4d Mirror Symmetry for Class-S Theories

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Joint work with Yiwen Pan

Mirror symmetry of 3d $\mathcal{N}=4$ [Intriligator - Seiberg] ...

$$\begin{tikzcd} & X & \\ T \arrow[ur, "\text{Higgs}"] \arrow[dr, "\text{Coulomb}"'] & & T^\vee \arrow[ul, "\text{Coulomb}"'] \arrow[dl, "\text{Higgs}"] \\ & Y & \end{tikzcd}$$

mirror pair: (X, Y)

hyper-Kahler cone

Symplectic duality:

algebra Geometry

$$\begin{tikzcd} \mathcal{A}_X \arrow[r, leftrightarrow] & Y \\ X \arrow[u, squiggly, "\text{quantization}"'] & \end{tikzcd}$$

c.f. [Braden-Licata-Proudfoot-Webster]

Moduli spaces of 4d $\mathcal{N}=2$ SCFTs

$T \xrightarrow{\text{Higgs}} \mathcal{M}_H$: hyper-Kahler cone

$\updownarrow$ relations? not mirror pair

$T \xrightarrow{\text{Coulomb}} \mathcal{M}_C$: special Kahler, SW geometry [Seiberg-Witten]

Algebra $\longleftrightarrow$ Geometry

relations?

4d SCFT / VOA : $T \longrightarrow \text{VOA } V_T \longrightarrow \text{Asso. Var. } X_{V_T}$

[BLLPRvR]

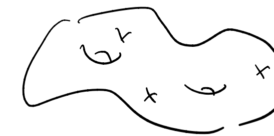
"chiral quantization"

X_{V_T} is $\mathcal{M}_H$

[Song-Xie-Y]
[Beem-Rastelli]
[Arakawa]

Algebra / Geometry correspondence

Need T s.t both are known



- class-S : 6d ADE (2,0) SCFT on $\mathbb{R}^3 \times S^1 \times \Sigma_{g,n}$

[Gaiotto] [Gaiotto-Moore-Neitzke] [Xie] [Xie-Wang] [Xie-Ye] ...

$T_{g,n} \xrightarrow{\text{Higgs}} \mathcal{M}_H \simeq X_{V(T)}$: $V(T)$ known in many cases

[Cordova-Shao] [Buican-Nishinaka] [Song] [Xie-Y] [Arakawa] ....

$T_{g,n} \xrightarrow{\text{Coulomb}} \mathcal{M}_C \simeq \mathcal{M}_{Hitchin}$: moduli space of Hitchin systems on $\Sigma_{g,n}$

[Gaiotto-Moore-Neitzke]

integrable

c.f. Etingof's lecture at BIMSA

4d mirror symmetry

algebra / geometry correspondence [Shan - Xie - Y 22] [Shan - Xie - Y 23]

Higgs/Algebra	Coulomb/Geometry
admissible W-algebras	elliptic affine Springer fiber
simple modules	fixed points $\mathcal{F}\ell_\gamma^*$
highest weight h	moment map μ
$SL(2,\mathbb{Z}) \circlearrowright$ Space of characters	$H^\bullet(\mathcal{F}\ell_\gamma^*) \circlearrowleft$ automorphism of DAHA
Zhu's C_2 alg	Cohomology ring

- gen. Argyres - Douglas [Fredrickson - Neitzke] [Fredrickson - Pei - Y - Ye] [SXY 22] [SXY 23]

  $\Sigma = \mathbb{P}^1$ with 1 regular and/or 1 irregular singularities

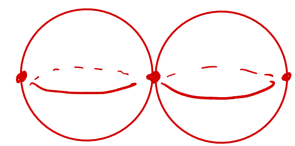
Example

- T: (A_1, D_3) AD theory

$L_{-\frac{4}{3}}(sl_2)$ $\qquad$ $\mathcal{M}_C$, $\mathcal{F}l_\gamma$

3 simple modules

$$\begin{cases} L(-\frac{4}{3}\Lambda_0) \\ L(-\frac{2}{3}\Lambda_1 - \frac{2}{3}\Lambda_0) \\ L(-\frac{4}{3}\Lambda_1) \end{cases}$$



