Bootstrapping the Abelian Lattice Gauge Theories

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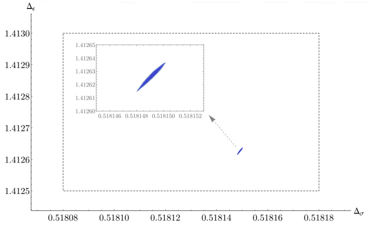
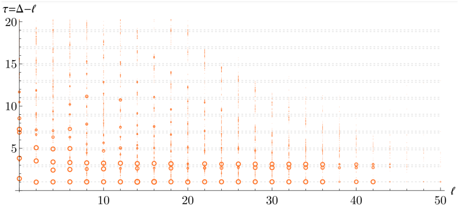
Based on a recent work with Shutong Zhou, arXiv: 2404.17071

Motivation: The renaissance of the bootstrap method

- Classical approach in QFT:
  1. UV Lagrangian
  2. Perturbative computations
  3. RG flow
- **Modern conformal bootstrap approach**:
  1. Consistency conditions, e.g., unitarity
  2. Crossing symmetry
  3. Positivity/Convexity

Remarkable successes: critical 3D Ising model





(*Simmons-Duffin, 2016*)

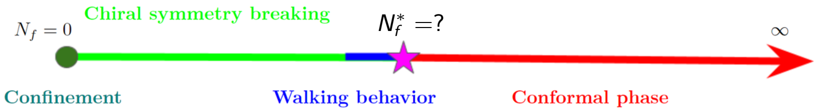
Motivation: Bootstrap strongly coupled gauge theories

Strong coupling phenomena in QCD_4:

- Conformal window, Confinement, Chiral symmetry breaking

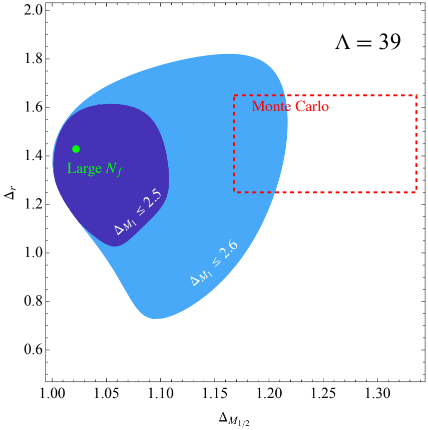
A simplified version– QED_3: $\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i \sigma^\mu (\partial_\mu + i A_\mu) \psi^i$

- The flavor number $N_f \in 2\mathbb{Z}$ to avoid parity anomaly (*Witten 2016*)
- QED_3 is asymptotically free (gauge coupling e^2 has mass unit)
- Large N_f: IR fixed point at $e_*^2 = 6\pi^2/N_f$ (*Appelquist et al 1988*)
- Small N_f: Chiral symmetry breaking (*Appelquist et al 1985*)
- $N_f = 0$: Gauge confinement (*Polyakov 1975, 77*)



Motivation: What bootstrap can do

- Isolate $N_f = 4$ QED_3 CFT data into a small island (with extra input):



- Fix the critical flavor number $N_f^* \in (2, 4)$ (combined with MC data).

$N_f^* \in 2\mathbb{Z}$ for QED_3	Method	Year and Reference
$2 < N_f^* < 4$	**conformal bootstrap**	**2018-22 Li & Li et al**
$\frac{64}{\pi^2} \approx 6.5$	Schwinger-Dyson equations	1984-88 Pisarski,Appelquist et al
8.6	Schwinger-Dyson equations	1996-97 Maris,Aitchison, et al
$\lesssim 3$	thermal free energy	1999-2004 Appelquist et al
$\lesssim 4$	hybrid Monte Carlo	2002-04 Hands et al
4.3	divergence of the chiral susceptibility	2002 Franz et al
8	covariant solutions for propagators	2004 Fischer et al
12	perturbative RG in the large-N_f limit	2004 Kaveh et al
$10 \dots 13$	comparison to the Thirring model	2007-12 Christofi,Janssen, et al
3	lattice simulations	2008 Strouthos et al
$8 \leq N_f^* \leq 20$	functional RG	2014 Braun et al
$\lesssim 8$	F-theorem	2015 Klebanov et al
$\lesssim 4$	one-loop ϵ-expansion	2015 Komargodski et al
5.7	$1/N_f$ expansion	2016 Gusynin et al
5.8	ϵ-expansion	2016 Herbut et al
< 2	lattice simulations	2017 Zi-Yang Meng et al
< 2	lattice simulations	2015-20 Narayanan et al

Motivation: What bootstrap can NOT do

- Hard to distinguish different conformal gauge theories:
  - Local gauge invariant operators:
    No explicitly information on gauge interactions
  - Bootstrap bound coincidence among different symmetries:
    Hidden algebraic structure in the conformal four-point crossing equation
- Non-conformal phase – confinement, chiral symmetry breaking, etc.

