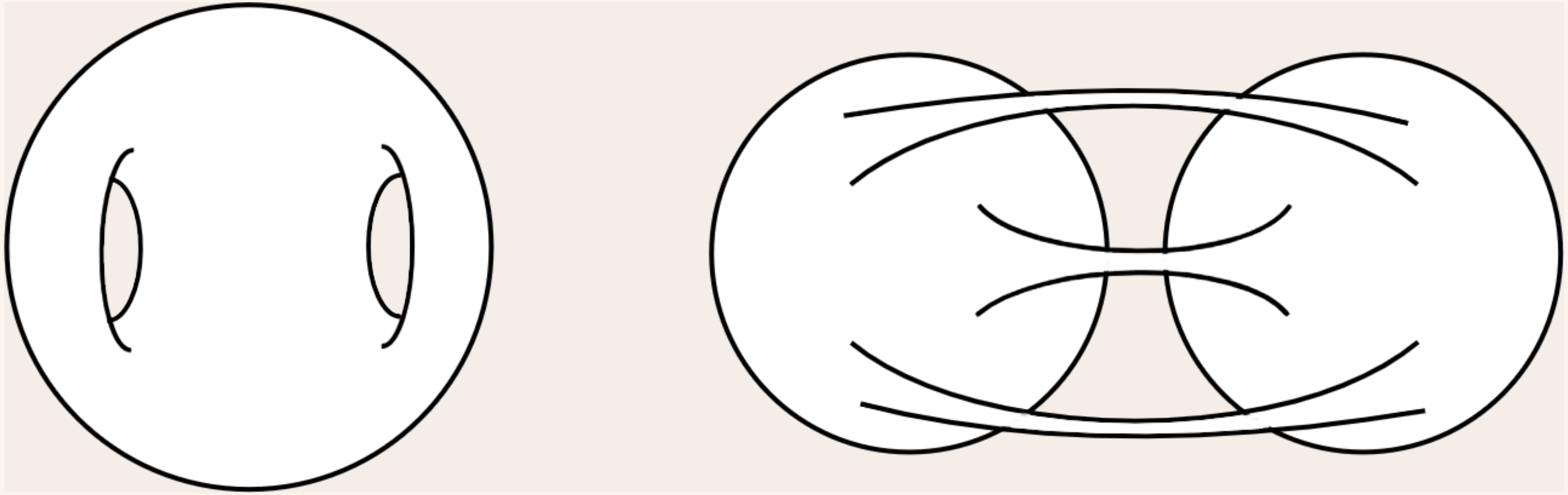


> `[Gaiotto]` `(Dan Xie's talk)`

- A_1: all **Lagrangian** theories with $SU(2)$ gauge groups, $\mathcal{T}_{g,n}$

A_1 class-$\mathcal{S}$ theories

- pants-decomposition: gauge theory description



- a long **tube**: an $SU(2)$ **gauge group**
- a **puncture**: an $SU(2)$ **flavor** symmetry

Indices of A_1 class-$\mathcal{S}$ theories

- Schur index with or without non-local BPS operators
  - No insertion: $\mathcal{I}_{g,n}(b_j)$, ($b_j = e^{2\pi i \mathfrak{b}_j}$)
  - Wilson or 't Hooft line: $\langle L \rangle_{g,n}$
  - Vortex surface defect: $\mathcal{I}^{\text{vortex}}_{g,n}(\kappa)$ (**vorticity** κ)

> [Cordova, Gaiotto, Shao] [Alday, et.al.] [Bianchi, Lemos] [Nishinaka, Sasa, Zhu] [Beem, Rastelli] [Xinan's talk]

Indices of A_1 class-$\mathcal{S}$ theories

- Contour integral

$$\mathcal{I} = \operatorname{Res} \oint \frac{da_i}{2\pi i a_i} R(a) \mathcal{Z}(a, b)$$

- **Closed form**: general integration formula $\Rightarrow$ finite sums and products of special functions

> `[YP, Peelaers]`; see also `[Bourdier, Drukker, Felix]` `[Hatsuda, Okazaki]` `[Du, Huang, Wang]` `...`

- Common ingredients (**twisted Eisenstein** $E_k\left[\begin{smallmatrix}\pm 1\\ b\end{smallmatrix}\right]$)

$$\frac{\eta(\tau)^{2g-2+n}}{\prod_{i=1}^{n} \vartheta_1(2\mathfrak{b}_i)}, \qquad \mathbf{E}_k^{\pm}(b) := \sum_{\alpha_j = \pm} \Big(\prod_{i=1}^{n} \alpha_i\Big) E_k \begin{bmatrix} \pm 1 \\ \prod_j b_j^{\alpha_j} \end{bmatrix}$$

