Dualities Through Bethe/Gauge Correspondence

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outline

Yang-Mills

- The YM action $S_{YM} = -\frac{1}{2g_0^2}\int d^4x F^{\mu\nu}F_{\mu\nu}$ (1954)
- (Anti-)Self-dual YM: $*F = \pm F$
- (A)SDYS: Instanton, Non-perturbative
- A. Polyakov (1975)
- G. 't Hooft (1976)
- M. Atiyah and R. Ward (1977): 4d self-dual YM, holomorphic condition
- Atiyah-Drinfeld-Hitchin-Manin: (ADHM Construction, 1978),
- S. Donaldson: 4d Geometry, Donaldson invariant (1983)
- Instanton No.: Topological invariant integer.

Super Extension of YM

- 4d $\mathcal{N} = 2$ YM
- Seiberg-Witten (1994),
- SW Simplifies the Donaldson Theory
- Low Energy Effective Theory
- N. Nekrasov: Nekrasov Partition Function (2002)
- IR and UV finite
- NPF Simplifies SW, the proof SW of Donaldson theory
- N. Nekrasov and A. Okounkov (2003): SW and Random Partitions
- Beth/Gauge Correspondence (Nekrasov 2008).

Correspondence

- The gauge theory dynamics in the vacuum sector $\rightarrow$ quantum many-body systems.
- The 2d pure $U(N)$ Yang-Mills theory the system $\rightarrow$ N free non-relativistic fermions on a circle
- The nilpotent supercharges $\mathcal{Q}$, $\mathcal{Q}^{\dagger}$-commutators form a (super)commutative chiral ring.
- Hamiltonian the spin chain $H \sim \mathcal{Q}\mathcal{Q}^{\dagger} + \mathcal{Q}^{\dagger}\mathcal{Q}$
- The supersymmetric vacua of the theory form a representation of the ring, and identified with the space of states of a quantum integrable system

The geometric Representation Theory for A-type

- 2d $\mathcal{N} = 2$ theory with $U(N)$ gauge group and L fundamental hypermultiplet with twisted masses.
- In certain limit, the theory reduces to the supersymmetric σ-model on the noncompact hyperkahler manifold, that of the cotangent bundle to the Grassmannian $Gr(N, L)$ of the N-dimensional complex planes in C^L.
- The equivariant quantum cohomology algebra of $T^*Gr(N,\ L)$ maps to the algebra of quantum integrals of motion of the $XXX_{1/2}$ spin chain.

Specific Correspondence

- The effective twisted superpotential W_{eff} of a gauge theory corresponds to the Yang-Yang function of a quantum integrable system.
- 3d $\mathcal{N} = 2$ Gauge theory $\rightarrow$ A-type closed XXZ (Nekrasov: *Prog. Theor. Phys. Suppl.* **177** (2009) 105)
- 3d $\mathcal{N} = 2$ Gauge theory $\rightarrow$ BD-type open XXZ (T. Kimura and R.D. Zhu, *JHEP* **03** (2021) 227).
- 2d $\mathcal{N} = 2$ Gauge theory $\rightarrow$ BCD-type open XXX (T. Kimura and R.D. Zhu, 2021)

Why not for C_n gauge theory?

- The relationship was known as Bethe/Gauge correspondence, which opens a new door for the relationship with quantum integrable system and the gauge theories.
- Why Bethe/Gauge not for 3d C_n ?
- From Lie theory, ADE series are self-dual.
- B_n and C_n Langlands dual to each other, but they belong to different branches.
- The Bethe equation for a given boundary parameter matches only one branch of the squared root of the vacuum equation.

Spin chains

- Hamiltonian

$$\mathcal{H} = \sum_{n=1}^{L} \left(J_x \sigma_x^{(n)} \sigma_x^{(n+1)} + J_y \sigma_y^{(n)} \sigma_y^{(n+1)} + J_z \sigma_z^{(n)} \sigma_z^{(n+1)} \right)$$

- Quantum Ingebrable Model
- XXX: if $J_x = J_y = J_z$, Yangian algebra $Y(g)$
- XXZ: if $J_x = J_y \neq J_z$, Quantum Affine Algebra $U_q(\hat{g})$
- XYZ: if $J_x \neq J_y \neq J_z$, Elliptic Quantum Group $U_{q,t}(g)$.