Goal of this work:

- Bootstrapping non-local gauge invariant operators
- Confined phase of lattice gauge theories

Why lattice?

Outline

1. Lattice gauge theories and loop equations
2. Positivities in lattice theories and bootstrap implementation
3. Where is the limit of the constraint from loop equations + positivity?
4. Bootstrapping the 3D $U(1)$ and $\mathbb{Z}_2$ lattice gauge theories
5. Towards bootstrapping the string tension and glueball mass

1. Lattice gauge theories and loop equations

Why lattice gauge theory?

To understand the dynamics of Wilson lines/loops (WLs):

- Effective field theory description?
- A widely open question: *What is the bootstrap setup for the WLs:*
  - Correlation functions of WLs?
  - Operator product expansion of WLs?

Difficulties for WLs in quantum gauge field theories, like QCD:

1. Infinite number of D.O.Fs, **divergence and RG flow**
2. Strongly coupled

Lattice gauge theories (LGTs):

Discretizing space or spacetime $\Longrightarrow$ Finite number of D.O.Fs.

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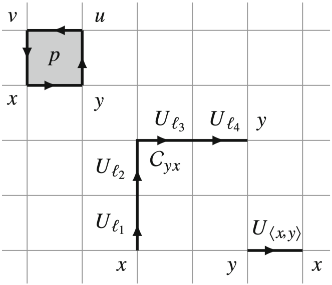
Gauge field on the lattice:



Link $\Leftrightarrow$ Parallel transport:

$$U_{\langle x,y\rangle} = P\exp\left(ig\int_y^x A_\mu(z)dz^\mu\right)$$

Parallel transport along a path:

$$C_{yx} = U_{\ell_1}U_{\ell_2}U_{\ell_3}U_{\ell_4}$$

Wilson loop: $W_C = \mathrm{tr}\left(\prod_{\ell_i\in C} U_{\ell_i}\right)$

Plaquette: a minimal closed path

$$U_P = \mathrm{tr}\left(U_{\langle x,y\rangle}U_{\langle y,u\rangle}U_{\langle u,v\rangle}U_{\langle v,x\rangle}\right)$$

Lattice gauge theory and Wilson action

Fundamental D.O.F in LGT with lattice length a:

$$\text{link } x \to x + \hat{\mu} : \quad U_\mu(x) = e^{iaA_\mu(x)}$$

Gauge transformation in continuum field theory and lattice:

$$\text{on lattice:} \;\; U_\mu(x) \to \Omega(x) U_\mu(x) \Omega(x + \hat{\mu})$$
$$\text{continuum:} \;\; U_{\langle x,y \rangle} \to \Omega(x) U_{\langle x,y \rangle} \Omega^\dagger(y)$$

Gauge invariant plaquette U_P:

$$U_P = \mathrm{tr}\left(U_\mu(x) U_\nu(x + \hat{\mu}) U^\dagger_\mu(x + \hat{\nu}) U^\dagger_\nu(x) \right) = -\frac{a^4}{2} \mathrm{tr} F_{\mu\nu} F_{\mu\nu} + \cdots$$

Wilson action for LGTs:

$$S_{\mathrm{Wilson}} = -\frac{1}{\lambda} \sum_P (U_P + P^\dagger_P)$$

Match the continuum Yang-Mills action to the order $O(a^2)$.

From Wilson action to Loop equations

The physical observables: averaged Wilson loops $W[C] \equiv \mathrm{tr}(\prod_{\ell \in C} U_\ell)$:

$$\langle W[C] \rangle = \frac{1}{Z} \int \prod_{x,\mu} dU_\mu(x) W[C] e^{-S_{\mathrm{Wilson}}}$$

Loop equations: Schwinger-Dyson equations for $\langle W[C] \rangle$.