Schur index of A_1 class-$\mathcal{S}$ theories

- Compact formula for all A_1 class-$\mathcal{S}$ theories

> `[YP, Peelaers]`

$$\mathcal{I}_{g,n\geq 1} = \frac{i^n}{2}\frac{\eta(\tau)^{n+2g-2}}{\prod_{j=1}^{n}\vartheta_1(2\mathfrak{b}_j)}\sum_{k=1}^{n+2g-2}\lambda_k^{(n+2g-2)}\mathbf{E}_k^{\pm}(b)\ ,$$

$$\mathcal{I}_{g\geq 2,0} = \frac{1}{2}\eta(\tau)^{2g-2}\sum_{k=1}^{g-1}\lambda_{2k}^{2g-2}\left(E_{2k} + \frac{B_{2k}}{(2k)!}\right)\ ,$$

where $b^\alpha := \prod_{j=1}^{n} b_j^{\alpha_j}$, λ's are rational numbers

Indices of A_1 class-$\mathcal{S}$ theories

- $\mathcal{I}_{g,n}$ and $\mathcal{I}^{\text{vortex}}_{g,n}(\kappa)$: linear combination of

$$\frac{\eta(\tau)^{2g-2+n}}{\prod_{i=1}^{n}\vartheta_1(2\mathfrak{b}_i)}, \qquad \frac{\eta(\tau)^{2g-2+n}}{\prod_{i=1}^{n}\vartheta_1(2\mathfrak{b}_i)}\mathbf{E}^{\pm}_{k}(b)$$

  with **rational numerical** coefficients

  > `[YP, Peelaers]`

- Wilson line index: linear combination of the same ingredients, with **rational functional** coefficients, like

$$\frac{b^m \pm b^{-m}}{q^m - q^{-m}} .$$

  > `[Guo, Li, YP, Wang]`; see also `[Hatsuda, Okazaki]`

Modular linear differential equation (MLDE)

- Chiral algebra $\mathbb{V}[\mathcal{T}]$: **quasi-lisse**
- There **exists** special **null state** $|\mathcal{N}_T\rangle$, conformal weight $2n_T$

$$L_{-2}^{n_T}|0\rangle = |\mathcal{N}_T\rangle + |\varphi\rangle , \qquad |\varphi\rangle \in C_2(\mathbb{V}[\mathcal{T}]) \ .$$

> $C_2(\mathbb{V}[\mathcal{T}])$ is spanned by elements of the form $a_{-h_a-1}|b\rangle$
>
> `[Beem, Rastelli]`

- **What is $|\mathcal{N}_T\rangle$ exactly** ? **Not much is known**

Modular linear differential equation (MLDE)

- Apply **Zhu's recursion formula** (Song He's talk) to $|\mathcal{N}_T\rangle$: **unflavored** Schur index satisfies an n_T-th order **unflavored MLDE**

$$\operatorname{str}\mathcal{N}_T(0)q^{L_0-\frac{c}{24}} = \left[D_q^{(n_T)} + \sum_{i=0}^{n_T-1}\phi_r(\tau)D_q^{(n-r)}\right]\mathrm{ch} = 0\ .$$

> [Arakawa] [Beem, Rastelli] [Kaidi, et.al.]

- **Serre derivative** $D_q^{(n)} := \partial_{(2n-2)}\cdots\partial_{(2)}\partial_{(0)}\ ,\ \partial_{(\ell)} := q\partial_q + \ell E_2(\tau)$
- **Modular form** (w.r.t. **modular group** Γ) as coefficients,

$$\phi_r\left(\frac{a\tau+b}{c\tau+d}\right) \to (c\tau+d)^{2r}\phi(\tau)\ , \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma \subset SL(2,\mathbb{Z})\ .$$

Modular linear differential equation (MLDE)

- All **unflavored** characters of $\mathbb{V}[\mathcal{T}]$ satisfy the same **unflavored** MLDE from $|\mathcal{N}_T\rangle$

  > "**ordinary**" module character

- The unflavored MLDE is **covariant** under suitable **modular group** Γ
- The space of solutions/characters form a representation of Γ
- **Other** nulls: additional order-n **unflavored** MLDEs
- **Minimal order** $n_{\min} \leq n_T$

  > may have **non-zero** Wronskian index, namely, ϕ may have **poles**

Flavored modular linear differential equation (FMLDE)

- Characters with **no** unflavoring limit (**non-ordinary** module)

$$\mathrm{ch}(q, b) := \mathrm{str}\, q^{L_0 - \frac{c}{24}} b_1^{f_1} \dots b_n^{f_n}$$

- $\Rightarrow$ Need **flavored** MLDEs (**FMLDEs**)

$$\sum_{\substack{k,\ell_1,\dots,\ell_r \\ 2k+\ell_1+\dots+\ell_r \le 2n}} \phi_{k,\ell_1,\dots,\ell_r}(\tau, b_1, \dots, b_r) D_q^{(k)} \prod_{j=1}^{r} D_{b_j}^{\ell_j} \mathrm{ch}_M = 0 \ ,$$