Brief recall

- 1931, H. Bethe' constructed N spin-1/2 Heisenberg chain wave functions.
- Calculating the spectrum of the Hamiltonian to solving a set of N coupled algebraic equations, the Bethe Ansatz Equations, or BAE.
- 1944, L. Onsager's, Onsager Algebra, Star-Triangle Relation
- 1969, C. N. Yang and C. P. Yang, Thermodynamics Bethe Ansatz (TBA), Yang-Yang Funct.
- 1980, L. D. Faddeev et al, Algebraic Bethe Ansatz (ABA), Yang-Baxter Eq.

R-matrix

The integrability of a spin chain is characterized by an R-matrix, $R(u) : V \otimes V \to V \otimes V$, satisfying the Yang-Baxter equation,

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)$$

where u, v are called the spectral parameters, R_{ij} are linear operators in the tensor product of the three linear space $V \otimes V \otimes V$ with $R_{12} = R(u) \otimes 1$, $R_{23} = 1 \otimes R(u)$, etc.

XXZ or 6-vertex Model

The most general R-matrix for a solvable XXZ spin chain model can be expressed [Baxter: 1982]

$$R^{\mathrm{XXZ}}(u) = \begin{pmatrix} [u+\eta] & 0 & 0 & 0 \\ 0 & [u] & [\eta] & 0 \\ 0 & [\eta] & [u] & 0 \\ 0 & 0 & 0 & [u+\eta] \end{pmatrix}$$

$[x] := \dfrac{\sin(\pi x)}{\sin(\pi\eta)}$, η is the crossing parameter. For R: $R(0) = P$, P is the permutation operator that acts as $P(x \otimes y) = y \otimes x$, $\forall x, y \in V$.

Closed Spin Chain

For a closed spin chain with periodic boundary condition, the monodromy matrix is

$$T_0(u) = R_{0L}(u-\vartheta_L)\cdots R_{01}(u-\vartheta_1) := \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$T(u) \in \mathrm{End}(V^{(0)} \otimes V^{\otimes L})$; $V^{(0)}$: Auxiliary space
$A(u), B(u), C(u), D(u) \in \mathrm{End}(V^{\otimes L})$; ϑ_j: inhomog. parameters.
The monodromy matrix satisfies the RTT-relation,

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

$$t(u) = \mathrm{tr}_0\, T_0(u) = A(u) + D(u)$$

Hamiltonian $H = \frac{\partial}{\partial u} log\, t(u) + const.$

Twisted periodic boundary condition

- Twisted(Gauge) Operator

$$\iota(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

- Twisted periodic boundary condition

$$\sigma_{x,y,z}^{L+1} = e^{\frac{i}{2}\theta\sigma_z}\sigma_{x,y,z}^{1}e^{-\frac{i}{2}\theta\sigma_z}$$

- The transfer matrix

$$t(u;\theta) := \mathrm{tr}_0 \iota_0(\theta) T_0(u) = A(u) + e^{i\theta} D(u)$$

The Bethe Ansatz State

- The ground state $\Omega = |\uparrow, \cdots, \uparrow\rangle$

$$A(u)\Omega = \delta_+(u)\Omega, \quad D(u)\Omega = \delta_-(u)\Omega, \quad C(u)\Omega = 0$$

$$\delta_+(u) = \prod_{a=1}^{L}[u + \frac{\eta}{2} + \eta s_a - \vartheta_a], \quad \delta_-(u) = \prod_{a=1}^{L}[u + \frac{\eta}{2} - \eta s_a - \vartheta_a]$$