Take the variation of the link variables

$$U_\mu \to (1 + i\epsilon) U_\mu, U_\mu^\dagger \to U_\mu^\dagger (1 - i\epsilon), S(W) \to S(W) + \delta_\epsilon S(W)$$

Invariance of the integral $\int DU \delta_\epsilon \left(W_C e^{-S} \right)$ leads to the loop equations

$$\sum_{|\alpha| \neq \mu} \langle W[\mu\alpha\bar{\mu}\bar{\alpha}\mu C] \rangle - \langle W[\alpha\mu\bar{\alpha} C] \rangle + n\lambda W[\mu C] = 0$$

Loop equations are (non-)linear for (non-)Abelian LGTs.

Comments on the loop equations in 2D

2D $U(1)$ LGT: After taking gauge fixing on the lattice, the fundamental Wilson loops are $w_m \equiv U_P^m$:

$$w_m - w_{m-2} + (m-1)\lambda w_{m-1} = 0, \quad w_0 = 1.$$

2D $\mathbb{Z}_2$ LGT ($J = 1/\lambda$): Since $U_\ell \in \{1, -1\}$, $U_\ell^2 = 1$, the fundamental Wilson loops satisfy $w_2 = 1$, there is only one independent observables w_1

$$-\frac{1}{2}\sinh(4J) + (1 + \cosh(4J))w_1 - \frac{1}{2}\sinh(4J)w_1^2 = 0,$$

which is solved by $w_1 = \tanh J$, the solution of the 1D Ising model.

Duality from loop equations: 2D $\mathbb{Z}_2$ LGT $\Longleftrightarrow$ 1D Ising model

2. Positivities in lattice theories and bootstrap implementation

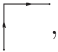
Square positivity in LGTs

Square positivity (*Anderson and Kruczenski, 2016*):
Consider the Hilbert space $\mathcal{H}$ spanned by the Wilson lines W_i, any state $\mathcal{O} = \sum_i c_i W_i \in \mathcal{H}$ has non-negative inner product:

$$\langle \mathcal{O} | \mathcal{O} \rangle = c_j^* \langle W_j | W_i \rangle c_i \geqslant 0, \Longleftrightarrow \langle W_j | W_i \rangle \succeq 0.$$

An illustrative example:

Consider two Wilson lines $W_{1/2}$ along a path $1/2$ from $0 \to x$:

$\mathrm{Path}_1 =$  , $\quad \mathrm{Path}_2 =$ 

$$\begin{array}{c c} & \begin{array}{cc} \mathrm{Path}_1 & \mathrm{Path}_2 \end{array} \\ \begin{array}{c} \mathrm{Path}_1^\dagger \\ \mathrm{Path}_2^\dagger \end{array} & \begin{pmatrix} 1 & u_P \\ u_P & 1 \end{pmatrix} \end{array} \succeq 0.$$

Then the square positivity condition $\langle W_j | W_i \rangle \succeq 0$ requires

$$1 - u_p^2 \geqslant 0.$$

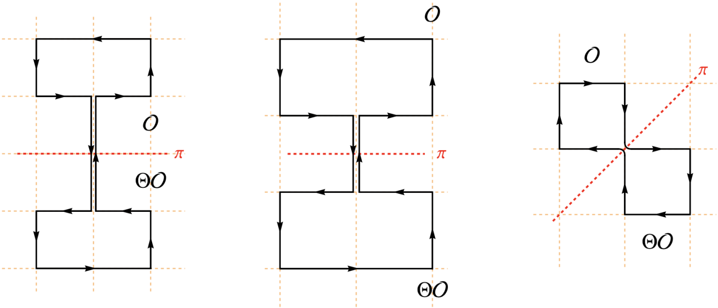
Reflection positivity in LGTs

Choose a direction ρ on the lattice as the "time" direction, the Hilbert space $\mathcal{H}$ is separated into two parts: $\mathcal{H}^+, \rho > \rho_0$ and $\mathcal{H}^-, \rho < \rho_0$.