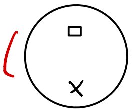
Example

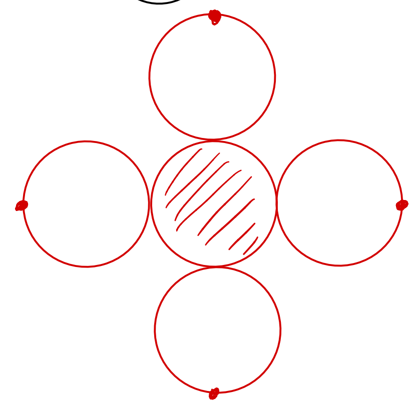
- T

$L_{-2}(D_4)$

5 simple modules [Perše]

$$\begin{tikzcd} & -2\Lambda_0 & \\ -2\Lambda_1 & -\Lambda_2 \arrow[l, "s_1."'] \arrow[u, "s_0."'] \arrow[r, "s_3."] \arrow[d, "s_4."] & -2\Lambda_3 \\ & -2\Lambda_4 & \end{tikzcd}$$

$\mathcal{M}_c = \mathcal{M}_{Hit}($  $)$ $\mathcal{F}\ell_\gamma$: D_4, $\nu = \frac{1}{4}$

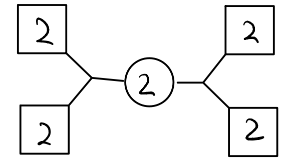


1 fixed manifold $\mathbb{P}^1$

4 fixed points

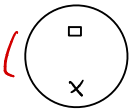
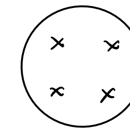
Example

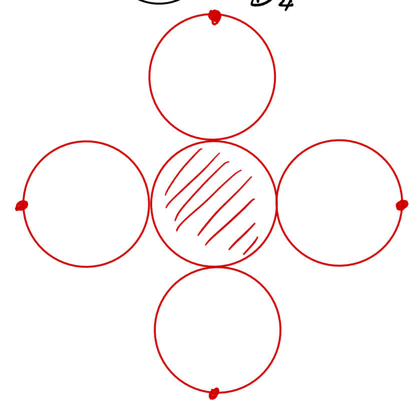
- T : 4d $\mathcal{N}=2$ $SU(2)$ $N_f = 4$



$L_{-2}(D_4)$

5 simple modules [Perše]

$\mathcal{M}_C = \mathcal{M}_{Hit}\Big($  $\Big)_{D_4}$ $\overset{[\text{Boalch}]}{\simeq}$ $\mathcal{M}_{Hit}\Big($  $\Big)_{A_1}$



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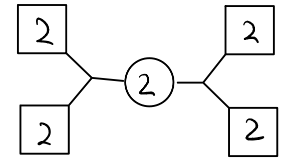


$L_{-2}(D_4)$

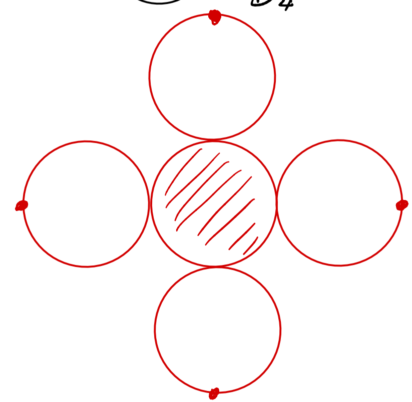
$\mathcal{M}_c = \mathcal{M}_{Hit}(\quad)_{D_4} \overset{[\text{Boalch}]}{\simeq} \mathcal{M}_{Hit}(\quad)_{A_1}$

$\boxed{\begin{array}{l} 5 \text{ simple modules [Perše]} \\ \qquad + \\ 1 \text{ log module} \end{array}}$

[Arakawa-Kawasetsu]

$\Downarrow$

$SL(2,\mathbb{Z}) \circlearrowleft V = \mathrm{span}_{\mathbb{C}}\{\chi_i\}$



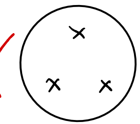
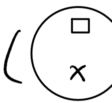
1 fixed manifold $\mathbb{P}^1$

4 fixed points

Example

T: T_3-theory (E_6-MN theory)

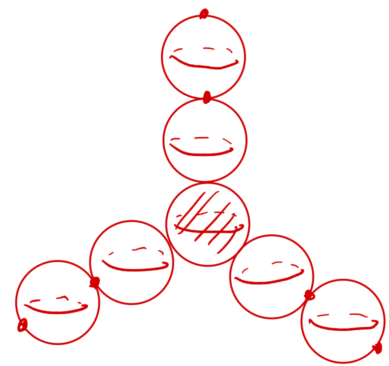
$L_{-3}(E_6)$

$\mathcal{M}_C \simeq \mathcal{M}^{A_2}_{Hit}\Big($  $\Big) \simeq \mathcal{M}^{E_6}_{Hit}\Big($  $\Big)$

7 simple modules [Arakawa-Moreau]

+

1 log-module [Arakawa-Kawasetsu]

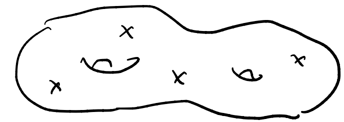


6 pts

1 $\mathbb{P}^1$

4d mirror for regular class-S ?