ϕ's are **not** modular form: **quasi-Jacobi**

- There maybe **several** null states $\Rightarrow$ system of FMLDEs
- All characters of $\mathbb{V}[\mathcal{T}]$ satisfy the same system of FMLDEs

Flavored modular linear differential equation (FMLDE)

- Solutions may have 4d physical origin
- Vortex surface defects are potential solutions: characters

$$\mathcal{I}^{\text{def}}_{g,n}(\kappa = \text{even})$$

- Modular transformation of $\mathcal{I}_{g,n}, \mathcal{I}^{\text{def}}_{g,n}(\kappa = \text{even})$ are also solutions
- Certain residues $\operatorname{Res} \mathcal{Z} \sim$ Gukov-Witten defect index
- $\langle W \rangle_{g,n}$ (stripping off rational functional coefficients)

> `[Zheng, YP, Wang]`

Modularity

- Three inter-connected objects:
  - The Schur **index** $\mathcal{I}$/vacuum character ch_0; other index/characters ch
  - The **FMLDEs** that constrain $\mathcal{I}$ or ch
  - **Null states** ($|\mathcal{N}_T\rangle$, etc.)
- **What is, and how to characterize $|\mathcal{N}_T\rangle$?**
- **Modularity of FMLDEs and implication on the nulls**
- **Which nulls or FMLDEs are most important?**
- **Modularity of $\mathcal{I}$ and solution space**

$|\mathcal{N}_T\rangle$ and FMLDE

> `work in progress`

- Modularity of A_1 class-$\mathcal{S}$
- Dimension and basis of modular orbit
- Modularity of $\mathcal{N} = 4$ theories

Property of $|\mathcal{N}_T\rangle$

- Consider Kac-Moody algebra $\widehat{\mathfrak{g}}_k$, with simple $\mathfrak{g}$
- Appear in SCFT/VOA: e.g.,

$$\mathbb{V}[(A_1, D_{2n+1})] = \widehat{\mathfrak{su}}(2)_{-\frac{4n}{2n+1}}$$

$$\mathbb{V}[\mathcal{T}_{0,4}] = \widehat{\mathfrak{so}}(8)_{-2}$$

$$\mathbb{V}[T_3] = (\hat{\mathfrak{e}}_6)_{-3}$$

> `[Minahan, Nemeschansky]` `[Argyres, Douglas]` `[Xie]` `[Beem, et.al.]` `[Beem, Rastelli]`

Property of $|\mathcal{N}_T\rangle$

- Defining property:

$$L_{-2}^{n_T}|0\rangle = |\mathcal{N}_T\rangle + |\varphi\rangle, \qquad |\varphi\rangle \in C_2 \ .$$

> n_T unknown, $|\varphi\rangle$ unknown

- $|\mathcal{N}_T\rangle$ is neutral under $\mathfrak{g}$
- conformal weight $= 2n_T$
- $|\mathcal{N}_T\rangle$ is null: descendant of singular vector

Property of $|\mathcal{N}_T\rangle$

- Preliminary observation: "**intrinsic**" constraints

$$J_0^a|\mathcal{N}_T\rangle = L_2|\mathcal{N}_T\rangle = h^i_{n\geq 2}|\mathcal{N}_T\rangle = 0 \ .$$

> Corollary
>
> $$J^a_{n\geq 2}|\mathcal{N}_T\rangle = L_{n\geq 1}|\mathcal{N}_T\rangle = 0 \ .$$

Property of $|\mathcal{N}_T\rangle$

- **Classify** Kac-Moody algebras
- Fix $n_T = 2, 3, 4, \ldots$, **solve** the constraints of $|\mathcal{N}_T\rangle$
- Apply **Zhu's recursion** to $|\mathcal{N}_T\rangle$: FMLDE of **weight**-$2n_T$

$$\mathcal{E}_T = (D_q^{(n_T)} + \phi_r(b) D_q^{\#} D_{b_i}^{\#} D_{b_j}^{\#} + \cdots)\,\mathrm{ch} = 0\,.$$

> ϕ_r denotes product of **twisted Eisenstein series** $E_k\begin{bmatrix}+1\\ b^{\#}\end{bmatrix}$
> $\mathrm{wt}(D_q^{(n)}) = 2n, \mathrm{wt}(D_{b_i}) = 1, \mathrm{wt}(E_k[\cdot]) = k$

- Apply **Zhu's recursion** to **Sugawara** construction $T - \frac{1}{2(k+h^\vee)} JJ = 0$

$$\mathcal{E}_\mathrm{S} = (D_q^{(1)} + \sum_{i,j}^{r} D_{b_i} D_{b_j} + \cdots)\,\mathrm{ch} = 0$$