Reflection map: $\Theta : \mathcal{H}^+ \to \mathcal{H}^-$:

$$\Theta \cdot W[C] = W[R \cdot C]^\dagger.$$

Reflection positivity: $\langle \Theta \cdot \mathcal{O} | \mathcal{O} \rangle \geqslant 0$ (*Osterwalder and Seiler 1978, Kazakov and Zheng, 2022*)



Algorithm for bootstrapping Abelian LGTs

Target: constraints on the averaged Wilson loops: $\langle W[C]\rangle$

Bootstrap computations:

1. Select Wilson lines W_i up to the maximum length $L \leqslant L_{\max}/2$;
2. Construct all the semi-positive matrices $M_a \succeq 0$ from square and reflection positivity

$$M_a^{ij} \equiv \langle W_i | W_j \rangle, \quad M_{a'}^{ij} \equiv \langle \Theta \cdot W_i | W_j \rangle;$$

3. Generate the loop equations for the Wilson loops in M_a;
4. Using the loop equations to reduce the free variables in M_a;
5. Extract the constraints on $\langle W[C]\rangle$ for each fixed gauge coupling λ

$$M_a \succeq 0 \implies x_{\min} \leqslant \langle W[C]\rangle \leqslant x_{\max}$$

Griffiths' inequalities: positivity before modern bootstrap

Positivity/convexity has been extensively studied long before the modern bootstrap endeavor!

Griffiths' inequalities are satisfied by correlators in many lattice theories

- Ferromagnets Ising model (Griffiths, 1967, 1969)
- More spin lattice models (Kelly and Sherman, 1968)
- General lattice theories including the LGTs (Ginibre, 1970)
- Their roles in bootstrap studies (Cho et al., 2022)

Griffiths' inequalities: general form

Consider a compact space X and a convex function space $\mathcal{Q}$ on X:

- For any $p(x), q(x) \in \mathcal{Q}$ and $a, b \geqslant 0$, the functions
$$ap(x) + bq(x), p(x)q(x), p^*(x) \in \mathcal{Q}.$$
- For any finite set of functions $p_i \in \mathcal{Q}$
$$\int d\mu(x)d\mu(y)\prod_{i=1}^{n}(p_i(x) \pm p_i(y)) \geqslant 0.$$

Then the functions $p_i(x) \in \mathcal{Q}$ satisfy: $\int d\mu(x)\prod_i p_i(x) \geqslant 0$.

Griffiths' inequalities: If a lattice action $-S(x) \in \mathcal{Q}$, then

$$I: Z_S = \int d\mu(x)e^{-S(x)} > 0, \langle p(x)\rangle = Z_S^{-1}\int d\mu(x)p(x)e^{-S(x)} \geqslant 0;$$
$$II: \langle p(x)q(x)\rangle - \langle p(x)\rangle\langle q(y)\rangle \geqslant 0.$$

Applications of the Griffiths' inequalities

Consider a finite-size lattice $\Lambda \subset \mathbb{Z}^D$, the action of the $U(1)$ LGT:

$$S_\Lambda = -\frac{2}{\lambda} \sum_{P \in \Lambda} \cos\theta_P, \quad -S \in \mathcal{Q}_{U(1)}.$$

For another lattice $\Lambda' = \Lambda + \bar{\Lambda}$ with action $S_\Lambda - S_{\Lambda'} = \beta \sum_{P \in \bar{\Lambda}} \cos\theta_P \in \mathcal{Q}_{U(1)}$

$$\frac{d}{d\beta}\langle \cos\theta_C \rangle = \frac{1}{Z_{\Lambda'}^2} \sum_{P \in \bar{\Lambda}} \langle \cos\theta_C \cos\theta_P \rangle - \langle \cos\theta_C \rangle \langle \cos\theta_P \rangle \geqslant 0.$$

Prove the thermodynamics limit:

$\langle \cos\theta_C \rangle$ is monotonic and bounded, so it has a unique limit $\Lambda \to \mathbb{Z}^D$.