- The Bethe ansatz state

$$\prod_{i=1}^{M} B(u_i)\Omega$$

The BAE

- The Bethe Ansatz Equation with twisted periodic boundary condition

$$\prod_{a=1}^{L} \frac{[u_i + \frac{\eta}{2} + \eta s_a - \vartheta_a]}{[u_i + \frac{\eta}{2} - \eta s_a - \vartheta_a]} = e^{i\theta} \prod_{j\neq i, j=1}^{M} \frac{[u_i - u_j + \eta]}{[u_i - u_j - \eta]}$$

- s_a: spin at the a-th site

Boundary Matrix

- Diagonal Boundary Matrix

$$K^{\mathrm{XXZ}}(u,\xi) = \begin{pmatrix} [u+\xi] & 0 \\ 0 & -[u-\xi] \end{pmatrix}$$

- The transfer matrix (Y. Wang, W. Yang et al, Springer 2015)

$$t(u) = \mathrm{Tr}_0 K_+(u) T_0(u) K_-(u) T_0^{-1}(-u)$$

- The Reflection Equation:

$$R_{12}(u-v)K_-^1(u)R_{21}(u+v)K_-^2(v)$$
$$= K_-^2(v)R_{21}(u+v)K_-^1(u)R_{12}(u-v)$$

- $K_-^1(u) := K_-(u) \otimes id_{V_2}, \cdots$

Open Chain Transfer Matrix

- Monodromy matrix

$$U_-(u) = T(u)K(u-\frac{\eta}{2},\xi_-)\sigma_y T^t(-u)\sigma_y := \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}$$

- Reflect. Eq.

$$\begin{aligned} &R_{12}(u-v)U_-^1(u)R_{12}(u+v-\eta)U_-^2(v) \\ &= U_-^2(v)R_{12}(u+v-\eta)U_-^1(u)R_{12}(u-v) \end{aligned}$$

- Transfer Matrix

$$\begin{aligned} t(u) &= \mathrm{Tr}\left(K_+(u+\eta,\xi_+)U_-(u)\right) \\ &= \frac{[2u+\eta][u-\frac{\eta}{2}+\xi_+]}{[2u]}\mathcal{A}(u) - \frac{[u+\frac{\eta}{2}-\xi_+]}{[2u]}\tilde{\mathcal{D}}(u) \end{aligned}$$

Open BAE

- $$K_+(u,\xi_+) = K(u+\frac{\eta}{2},\xi_+) := \begin{pmatrix} [u+\frac{\eta}{2}+\xi_+] & 0 \\ 0 & -[u+\frac{\eta}{2}-\xi_+] \end{pmatrix}$$
- If $\prod_{i=1}^{M} \mathcal{B}(u_i)\Omega$ are eigenstates, then

$$\frac{\sin[\pi(u_i-\frac{\eta}{2}+\xi_+)]}{\sin[\pi(u_i+\frac{\eta}{2}-\xi_+)]}\frac{\sin[\pi(u_i-\frac{\eta}{2}+\xi_-)]}{\sin[\pi(u_i+\frac{\eta}{2}-\xi_-)]}$$

$$\times\prod_{a=1}^{L}\frac{\sin[\pi(u_i+\frac{\eta}{2}+\eta s_a-\vartheta_a)]\sin[\pi(-u_i+\frac{\eta}{2}-\eta s_a-\vartheta_a)]}{\sin[\pi(-u_i+\frac{\eta}{2}+\eta s_a-\vartheta_a)]\sin[\pi(u_i+\frac{\eta}{2}-\eta s_a-\vartheta_a)]}$$

$$\times\prod_{j\neq i,j=1}^{M}\frac{\sin[\pi(u_j+u_i-\eta)]\sin[\pi(u_j-u_i-\eta])}{\sin[\pi(u_j-u_i+\eta)]\sin[\pi(u_j+u_i+\eta)]}=1$$