Quasi-modularity

- $E_k\left[{\pm 1 \atop b^{\#}}\right]$ are **not** modular: **quasi-Jacobi**
- $\mathcal{E}_T$ is **not** modular: **quasi-modular**

$$\mathcal{E}_T \xrightarrow{S} \tau^{2n_T}\mathcal{E}_T + \tau^{2n_T-1}\sum_{i=1}^{r}\mathfrak{b}_i\mathcal{E}_i + \tau^{2n_T-2}\sum_{i,j=1}^{r}\mathfrak{b}_i\mathfrak{b}_j\mathcal{E}_{ij} + \cdots + \sum_{i_1\ldots i_{2n_T}}\mathfrak{b}_1\cdots\mathfrak{b}_{2n_T}\mathcal{E}_{i_1\ldots i_{2n_T}} .$$

- $\mathrm{wt}(\mathcal{E}_{i_1\ldots i_\ell}) = 2n_T - \ell$
- $\mathcal{E}_{i_1\ldots i_\ell}$ also transforms into $\mathcal{E}_{i_1\ldots i_\ell i_{\ell+1}\ldots}$ (lower weights)

Quasi-modularity

- All $\mathcal{E}_{i_1\dots}$ correspond to additional states

$$\mathcal{E}_{i_1 i_2 \dots i_\ell} \leftrightarrow \frac{1}{\ell!} h_1^{i_1} \dots h_1^{i_\ell} |\mathcal{N}_T\rangle \ , \qquad \mathcal{E}_T \leftrightarrow |\mathcal{N}_T\rangle$$

- **Modularity** of null states

$$S|\mathcal{N}_T\rangle = \sum_{\ell,\vec{i}} \frac{1}{\ell!} h_1^{i_1} \dots h_1^{i_\ell} |\mathcal{N}_T\rangle := \sum_{\ell,\vec{i}} \frac{1}{\ell!} |\mathcal{N}^{i_1 \dots i_\ell}\rangle$$

$$S|\mathcal{N}^{i_1 \dots i_\ell}\rangle = \sum_{\ell',\vec{j}} \frac{1}{\ell'!} h_1^{j_1} \dots h_1^{j_{\ell'}} |\mathcal{N}^{i_1 \dots i_\ell}\rangle$$

Quasi-modularity

- $|\mathcal{N}^{i_1\ldots i_\ell}\rangle$: **additional** null states
- **Additional PDEs** $\mathcal{E}_{i_1\ldots i_\ell}$
- $\mathcal{E}_{i_1\ldots i_{2n_T}}, \mathcal{E}_{i_1\ldots i_{2n_T}-1} = 0$ fix the algebra $\mathfrak{g}$ and level k
- $\mathcal{E}_{i_1\ldots i_{2n_T}-2}, \ldots, \mathcal{E}_T, \mathcal{E}_{\mathrm{S}} = 0$ fix **all** the flavored **characters**

Examples

- $n_T = 2$,

$$\mathfrak{g} = \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{g}_2, \mathfrak{d}_4, \mathfrak{f}_4, \mathfrak{e}_{6,7,8}, \quad k = 1, \text{or}, -\frac{h^\vee}{6} - 1$$

> unitary/non-unitary Deligne-Cvitanovic exceptional series
> `[Perše]` `[Axtell, Lee]` `[Arakawa, Moreau]` `[Beem, Rastelli]`

- Equation $\mathcal{E}_{i_1 i_2 i_3 i_4} = 0$

$$\Rightarrow (\lambda, \lambda)^2 = N \sum_{\alpha \in \Delta} (\alpha, \lambda)^4 , \qquad \forall \lambda , \quad \text{some } N \in \mathbb{R}$$

- $n_T = 3, 4, 5, \mathfrak{g} = \mathfrak{su}(2)$

n_T	k
2	$k = 1, -4/3$
3	$k = 2, -8/5, -1/2$
4	$k = 3, -12/7$
5	$k = 4, -16/9, -5/4$

- Simple relation with the **singular vector** $|v\rangle$ (and its $\mathfrak{g}$ descendants $|v^{B_1 \ldots B_n}\rangle$)

$$|\mathcal{N}_T\rangle = C_{A_1 \ldots A_n B_1 \ldots B_n}(k) J_{-1}^{A_1} \ldots J_{-1}^{A_n} |v^{B_1 \ldots B_n}\rangle$$

> C is some combination of the Killing form.