$T_{g,n}$: 6d (2,0) of SCFT on $\Sigma_{g,n}$



[Arakawa]

[GMW]

$$V(\Sigma_{g,n}) \longleftrightarrow \mathcal{M}_{Hit}(\Sigma_{g,n})$$

representations

fixed manifolds

- $V(\Sigma_{g,n})$ defined abstractly, few cases identified with known VOA
- littles is known about their representation theories.

- Well-studied in A_1, A_2,
- Some results in A_{N-1}

Modular invariant representations of $V(\Sigma_{g,n})$

- $X_{V(\Sigma_{g,n})} \simeq \mathcal{M}_H(T_{g,n})$: finitely many symplectic leaves

$\Rightarrow$ $V(\Sigma_{g,n})$ is quasi-lisse [Arakawa-Kawasetsu]

RMK: V_T from 4d T is conjectured to be quasi-lisse.

- Theorem (Arakawa-Kawasetsu)

  V quasi-lisse. vacuum character χ_0 is solution of a modular differential equation (MDE) $\Rightarrow$ $\chi_0 \in \mathbb{V} \circlearrowleft SL(2,\mathbb{Z})$

Ex. $L_{-2}(D_4)$

$$f''(\tau) - \frac{1}{6}E_2 f'(\tau) - \frac{35}{144}E_4 f(\tau) = 0$$

Modular invariant representations of $V(\Sigma_{g,n})$

- Generalization to flavored cases: [Mason] [Pan-Wang-Zheng] [Pan-Wang] ...

flavored MDE (FMDE) $\leftarrow$ singular vectors $\Big\}$ difficult to compute when $g, n \gg 1$

Solutions : non-log $\qquad\qquad$ log

$\qquad\qquad\qquad\downarrow \qquad\qquad\qquad\qquad \downarrow$

char : Simple modules $\qquad$ log modules

$$\underbrace{\text{Simple modules} \qquad \text{log modules}}_{\text{modular } SL(2,\mathbb{Z}) \text{ rep. } \mathbb{V}}$$

$T \to e$

$STS \to f$

- 4d/VOA : vacuum char = Schur index $\mathcal{I}$

$\mathcal{I}$, $T^2\mathcal{I}$, $(STS)T^2\mathcal{I}$, $T^4\mathcal{I}$, $(STS)T^4\mathcal{I}$ ...

$$\begin{tikzcd} \bullet \arrow[r, "T^2"] & \bullet \arrow[l, bend left, "STS"] \arrow[r] & \bullet \arrow[l, bend left] \arrow[r] & \bullet \arrow[l, bend left] & \cdots \end{tikzcd}$$

Fixed manifolds of $\mathcal{M}_{Hit}(\Sigma_{g,n})$

$\mathfrak{g} = A_1$ (rank 2 Higgs bundle) $\qquad \mathcal{M}_{Hit}(\Sigma_{g,n}) \circlearrowleft U(1) \qquad \mu$: moment map

- $g \geq 2$, $n = 0$ [Hitchin]

  2 types $\begin{cases} \mathcal{M}_0 \simeq \text{moduli space of stable rk 2 bundle over } \Sigma_{g,0} . & \mu = 0 \\ \{\mathcal{M}_d\}_{1 \leq d \leq g-1} : \ \mathcal{M}_d \simeq 2^{2g} \text{ cover of } S^{2g-2d-1}\Sigma_{g,0} . & \mu = d - \frac{1}{2} \end{cases}$

- $g = 0$, $n \geq 3$ or $g \geq 1$, $n \geq 1$ [Boden - Yokogawa]

  $0 < \alpha_1 < \alpha_2 \cdots < \alpha_n < \frac{1}{2}$: parabolic structure on each puncture

  $e = (e_1 \cdots e_n) \quad e_i = 0 , 1 . \qquad \beta = (\beta_1 \cdots \beta_n) \qquad \beta_i = e_i + (-1)^{e_i}\alpha_i$

  2 types $\begin{cases} \overset{(g \geq 1) \text{ only}}{\mathcal{M}_0} \simeq \text{moduli space of stable rk 2 bundle over } \Sigma_{g,0} . \quad \mu = 0 \\ \{\mathcal{M}^{\alpha}_{d,e}\} \quad -\sum_{i=1}^{n} \beta_i < d \leq g - 1 - \frac{|e|}{2} , \quad \mu_{d,e} = d + \sum \beta_i > 0 \end{cases}$