Instantons

- Curvature: $F_A = dA + A \wedge A$
- Topological instanton number: $k = \frac{1}{8\pi^2} \int F_A \wedge F_A$.
- Yang-Mills action: $S_{YM} = -\frac{1}{2{g_0}^2} \int F_A \wedge {}^* F_A$
- Instantons on R^4 are solutions of the self-dual instanton equation: $F_A^+ = 0$.
- Atiyah-Drinfeld-Hitchin-Manin construction, ADHM

$\mathcal{N} = 4$ Super Yang-Mills

- a $\mathcal{N} = 4$ gauge theory is characterized purely a gauge group G, and a complexified gauge coupling τ.
- τ be invariant under $SL(2, Z)$.
- T^2 with complex structure parameter τ
- Modular properties of instanton partition function for gauge group G is a character for the affine Lie algebra $\hat{g}$
- This character also appears as the partition function of a 2d CFT with Kac-Moody algebras on the T^2.

$\mathcal{N} = 2$ Super Yang-Mills

- $\mathcal{N} = 2$ gauge theories are much richer than their $\mathcal{N} = 4$ counterparts.
- Different kinds of matter multiplets be added.
- Even for conformal theories the gauge coupling receives a finite renormalization
- In the low energy limit, the gauge theory is characterized by a Seiberg-Witten curve.
- SW geometry captures the prepotential of the gauge theory as well as the masses of BPS particles.

$\mathcal{N} = 2$ Super Yang-Mills

- $\mathcal{N} = 2$ gauge theories duality to 2d CFT: the Alday-Gaiotto-Tachikawa (AGT) correspondence
- U(2) instanton partition functions and Virasoro conformal blocks.
- $F^{+}_{A,\mu\nu} + \frac{i}{2}\bar{q}_\alpha \Gamma_{\mu\nu}{}^{\alpha}{}_{\beta} q^{\beta} = 0$
- Γ^μ are the Clifford matrices.
- $q = exp 2 i\pi\tau$.

Instanton compactness

- Instantons can become arbitrary small, or move away to infinity in R^4
- UV and IR non-compactness.
- Ω-background: $T^2_{\epsilon_1,\epsilon_2} = U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$
- $R^4 = C \oplus C$ a rotation $(z_1, z_2) \rightarrow (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$
- If we localize the instanton partition function equivariantly with respect to the T^2, only instantons at the fixed origin will contribute, so that we can ignore the instantons that run off to infinity.
- The UV non-compactness can be cured for gauge group $U(N)$ by turning on a Fayet-Illiopoulos (FI) parameter.

Moment map

- The Instantons moduli space $\mathcal{M}_k^G$ can be regraded as the quotient of the solutions of the ADHM equations:
- A moment map for G is a smooth map μ: $M \to g^\star$.
- Three real moment maps:
  $\mu_R = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - JJ^\dagger$.
  $\mu_C = [B_1, B_2] + IJ$.
- The dual group: $M_k^{U(N)} = \vec{\mu}^{-1}(0)/U(k)$.

Hyperkahler quotient

- Dual group G_k^D with Cartan torus $T^k_{\Phi_i}$ whose weights we will call Φ_i.
- Natural action of the Cartan torus $T^N_{\mathbf{a}}$ of the framing group G on the ADHM solution space with weights are given by the Coulomb branch parameters $\mathbf{a}$.
- Action of the Cartan $T^{N_f}_{\mathbf{m}}$ of the flavor symmetry group acting on M, whose weights correspond to the masses $\mathbf{m}$ of the hypers.
- The instanton partition function equivariantly with respect to the torus: $T = T^2_{\epsilon_1,\epsilon_2} \times T^k_{\Phi_i} \times T^N_{\mathbf{a}} \times T^{N_f}_{\mathbf{m}}$.