Modularity

> `[YP, Yang]`

- Modularity of A_1 class-$\mathcal{S}$
- Dimension and basis of modular orbit
- Modularity of $\mathcal{N} = 4$ theories

Schur index of A_1 class-$\mathcal{S}$ theories

- Compact formula for all A_1 class-$\mathcal{S}$ theories

> `[YP, Peelaers]`

$$\mathcal{I}_{g,n\geq 1} = \frac{i^n}{2}\frac{\eta(\tau)^{n+2g-2}}{\prod_{j=1}^{n}\vartheta_1(2\mathfrak{b}_j)}\sum_{k=1}^{n+2g-2}\lambda_k^{(n+2g-2)}\mathbf{E}_k^{\pm}(b)\ ,$$

$$\mathcal{I}_{g\geq 2,0} = \frac{1}{2}\eta(\tau)^{2g-2}\sum_{k=1}^{g-1}\lambda_{2k}^{2g-2}\left(E_{2k}+\frac{B_{2k}}{(2k)!}\right)\ ,$$

where $b^\alpha := \prod_{j=1}^{n} b_j^{\alpha_j}$, λ's are rational numbers

A_1 class-$\mathcal{S}$ theories

- Modular transformation of $\mathcal{I}_{g,n}$ ($\tau = \frac{1}{2\pi i}\log q$, $\mathfrak{b}_j = \frac{1}{2\pi i}\log b_j$)

$$\mathcal{I}_{g,n} \xrightarrow{S} \frac{(-i\tau)^{g-1}\eta(\tau)^{2g-2+n}}{(-i)^n e^{\frac{4\pi i}{\tau}\sum_{j=1}^n \mathfrak{b}_j^2}\prod_{j=1}^n \vartheta_1(2\mathfrak{b}_j)} \times \sum_{k=1}^{2g-2+n} \lambda_k^{(2g-2+n)} \sum_{\alpha_j=\pm}\left(\prod_{j=1}^n \alpha_j\right) \times \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell}}{(k-\ell)!}\left(\sum_{j=1}^n \alpha_j \mathfrak{b}_j\right)^{k-\ell} \tau^\ell E_\ell\left[\begin{matrix}1\\(-1)^n b^\alpha\end{matrix}\right].$$

> Implicit $\prod_{j=1}^n y_j^{-2}$ factor to soak up $\exp\left(\frac{4\pi i}{\tau}\sum_{j=1}^n \mathfrak{b}_j^2\right)$

A_1 class-$\mathcal{S}$ theories

- Structure: $\tau^{g-1} \times (\tau^0 + \tau + \cdots + \tau^{2g-2+n})$
- **Highest** power in τ: $3g - 3 + n = \dim \mathcal{M}_{g,n}$
- **Lowest** power in τ: $g = 0$, potential τ^{-1} term, **dangerous**
- Luckily, τ^{-1} term always **vanishes**,

$$\sum_{\alpha_j = \pm} \left(\prod_{j=1}^{n} \alpha_j\right) \left(\sum_{j=1}^{n} \alpha_j \mathfrak{b}_j\right)^k = 0\,, \quad k = 0, 1, 2, \ldots, n-1\,.$$

A_1 class-$\mathcal{S}$ theories

- Modular orbit $SL(2,\mathbb{Z}) \cdot \mathcal{I}_{g,n}$ or $\Gamma^0(2) \cdot \mathcal{I}_{g,n}$
- **No negative** power in τ: **finitely** many linear independent expressions
- Spanning a **linear space** $\mathcal{V}_{g,n}$

A_1 class-$\mathcal{S}$ theories

- Compute $\dim \mathcal{V}_{g,n}$ (assuming n **positive even**, $g > 0$)
  - The structure of $\mathcal{I}_{g,n}$

$$\frac{\eta(\tau)^{2g-2+n}}{\prod_j \vartheta_1(2\mathfrak{b}_j)}\left(\mathbf{E}_2^+(b) + \mathbf{E}_4^+(b) + \cdots + \mathbf{E}_{2g-2+n}^+(b)\right) .$$

  - $\mathbf{E}_k^+(b)$ generates a $k+1$ dimensional space
  - η/ϑ_1 factor **increases** each dimension by $\frac{2g-2+n-n}{2} = g-1$
  - Total dimension

$$\dim \mathcal{V}_{g,n} = \sum_{\substack{k=2 \\ k=n \bmod 2}}^{2g-2+n} (k+1+g-1) = \sum_{\substack{k=2 \\ k=n \bmod 2}}^{2g-2+n} (k+g) .$$

A_1 class-$\mathcal{S}$ theories

- All cases

$$\dim \mathcal{V}_{g,n} = \sum_{\substack{k=1 \\ k=n \bmod 2}}^{2g-2+n} (k+g) \ , \qquad \text{when } n > 0 \ ,$$

$$\dim \mathcal{V}_{g,n} = \sum_{\substack{k=0 \\ k=n \bmod 2}}^{2g-2+n} (k+g) \qquad \text{when } n = 0 \ .$$

- Agrees with the **unflavored** orbit dimension predicted in

> `[Beem, Singh, Razamat]`

$$\delta_{g\neq 0}(2g+n-1)g + \delta_{n\neq 0}\left\lfloor \frac{n-1}{2} \right\rfloor \left(g+n-1-\left\lfloor \frac{n-1}{2} \right\rfloor\right) .$$