Bound $\langle W[C] \rangle$ in general D:

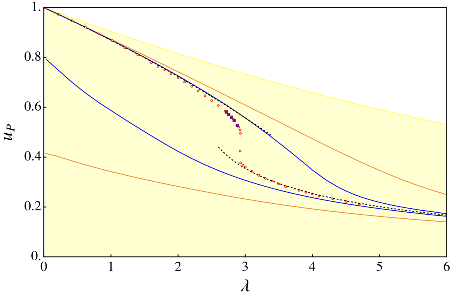
Take $\Lambda = \mathbb{Z}^{D-1}, \Lambda' = \mathbb{Z}^D$, since $S_{\Lambda'}|_{\beta=0} = S_\Lambda$, the monotonicity gives

$$\langle W[C] \rangle_D \geqslant \langle W[C] \rangle_{D-1} \geqslant \ldots \geqslant \langle W[C] \rangle_{D=2}.$$

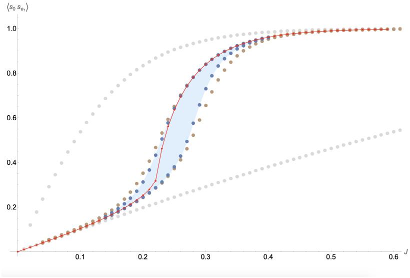
3. Where is the limit of the constraint from loop equations + positivity?

Lattice bootstrap bounds and Wilson line truncation $L_{\max}$

Lattice bootstrap bounds: strong but not enough



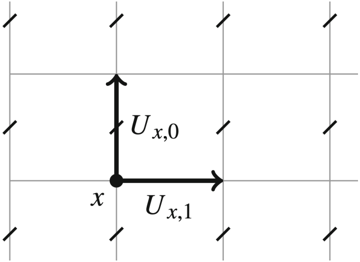
$SU(\infty)$ LGT bootstrap bounds $L_{\max} \leqslant 16$
(*Kazakov and Zheng, 2022*)



3D Ising nearest-neighbor correlator
(*Cho et al., 2022*)

Question: can the bound converge to high precision with larger $L_{\max}$? or, are the loop equations and positivity sufficient to pin down the LGTs?

Explore the limit of bootstrap bound: a 2D example



Take temporal gauge $\theta_0(n) = 0$, $U_{x,0} = 1$, phase of Wilson loop $W[C] = e^{i\theta_C}$ satisfies

$$\theta_C = \sum_{\ell \in C} \theta_\ell = \sum_{\{P_C\}} \theta_P(n).$$

Variable change in the path integral:

$$d\theta_\ell \to d\theta_P(n).$$

The path integral with variable θ_P factorized to incomplete Bessel functions:

$$I_m(2/\lambda) = \int_0^{2\pi} d\theta_P e^{\frac{2}{\lambda}\cos\theta_P} \cos m\theta_P$$

and the Wilson loop averages are

$$w_m \equiv W[P^m] = I_m/I_0.$$

Bootstrap setup of the 2D $U(1)$ LGT

Take the Wilson lines $W_i = P^m, m = 1, 2, ..., n$, its positivity matrix is

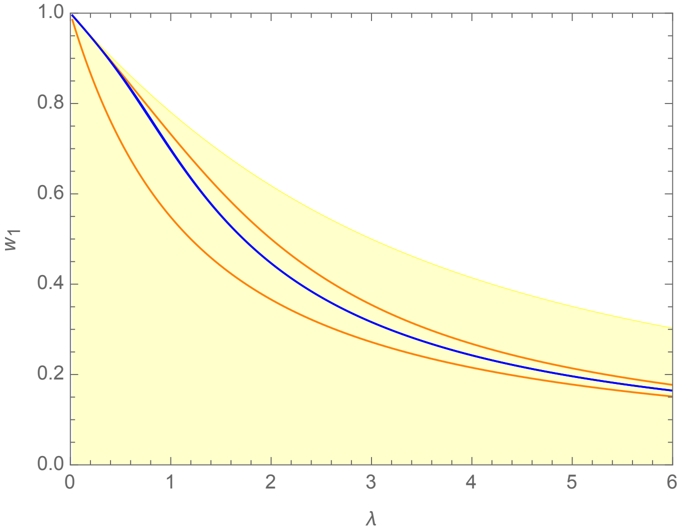
$$\mathcal{M}_{2D} = \begin{pmatrix} w_0 & w_1 & w_2 & \cdots & w_n \\ w_1 & w_0 & w_1 & \cdots & w_{n-1} \\ w_2 & w_1 & w_0 & \cdots & w_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_n & w_{n-1} & w_{n-2} & \cdots & w_0 \end{pmatrix} \succeq 0,$$

associated with the loop equations and boundary condition

$$w_m - w_{m-2} + (m-1)\lambda w_{m-1} = 0, \quad w_0 = 1.$$

The averaged Wilson loops w_m can be restricted to extremely high precision!