Torus action

- Localization: Duistermaat-Heckman formula as a particular case of Atiyah-Bott.
- $Z^{inst}k = \int \Pi_i d\phi_i z_k^{gauge}(\phi_i, \mathbf{m}, \mathbf{a}, \epsilon_1, \epsilon_2) z_k^{matter}(\phi_i, \mathbf{m}, \mathbf{a}, \epsilon_1, \epsilon_2)$.
- For $U(N)$ theory the poles are labeled by N Young diagrams with in total k boxes.
- $U(N)$ instanton splits into N non-commutative $U(1)$ instantons.
- Nekrasov partition function: $Z^{Nek} = Z^{clas} Z^{1-loop} Z^{inst}$

Effective Potential

In terms of the general Ω-backgrounds, set to zero only one equivariant parameter, $\epsilon_2 = 0$, while keeping the parameter $\epsilon_1 = \epsilon$ finite. The effective twisted superpotential

$$W^{\text{eff}}(a, \mathbf{m}; q, \epsilon) \sim \epsilon \sum \text{Li}_2\left(e^{\frac{\text{linear}(a,\mathbf{m})}{\epsilon}}\right) - \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z(a, \mathbf{m}; q; \epsilon_1 = \epsilon, \epsilon_2)$$

- q: The set of gauge coupling constants of the theory.
- $\mathbf{m}$: The set of masses of the hypermultiplets fields.

Effective Low-energy Theory

- a: The set of the flat special coordinates on the moduli space of vacua on the Coulomb branch of the theory.
- The latter is identified with the asymptotics of the scalar fields of $N = 2$ vector multiplets in Euclidean space-time.
- The $\mathrm{Li}_2(z)$ is called the dilogarithm,

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

NS Rep. *Prog. Theor. Phys. Suppl.* **177** (2009) 105

For 2d $\mathcal{N} = (2,\ 2)$ gauge theory, $G = U(N)$, if we considered the representation

$$\mathcal{R} = V \otimes V^* \otimes \mathcal{L} \oplus V \otimes \mathcal{F} \oplus V^* \otimes \tilde{\mathcal{F}}$$

- The global symmetry group: $H^{\max} = U(L) \times U(L) \times U(1)$.
- $V = \mathbf{C}^N$ is the N-dimensional fundamental representation
- $\mathcal{F} \approx \mathbf{C}^L$, $\tilde{\mathcal{F}} \approx \mathbf{C}^L$ are the L-dimensional fundamental representations of the first and second $U(L)$ factors in the flavour group.
- $\mathcal{L}$ is the standard one-dimensional representation of the global group $U(1)$.

$U(N)$ Effective Potential

- The root system be the set of all vectors $\alpha \in \mathbf{E}$, $(\alpha, \alpha) = 2$.
- The root set consists of all $\{\sigma_i - \sigma_j\}$, $i \neq j$.
- The effective twisted superpotential is

$$
\begin{aligned}
W_{\text{eff}}^{2d}(\sigma) = \sum_{j=1}^{N} \sum_{a=1}^{L} \Big[& (\sigma_j + m_a^f)(\log(\sigma_j + m_a^f) - 1) \\
& + (-\sigma_i + m_a^{\bar{f}})(\log(-\sigma_j + m_a^{\bar{f}}) - 1) \\
& + \sum_{j,k=1}^{N} (\sigma_j - \sigma_k + m_{\text{adj}})(\log(\sigma_j - \sigma_k + m_{\text{adj}}) - 1) \\
& - 2\pi i \sum_{j=1}^{N} (t + j - \frac{1}{2}(N+1))\sigma_j
\end{aligned}
$$

Vacuum Equation

- σ_j are the eigenvalues of the complex scalar in the vector multiplet.
- m is the twisted mass parameters, t is a complex coupling number.
- The last term is the Fayet-Illiopoulos term.
- The vacuum equations for A-type gauge theory:

$$\exp\left(\frac{\partial W_{\text{eff}}(\sigma)}{\partial \sigma_i}\right) = 1$$

The vacuum equations of A-type 2d gauge theory are

$$\prod_{a=1}^{L} \frac{\sigma_j + m_a^f}{\sigma_j - m_a^{\bar{f}}} = -e^{2\pi i t} \prod_{k=1}^{N} \frac{\sigma_j - \sigma_k - m_{\text{adj}}}{\sigma_j - \sigma_k + m_{\text{adj}}}, \qquad j = 1, \cdots, N$$