A_1 class-$\mathcal{S}$ theories: observations

- $\dim \mathcal{V}_{g,n}^{\text{flavored}} = \dim \mathcal{V}_{g,n}^{\text{unflavored}}$
- $=$ **minimal** order n_{min} of the **unflavored** MLDE of $\mathcal{T}_{g,n}$
- $=$ number of **rational** indicial root of the $\mathcal{N}_T$ equation
- The closed-form $\mathcal{I}_{g,n}$ **transparently** encodes modular structure
- $\{k+g\}$ in the sum gives sizes of **Jordan-blocks** of T, S (or T^2, STS for $\Gamma = \Gamma^0(2)$)
- Number of $\mathbf{E}_k$ = number of **Jordan-blocks** = number of **non-log unflavored** solutions
- Encodes number/dimensions of fixed points/moment maps in CB Hitchin system

> `[Fredrickson, Pei, Yan, Ye]` `[Wenbin's talk]`

A_1 class-$\mathcal{S}$ theories

- basis of $\mathcal{V}_{g,n}$ for **even** n,

$$\left\{(\mathbf{T}^{\ell}_{g,n}S)\mathcal{I}_{g,n}\right\}_{\ell=0}^{g+1} \cup \left\{(\mathbf{T}^{\ell}_{g,n}S)(\mathbf{T}^{g+1}_{g,n}S)\mathcal{I}_{g,n}\right\}_{\ell=0}^{g+3}$$
$$\cup \left\{(\mathbf{T}^{\ell}_{g,n}S)(\mathbf{T}^{g+3}_{g,n}S)(\mathbf{T}^{g+1}_{g,n}S)\mathcal{I}_{g,n}\right\}_{\ell=0}^{g+5} \cup \cdots$$
$$\cdots \cup \left\{(\mathbf{T}^{\ell}_{g,n}S)(\mathbf{T}^{3g-3+n-2}_{g,n}S)\cdots(\mathbf{T}^{g+1}_{g,n}S)\mathcal{I}_{g,n}\right\}_{\ell=0}^{3g-3+n} .$$

- Similar basis for odd n
- S, T or STS, T^2 matrices can be computed easily

A_1 class-$\mathcal{S}$ theories

- The $\mathcal{V}_{g,n}$ is a **subspace** of the full space of solutions of the FMLDEs

  All elements in $\mathcal{V}_{g,n}$ have **unflavoring** limit

- There are additional solutions **outside** of the modular orbit

  e.g., $\dim \mathcal{V}_{0,4} = 2$, spanned $\mathcal{I}_{0,4}, S\mathcal{I}_{0,4}$.

  Additional solutions with **no** unflavoring limit

$$R_j := \frac{i}{2}\frac{\vartheta_1(2\mathfrak{m}_j)}{\eta(\tau)}\prod_{\substack{\ell=1\\ \ell\neq j}}^{4}\frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j+\mathfrak{m}_\ell)}\frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j-\mathfrak{m}_\ell)}, \qquad j = 1, 2, 3, 4\,.$$

  each R_j forms a **1d rep** of $SL(2,\mathbb{Z})$, correspond to four highest weight module of $\mathfrak{so}(8)_{-2}$ by `[Perše]` `[Arakawa, Moreau]`

A_1 class-$\mathcal{S}$ theories

- Defect index?
- Vortex surface defect index $\mathcal{I}^{\text{vortex}}_{g,n}(\kappa)$, Wilson line index $\langle W_j \rangle$: known in closed-form

> `[YP, Peelaers]` `[Guo, Li, YP, Wang]` `[Hatsuda, Okazaki]`

- Results:
  - $\mathcal{I}^{\text{vortex}}_{g,n}(\kappa) \in \mathcal{V}_{g,n}$, for all even κ
  - $\mathcal{I}^{\text{vortex}}_{g,n}(\kappa)$ with odd κ, solve **twisted** FMLDEs, form a separate linear space $\mathcal{V}^{\text{twisted}}_{g,n}$
  - after **extracting** rational function coefficients, $\langle W_j \rangle \in \mathcal{V}_{g,n}$

$\mathcal{N}=4$ SU Theories

- $SU(2N+1)$:
  - **unflavored** MLDE is $SL(2,\mathbb{Z})$-modular, $SL(2,\mathbb{Z})$-orbit dimension

$$1+3+5+\cdots+(2N+1)=(N+1)^2\ .$$

  - **flavored** MLDE is $\Gamma^0(2,\mathbb{Z})$ quasi-modular, $\Gamma^0(2)$-orbit dimension $=(N+1)^2$, $SL(2,\mathbb{Z})$ orbit dimension $3(N+1)^2$
- $SU(2N)$: unflavored(and flavored) MLDE both $\Gamma^0(2)$ (quasi-)modular, orbit dimension

$$1+\sum_{k=2}^{N}(2k)+1=N(N+1)\ .$$

- General basis have been proposed

High temperture

- High temp asymptotics
- 3d Mirror
- Defect and Wilson line

High temperature asymptotics

- Unflavored results in [Ardehali, Martone, Rosselo]
- Express $\mathcal{I}_{g,n}$ in **dual variables**

$$\tilde{\tau} = -\frac{1}{\tau}, \qquad \tilde{\mathfrak{b}}_i = \frac{\mathfrak{b}_i}{\tau}, \qquad \tilde{b}_i = e^{2\pi i \tilde{\mathfrak{b}}_i}.$$