Bootstrap bounds on the 2D $U(1)$ LGT



Yellow region: $L_{\max} = 8$; orange lines: $L_{\max} = 12$; blue line(s): $L_{\max} = 24$.

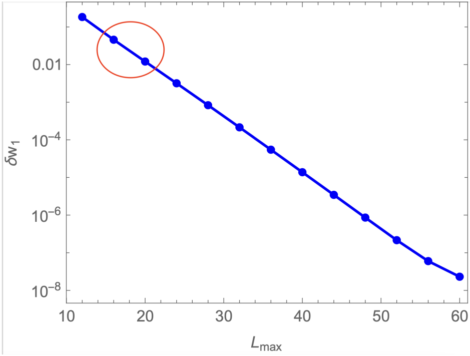
Bootstrap bounds with $L_{\max} = 60$ on the 2D $U(1)$ LGT

The first 10 digits of the bootstrap bounds on w_1 in 2D $U(1)$

λ	Lower bound	Upper bound	Exact value
0.01	0.9974968514	0.9974968595	0.9974968592
0.1	0.9746705040	0.9746705172	0.9746705078
0.2	0.9485998188	0.9485998342	0.9485998259
0.4	0.8933831327	0.8933831412	0.8933831370
0.6	0.8319000705	0.8319000717	0.8319000711
0.8	0.7649967470	0.7649967481	0.7649967475
1.0	0.6977746575	0.6977746583	0.6977746579
2.0	0.4463899656	0.4463899661	0.4463899659
4.0	0.2424996125	0.2424996126	0.2424996125
6.0	0.1643939155	0.1643939155	0.1643939155

What we learn from the 2D $U(1)$ LGT bootstrap

- **Where is the limit of the lattice bootstrap?** For $L_{\max} \simeq 60$, the lattice bootstrap can numerically pin down the 2D $U(1)$ LGT!
- **"Scaling law"** in lattice bootstrap: The precision of the bootstrap bounds $\ln \delta w_1$ is improved linearly with increasing $L_{\max}$.



4. Bootstrapping the 3D $U(1)$ and $\mathbb{Z}_2$ lattice gauge theories

A short review of the 3D $U(1)$ LGT

3D $U(1)$ **LGT:** one of the simplest examples for **gauge confinement**!

- On the lattice, the compact gauge group $U(1)$ can generate $U(1)$ monopoles;
- In the long distance limit, the monopole gas can screen the electric interaction, lead to the **gauge confinement**! (*Polyakov, 1975, 77*);
- One of the very few examples for which the confinement can be understood analytically;
- An important target for quantum simulation (e.g., *Zohar et al., 2012, Paulson et al., 2021*).

What we know about 3D $U(1)$ LGT

- The theory is confined in the long distance for general λ;
- Plaquette average in the weak coupling limit (Horsley and Wolff, 1981)

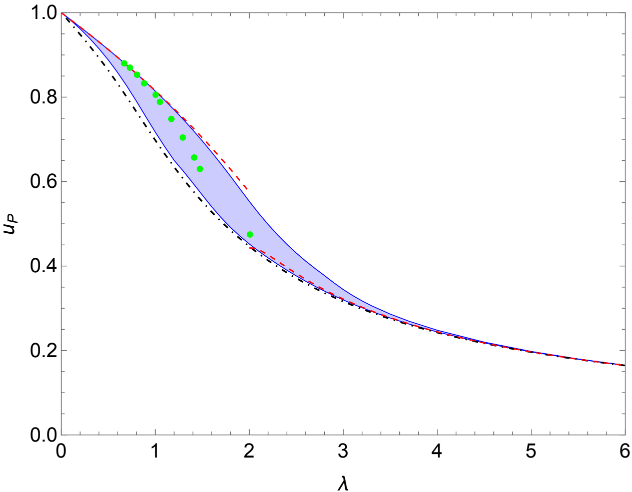
$$u_P = 1 - \frac{1}{6}\lambda - \frac{1}{72}\lambda^2 - 0.0041375\lambda^3 - 0.000175\lambda^4 + \cdots .$$

- Plaquette average in the strong coupling limit (Balian et al., 1975)

$$u_P = \frac{1}{\lambda} - \frac{1}{2\lambda^3} + \frac{7}{3\lambda^5} - \frac{395}{48\lambda^7} + \frac{1173}{40\lambda^9} - \frac{507803}{4320\lambda^{11}} + \frac{7352027}{15120\lambda^{13}} - \frac{443004913}{215040\lambda^{15}} + \cdots .$$

- Monte Carlo simulation with intermediate λ (Loan, et al., 2002);
- Effective string and flux tube description of the confinement (Caselle, et al., 2014, 2016);
- Monte Carlo simulation for the glueball mass spectrum and string tension (Athenodorou and Teper, 2019).