  $\mathcal{M}^{\alpha}_{d,e} \simeq 2^{2g}$ cover of $S^{h_{d,e}}\Sigma_{g,n} \qquad h_{d,e} = 2g-2-2d-|e|$

Mirror symmetry for class-S

- $V(\Sigma_{g,n})$: χ (simple, log), conformal weight. S. T
- $\mathcal{M}_{Hit}(\Sigma_{g,n})$: # of fixed manifolds, μ, dimension

Conjectures [Pan-Y]

1. $$\{\text{fixed points}\} \xrightarrow{\sim} \{\text{simple modules}\}$$
$$\mathcal{M}_i \longmapsto L_i$$

2. $$\mu(\mathcal{M}_i)\big|_{\alpha_i=0} - \mu_{max}\big|_{\alpha_i=0} - s(\mathcal{M}_i) = h(L_i)$$
slope of bundles

3. Jordan class of T-matrix

# of Jordan blocks = # of fixed points

Size of Jordan block $= \begin{cases} g & \mathcal{M}_0 \\ \dim \mathcal{M}_i + g + 1 & \text{others} \end{cases}$

Examples: $n=0 \quad g\geq 2$

$\dim \mathcal{M}_H = 0$. MDE is enough

Ex 1 $\quad g=2 \quad$ 6th order MDE [Kiyoshige-Nishinaka, Beem-Rastelli]

$$\boxed{\left[D_q^{(6)} - 305 E_4 D_q^{(4)} - 4060 E_6 D_q^{(3)} + 20275 E_4^2 D_q^{(2)} + 2100 E_4 E_6 D_q^{(2)} - 68600 \left(E_6^2 - \tfrac{49}{125} E_4^3\right)\right] \chi = 0}$$

Solution: 2 non-log, 4 log. $\quad \dim V = 6$

non-log:

$$\begin{array}{llll} & & h & \mu \\ \chi_0 = \frac{1}{2}\eta^2\left(E_2 + \frac{1}{12}\right) & \longrightarrow \mathcal{M}_1 & 0 & 1/2 \\ \chi_1 = \eta^2 & \longrightarrow \mathcal{M}_0 & -1 & 0 \end{array}$$

Jordan of $T = [4, 2]$

4: $\dim \mathcal{M}_1 + g + 1$

2: g

Examples: $n=0,\ g\geq 2$

Ex 2 $\quad g=3 \quad$ 15 th order MDE

Sol: $\quad$ 3 non-log $\quad$ 12 log

	$\mathcal{M}_0$	$\mathcal{M}_1$	$\mathcal{M}_2$
μ	0	$\frac{1}{2}$	$\frac{3}{2}$
h	-2	-1	0
dim		3	1
Jordan(T)	3	7	5

RMK: Similarly for higher g

Examples: $n > 0$

Ex: $g = 0$ $n = 4$ $V(\Sigma_{0,4}) = L_{-2}(D_4)$

5 non-log, 1 log, $\mathrm{Jordan}(T) = [2, 1, 1, 1, 1]$

Ex: $g = 1$ $n = 1$ $V(\Sigma_{1,1})$ = 2d small $N=4$ SCA

	$\mathcal{M}_0$	$\mathcal{M}_{0,0}$
μ	0	α
h	$-\frac{1}{2}$	0
dim		1
Jordan(T)	1	2

$\longrightarrow$ Matches classification of simple modules by Adamovic

Summary

Algebra / Geometry correspodence for 4d class-S

$V(\Sigma_{g,n})$	$\mathcal{M}_{Hit}(\Sigma_{g,n})$
simple modules	fixed manifolds
conformal weight	moment map
$\left.\begin{array}{l}\text{log modules} \\ \text{Jordan of } T\end{array}\right\}$	dim of fixed manifolds

RMK: geometry $\xrightarrow{\text{predict}}$ representation theory of VOA

e.g. $SU(3)$ $\mathcal{N}=2^*$ $\rightsquigarrow$ VOA from Hilbert scheme: 3 simple modules
[Arakawa]

Outlook

1. Understanding log-modules in geometry?

   $g=0$. non-log + log $\xrightarrow{\sim}$ cohomology ring

   $g>0$?

2. $SL(2,\mathbb{Z})$ action in geometry?

3. Generalization to arbitrary $\mathfrak{g}$

4. Geometry / Geometry correspondence?

   [Shan-Y-Zhao, WIP]: generalized AD

$$\mathcal{M}_c \xrightarrow[\text{central fiber}]{} \mathcal{F}\ell_\gamma \xrightarrow{R\,/_{\overline{\min}}} \mathcal{O}_x \xrightarrow{d} \mathcal{M}_H$$

Thank you!

Thank all organizers!