- $\tau \to +i0$ asymptotics: $\tilde{\tau}$, $\tilde{\mathfrak{b}}$ expansion

$$\mathcal{I}_{g,n} = \frac{1}{2}(-i\tilde{\tau})^{g-1}\frac{\eta(\tilde{\tau})^{2g-2+n}}{\prod_{j=1}^{n}\vartheta_1(2\tilde{\mathfrak{b}}_j|\tilde{\tau})}\sum_{k=2}^{2g-2+n}\lambda_k^{(2g-2+n)}\Bigg(\sum_{\ell=0}^{k}\frac{1}{\ell!}\sum_{\alpha_j=\pm}\Big(\prod_{j=1}^{n}\alpha_j\Big)\Big(\sum_{j=1}^{n}\alpha_j\tilde{\mathfrak{b}}_j\Big)^{\ell}(-\tilde{\tau})^{k-\ell}E_{k-\ell}\begin{bmatrix}1\\ \prod_j \tilde{b}^{\alpha_j}\end{bmatrix}\Bigg).$$

Analytic generalization of [Ardehali, Martone, Rosselo]

High temperature asymptotics

- Example: $\mathcal{N} = 4\ SU(2)$ theory,

$$\mathcal{I}_{\mathcal{N}=4\ SU(2)} = -\frac{\tilde{\mathfrak{b}}\vartheta_2(\tilde{\mathfrak{b}}|\tilde{\tau})}{\vartheta_1(2\tilde{\mathfrak{b}}|\tilde{\tau})} + \frac{i\tilde{\tau}}{2\pi}\frac{\vartheta_2'(\tilde{\mathfrak{b}}|\tilde{\tau})}{\vartheta_1(2\tilde{\mathfrak{b}}|\tilde{\tau})}$$

$$\overset{b\to 1}{=} -\frac{1}{2\pi} - \frac{2\tilde{q}}{\pi} - \frac{6\tilde{q}^2}{\pi} + \dots$$

$$-\frac{\ln\tilde{q}}{8\pi} - \frac{3\tilde{q}\ln\tilde{q}}{2\pi} - \frac{9\tilde{q}^2\ln\tilde{q}}{2\pi} - \frac{27\tilde{q}^3\ln\tilde{q}}{2\pi} + \dots .$$

agrees with `[Ardehali, Martone, Rossello]`.

High temperature asymptotics

- Example: $SU(2)$ SQCD

$$\mathcal{I}_{0,4} = \frac{1}{2}\frac{1}{-i\tilde{\tau}}\frac{\eta(\tilde{\tau})^2}{\prod_{j=1}^4 \vartheta_1(2\tilde{\mathfrak{b}}_j)}$$

$$\times \sum_{\alpha_j=\pm}\left(\prod_j \alpha_j\right)\left[-\frac{(2\pi i)^2}{2}(\alpha\cdot\tilde{\mathfrak{b}})^2 - 2\pi i\tilde{\tau}(\alpha\cdot\tilde{\mathfrak{b}})E_1\begin{bmatrix}1\\ \tilde{b}^\alpha\end{bmatrix}(\tilde{\tau}) + \frac{1}{2}\tilde{\tau}^2 E_2\begin{bmatrix}1\\ \tilde{b}^\alpha\end{bmatrix}(\tilde{\tau})\right].$$

$$= \frac{\tilde{q}^{-\frac{5}{12}}}{120\pi}\Big[1 + 250\tilde{q} + 4625\tilde{q}^2 + 44250\tilde{q}^3 + 305750\tilde{q}^4 + 1703752\tilde{q}^5$$

$$+ 8150375\tilde{q}^6 + 34673250\tilde{q}^7 + \ldots + 60\ln\tilde{q}\mathcal{I}_{0,4}(\tilde{q})\Big].$$

agrees with `[Ardehali, Martone, Rossello]`.