3d Gauge Theor. Part. Funct. (Yoshida et al : *PTEP* **2020**,113B02)

The geometry of $D^2 \times S^1$ is parameterized as

$$ds^2 = l^2(d\theta^2 + r^2 \sin^2\theta d\phi^2) + d\tau^2,$$

The S^1 circle has a periodicity βl. The index on $D^2 \times S^1$ is given by the following integral,

$$\mathcal{I} = \frac{1}{|W_G|} \int \frac{d^N}{(2\pi)^N} e^{-S_{cl}} Z_{\text{vec}} Z_{\text{chi}} Z_{\text{bd}},$$

W_G is the Weyl group of G.

The vector multiplet

- The vector multiplet

$$Z_{\text{vec}} = \prod_{\alpha\in\Delta} e^{\frac{1}{8\beta_2}(\alpha\cdot\sigma)^2}\left(e^{i\alpha\cdot\sigma};q^2\right)_\infty$$

- Δ: The set of the roots of G
- The One-loop determinant of the chiral multiplet with Neumann boundary condition

$$Z_{\text{chi}}^{\text{Neu}} = \prod_{w\in\mathcal{R}} e^{\mathcal{E}(iw\cdot\sigma+r\beta_2+im)}\left(e^{-iw\cdot\sigma-im}q^r;q^2\right)_\infty^{-1}, \qquad q=e^{-\beta_2}$$

- $\mathcal{R}$: The weight's set of the corresponding representation.

Partition Function

- r : The R-charge of the scalar in the chiral multiplet.
- $$\mathcal{E}(x) = \frac{1}{8\beta_2}x^2 - \frac{1}{4}x + \frac{\beta_2}{12}$$

  β_1:the fugacity of the rotation along S^1; β_2: the $U(1)$ charge fugacity; $\beta l = (\beta_1 + \beta_2)l$: the circumference of S^1.
- The one-loop contribution of chiral multiplet with Dirichlet boundary condition is

$$Z^{\text{Dir}}_{\text{chi}} = \prod_{w \in \mathcal{R}} e^{\mathcal{E}(-iw\cdot\sigma + (2-r)\beta_2 - im)} (e^{iw\cdot\sigma + im} q^{2-r}; q^2)_\infty, \qquad q = e^{-\beta_2}$$

Duality

- Only one-loop determinants Z_{vec} and Z_{chi} in $\mathcal{I}$ are needed to get W^{3d}_{eff}, with the rescaling $\beta_2 \to \beta_2\epsilon/2$ and $r \to 1 + \frac{2i}{\epsilon}\tilde{c}$.
- The effective potential is obtained by taking limit $\epsilon \to 0$
- $$\mathcal{I} \sim \exp\left(\frac{1}{\epsilon} W_{\text{eff}}(\sigma, m)\right)$$
- $$\exp\left(\beta_2 i \frac{\partial}{\partial \sigma} W^{3d}_{\text{eff}}(\sigma, m)\right) = 1$$

Adjoint Representaion (Fulton and Harris, GTM129, 2004)

- The adjoint representation of $SO(M)$ is isomorphic to the wedge product $\bigwedge^2 V$.
- In the case $M = 2N$, since the weights of V are $\pm e_I$, it follows that the roots of $SO(2N)$ are just the pairwise distinct sum $\pm e_i \pm e_j$.
- In the odd case $M = 2N + 1$, the weights of the standard representation V are $\{\pm e_i\} \cup \{0\}$ and the weights of the adjoint representation are $\{\pm e_i \pm e_j\} \cup \{\pm e_i\}$.

Adjoint Rep. -2

- To make a comparison with the Lie algebra $Sp(2N)$, we can say that the root diagram of $SO(2N)$ looks like that of $Sp(2N)$ with the roots $\pm 2e_i$ removed.
- The root diagram of $SO(2N+1)$ looks like that of $Sp(2N)$ with the roots $\pm 2e_i$ replaced by $\pm e_i$.