Bootstrap bounds with $L_{\max} = 16$ on the 3D $U(1)$ LGT



Black dot-dashed line: 2D $U(1)$ lattice gauge theory w_1; red dashed lines: strong and weak coupling expansions; green dots: Monte Carlo results.

Summary of the bootstrap bounds

- **Strong/weak couplings:** the two-sided bootstrap bounds converge quickly and consistent with the perturbative results;
- **Intermediate coupling:** the two-sided bootstrap bounds well agree with the Monte Carlo results, but precision is weaker;
- **Universal behavior in confinement:** with large λ, both upper and lower bounds quickly converge to the 2D solution!
  Bootstrap constraints $\Longleftrightarrow$ Griffiths' inequalities

A short review of the 3D $\mathbb{Z}_2$ LGT

- Dual to the 3D Ising model, 3D version of the Kramers-Wannier duality, provides phase transition with topological order;
- Plaquette average in the strong coupling limit (Balian et al., 1975):

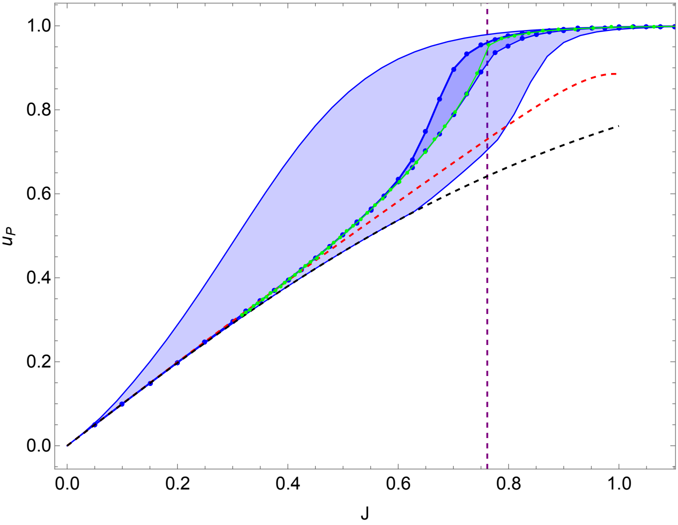
$$u_P = J - \frac{J^3}{8} + \frac{7J^5}{48} - \frac{395J^7}{3072} + \frac{1173J^9}{10240} - \frac{507803J^{11}}{4423680} + \frac{7352027J^{13}}{61931520} - \frac{443004913J^{15}}{3523215360} + \cdots$$

- Monte Carlo results on 3D Ising model can be transfered to the 3D $\mathbb{Z}_2$ LGT through transformation

$$F_{\text{Ising}}(J') = F_{\mathbb{Z}_2\ \text{gauge}}(J) - \frac{3}{2}\ln\sinh(2J) + \frac{1}{2}\ln 2.$$

- Second order phase transition at $J = 0.761413292(11)$ (Ferrenberg et al., 2018)
- Important applications in quantum computing, higher form symmetry, etc.

Bootstrap bounds for the 3D $\mathbb{Z}_2$ LGT with $L_{\max} = 16$



Light/dark blue region: $L_{\max} = 12/16$; green dots: MC data; red dashed line: strong coupling expansion; black dashed line: solution of 2D $\mathbb{Z}_2$ LGT.

Summary of the bootstrap bounds

- **Strong/weak couplings:** the two-sided bootstrap bounds converge quickly and consistent with the perturbative/Monte Carlo results;
- **Similar interesting observation:** with large λ, both upper and lower bounds quickly converge to the 2D solution! Why?
- **Numerical precision**: Even with $L_{\max} = 16$, the numerical precision is notably better than the 3D Ising model bootstrap bounds by the Harvard group (Cho et al., 2022)!