3d Mirror

- Consider $g = 0, n > 3$
- High temperature limit $\tau \to +i0 \Rightarrow S^3$-partition function of the $SU(2)^{n-3}$ linear quiver in **closed form**,

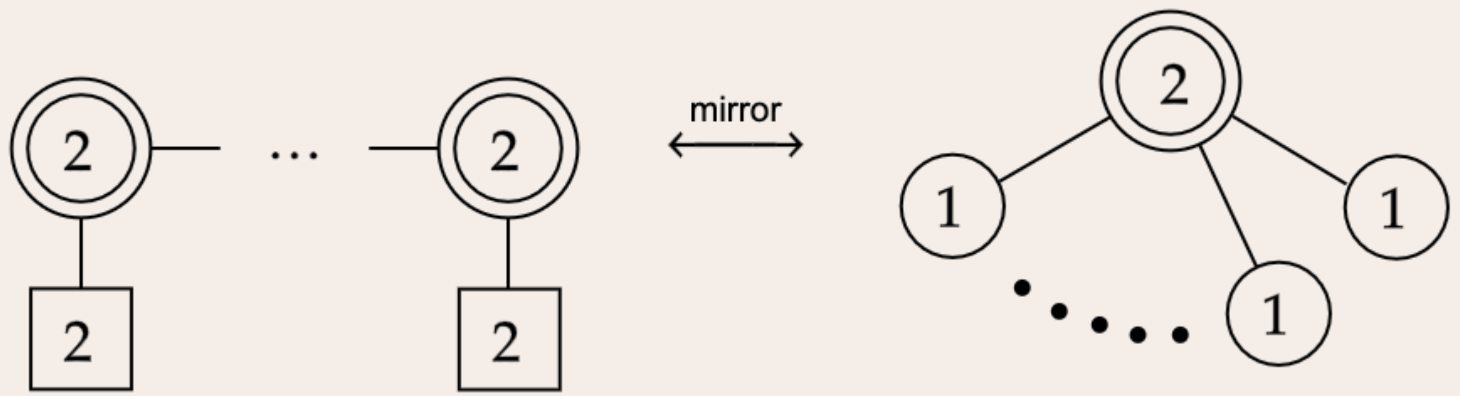
$$
\begin{aligned}
&Z^{S^3}_{SU(2)^{n-3}\text{ linear quiver}} \\
&= \frac{(-i)^{n+1}}{2\prod_{j=1}^{n}(\tilde{b}_j - \tilde{b}_j^{-1})} \sum_{\alpha_j = \pm} \frac{1}{(n-3)!} \left(-\frac{1}{2} - \frac{1}{1 + (-1)^n \prod_j^n \tilde{b}_j^{\alpha_j}} \right) \\
&\times \prod_{k=-\frac{n-4}{2}}^{k=\frac{n-4}{2}} \left(k + \alpha \cdot \tilde{\mathfrak{b}}\right) .
\end{aligned}
$$

3d Mirror

- Checked agreement with **mirror dual** partition function ($\xi_i = -2i\tilde{\mathfrak{b}}_i$)

$$Z^{S^3}_{SU(2)\times U(1)^n} = \int da \frac{1}{(2\sinh(2\pi a))^{n-2}} \prod_{i=1}^{n} \frac{2\sin(2\pi\xi_i a)}{2\sinh(\pi\xi_i)} \ .$$

- Generalize closed-form result in [Benvenuti, Pasquetti]
  > [Intriligator, Seiberg] [Benini, Tachikawa, Xie] (Dan Xie's talk)



Defect index and Wilson operator

- High temperature asymptotics for $\mathcal{I}^{\text{vortex}}_{g,n}(\kappa)$
- **2d q-YM description** predicts mirror S^3-partition function with a Wilson operator in spin-κ $SU(2)$ irrep

> `[Gadde, Rastelli, Razamat, Yan]`
> `[Gaiotto, Rastelli, Razamat]` `[Alday, et.al.]`

$$\mathcal{I}^{\text{vortex}}_{g,n}(\kappa) = q^{-\frac{c_{\text{2d}}}{24}} \sum_{j\in\frac{1}{2}\mathbb{N}} S_{\kappa j} C_j(q)^{-2+n} \prod_{i=1}^{n} \psi_j(b_i)$$

- **Formal** high temperature limit

$$\mathcal{I}^{\text{vortex}}_{g,n}(\kappa) \xrightarrow{\tau\to+i0,\mathfrak{b}=\tau\mathfrak{b}^{\text{3d}}\to 0, j=i\sigma/\tau} Z^{S^3}_{SU(2)\times U(1)^n}[W_\kappa]$$

Defect index and Wilson operator

- Caveat: given n, **not all** $Z^{S^3}_{SU(2)\times U(1)^n}[W_\kappa]$ **converge**
- $\kappa = 0, 1, \ldots, n-3$: **linear independent** defect indices
- $\kappa = 0, 1, \ldots, n-3$: **convergent** Wilson line partition function
- For these κ: $\mathcal{I}^{\text{vortex}}_{g,n}(\kappa) \xrightarrow{\tau\to 0} Z^{S^3}_{SU(2)\times U(1)^n}[W_\kappa]$

Outlook

- Other class-$\mathcal{S}$ theory, other defects
- FMLDEs in classfication of chiral algebras
  - role of $\mathcal{N}_T$?
  - modular bootstrap with quasi-Jacobi forms?
- More on modular data and Coulomb branch physics
- Null states and giant graviton expansion `[Gaiotto, Lee]` `[Beccaria, Cabo-Bizet]`

Thank you!