Refined Rep. (I) (D. and Zhang, JHEP04(2023)036)

- For 3d $\mathcal{N}=2$ $U(N)$ gauge theory, we choose the representation

$$\mathcal{R}^{ref} = V \otimes V^* \oplus V \otimes \mathcal{F} \oplus V \otimes \mathcal{F} \oplus V^* \otimes \tilde{\mathcal{F}} \oplus V^* \otimes \tilde{\mathcal{F}}$$

- Global symmetry group $H^{\max} = U(L) \times U(L) \times U(L) \times U(L)$.
- $V = \mathbf{C}^N$: N-dimensional fund. rep.,
- V^*: The dual of V.
- $\mathcal{F} \approx \mathbf{C}^L$, $\tilde{\mathcal{F}} \approx \mathbf{C}^L$: L-dim. fund. rep.

The $U(N)$ effective superpotential

$$
\begin{aligned}
W_{\text{eff}}^{3d} =& \frac{2}{\beta_2}\sum_{j\neq k}^{N}\mathrm{Li}_2(e^{-i(\sigma_j-\sigma_k)-im_{\text{adj}}}) - \frac{1}{2\beta_2}\sum_{j\neq k}^{N}(\sigma_j-\sigma_k+m_{\text{adj}})^2 \\
&+ \frac{2}{\beta_2}\sum_{j=1}^{N}\sum_{a=1}^{N_f}\mathrm{Li}_2(e^{-i\sigma_j-im_a}) - \frac{1}{2\beta_2}\sum_{j=1}^{N}\sum_{a=1}^{N_f}(\sigma_j+m_a)^2 \\
&+ \frac{2}{\beta_2}\sum_{j=1}^{N}\sum_{a=1}^{N_f'}\mathrm{Li}_2(e^{i\sigma_j-im_a'}) - \frac{1}{2\beta_2}\sum_{j=1}^{N}\sum_{a=1}^{N_f'}(\sigma_j-m_a')^2 \\
&- \frac{2}{\beta_2}\sum_{j\neq k}^{N}\mathrm{Li}_2(e^{i(\sigma_j-\sigma_k)}) + \frac{1}{2\beta_2}\sum_{j\neq k}^{N}(\sigma_j-\sigma_k)^2
\end{aligned}
$$

Then the vacuum equation

- Vacuum Equation of gauge theory

$$\frac{\prod_{a=1}^{N_f} \sin^2(\sigma_j + m_a)}{\prod_{a=1}^{N_f'} \sin^2(\sigma_j - m_a')} = \prod_{k\neq j}^{N} \frac{\sin^2(\sigma_j - \sigma_k + m_{\text{adj}})}{\sin^2(\sigma_j - \sigma_k - m_{\text{adj}})}$$

- if $N_f = N_f'$, then

$$\frac{\prod_{a=1}^{N_f} \sin(\sigma_j + m_a)}{\prod_{a=1}^{N_f'} \sin(\sigma_j - m_a')} = \pm \prod_{k\neq j}^{N} \frac{\sin(\sigma_j - \sigma_k + m_{\text{adj}})}{\sin(\sigma_j - \sigma_k - m_{\text{adj}})}$$

Duality Condition

The dictionary to Spin Chain:

- $\pi u \longleftrightarrow \sigma, \quad \pi\eta \longleftrightarrow m_{\text{adj}}.$
- $M \longleftrightarrow N, \quad L \longleftrightarrow N_f = N_f'.$
- $\{-\pi\eta s_a - \frac{\pi\eta}{2} + \pi\vartheta_a, -\pi\eta s_a + \frac{\pi\eta}{2} - \pi\vartheta_a\} \longleftrightarrow m_a.$

General Expression

- For BCDEFG gauge theory, the refined Representation.

$$\mathcal{R}^{ref} = V \otimes V^* \oplus V \otimes \mathcal{F} \oplus V \otimes \mathcal{F}'$$

- V is the standard representation of $SU(N)$
- V^* is the dual of V
- $\mathcal{F}$: The fund. rep.
- $\tilde{\mathcal{F}}$: Anti-fund. rep.