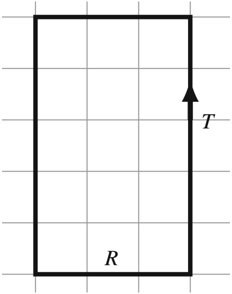
5. Towards bootstrapping the string tension and glueball mass

Confinement: string tension

Confinement: key problem for QCD. Can bootstrap do more?

String tension σ: large Wilson loop and area law

$$\langle W[C]\rangle \equiv \langle W[R,T]\rangle \propto e^{-\sigma RT-\gamma(R+T)-a}.$$



Potential between static quark pair and string tension:

$$V(R) = -\lim_{T\to\infty}\frac{1}{T}\log\langle W[R,T]\rangle,\ \ \sigma = \lim_{R\to\infty}\frac{V(R)}{R}.$$

Static quark pair with distance R:

Confinement:	$\langle W[R,T]\rangle \sim e^{-TV(R)} \sim e^{-\sigma\times\text{area}}$
Deconfinement:	$\langle W[R,T]\rangle \sim e^{-TV(R)} \sim e^{-\gamma\times\text{perimeter}}$

Bootstrap estimation of the string tension σ

Bootstrap: estimate σ with Wilson loops at finite lengths.
Creutz ratio: cancel the linear and constant terms in $V(R)$:

$$\chi(I,J) = -\ln\left(\frac{W[I,J]\ W[I-1,J-1]}{W[I,J-1]\ W[I-1,J]}\right).$$

Strict bounds:

λ	$W[3,2]$	$W[3,1]$	$W[2,2]$	$W[2,1]$	χ_{ext}	MC
0.6	0.57(16)	0.71(8)	0.66(10)	0.79(6)	0.04	N.A.
0.8	0.44(26)	0.60(14)	0.53(17)	0.70(11)	0.05	0.010
1.0	0.34(34)	0.49(20)	0.42(23)	0.61(16)	0.06	0.050

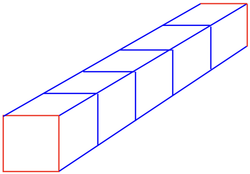
Glueball mass spectrum

The millennium problem: Mass gap in Yang-Mills theories

Can bootstrap, or positivity play a substantial role for mass gap problem?

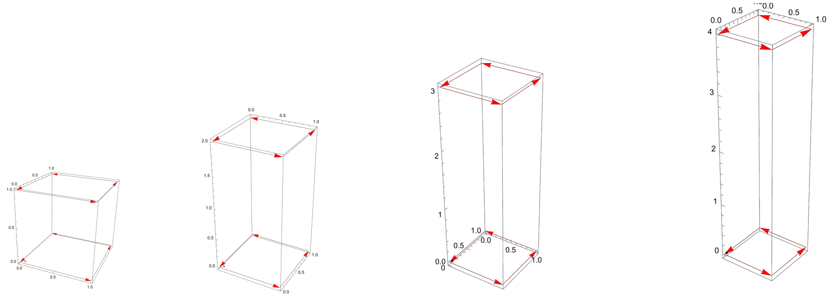
Consider two parallel Wilson loops with distance R, the correlation function is dominated by the lowest exciting states:

$$\langle W[P]W[P']\rangle_{\text{connect}} \simeq \sum_i e^{-m_i R}$$



Primary bootstrap results

Parallel plaquettes with distance $T = 1, 2, 3, 4$:



	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$
$\langle P(\tau)P(0)\rangle$	0.56(12)	0.58(14)	0.59(15)	0.61(22)
$\langle P(\tau)^* P(0)\rangle$	0.64(15)	0.60(15)	0.59(15)	0.63(22)

Comments: Nontrivial constraints but need one order higher precision to evaluate glueball mass.

Summary

1. *Can the loop equations and positivity be enough to pin down a LGT?*
   Yes for 2D $U(1)$ LGT! With $L_{\max} = 60$, the precision $\sim 10^{-8}$!
2. Strong two-sided bootstrap bounds on 3D $U(1)$ and $\mathbb{Z}_2$ LGT
3. Modern bootstrap constraints $\Longleftrightarrow$ Classical Griffiths' inequalities!
   Generalize to the Yang-Mills theories?
4. Bootstrapping the string tension and glueball mass spectrum.
   Needs higher precision. To be continue...

Can we solve the *gauge confinement problem* in QCD using **bootstrap**?
We will see...

Thank you!