3d BCDEFG Gauge Theories Potential

- Effective Potential

$$W^{3d}_{\text{eff}}(\sigma, m) = \frac{1}{\beta_2}\sum_{w\in\mathcal{R}}\sum_{a=1}^{N_f}\text{Li}_2(e^{-iw\cdot\sigma - im_a - i\beta_2\tilde{c}}) - \frac{1}{4\beta_2}\sum_{w\in\mathcal{R}}\sum_{a=1}^{N_f}(w\cdot\sigma + m_a + \beta_2\tilde{c})^2 + \frac{1}{\beta_2}\sum_{w\in\mathcal{R}}\sum_{a=1}^{N_f'}\text{Li}_2(e^{iw\cdot\sigma - im_a' - i\beta_2\tilde{c}}) - \frac{1}{4\beta_2}\sum_{w\in\mathcal{R}}\sum_{a=1}^{N_f'}(w\cdot\sigma - m_a' + \beta_2\tilde{c})^2$$

continue-

$$-\frac{1}{\beta_2}\sum_{\alpha\in\Delta}\frac{4}{\alpha_i^2}\mathrm{Li}_2(e^{i\alpha\cdot\sigma})+\frac{1}{4\beta_2}\sum_{\alpha\in\Delta}\frac{4}{\alpha_i^2}(\alpha\cdot\sigma)^2$$

- $\frac{4}{\alpha_i^2}$ times adjoint chiral multiplets.
- The only difference is the root set
- $N_f = N_f'$ will always be assumed for the duality with Spin Chain.

$SO(2n+\chi)$

- The bound. cond. $\xi_+ = \xi_- = -\frac{\eta}{2}$, for $\chi = 1$;
- The bound. cond. $\xi_+ = \xi_- = i\infty$, for $\chi = 0$.
- $N_f = N_f' = 2L$ is an even integer and the map is

$$\pi u \longleftrightarrow \sigma, \quad \pi\eta \longleftrightarrow m_{\mathrm{adj}}$$

$$M \longleftrightarrow N, \quad 2L \longleftrightarrow N_f$$

$$\{-\pi\eta s_a - \frac{\pi\eta}{2} + \pi\vartheta_a, -\pi\eta s_a + \frac{\pi\eta}{2} - \pi\vartheta_a\} \longleftrightarrow m_a$$

$Sp(2N)$

- $$\frac{\sin(\sigma_j - \frac{m_{\text{adj}}}{2})\cos(\sigma_j - \frac{m_{\text{adj}}}{2})}{\sin(\sigma_j + \frac{m_{\text{adj}}}{2})\cos(\sigma_j + \frac{m_{\text{adj}}}{2})} \prod_{j\neq k}^{N} \frac{\sin(\sigma_j \pm \sigma_k - m_{\text{adj}})}{\sin(-\sigma_j \pm \sigma_k - m_{\text{adj}})} \times \prod_{a=1}^{N_f} \frac{\sin(\sigma_j - m_a)}{\sin(-\sigma_j - m_a)} = -1$$
- Bond.Cond.: $\xi_+ = \frac{1}{2}, \xi_- = 0$; or $\xi_+ = 0, \xi_- = \frac{1}{2}$
- Dual Dictionary:

$$\pi u \longleftrightarrow \sigma,\ \pi\eta \longleftrightarrow m_{\text{adj}};\ M \longleftrightarrow N,\ 2L \longleftrightarrow N_f$$

$$\{-\pi\eta s_a - \frac{\pi\eta}{2} + \pi\vartheta_a, -\pi\eta s_a + \frac{\pi\eta}{2} - \pi\vartheta_a\} \longleftrightarrow m_a$$

Dual representation (II) for A_N

- $\mathcal{R} = V \otimes V^* \oplus V \otimes \mathcal{F} \oplus V \otimes \tilde{\mathcal{F}}$.
-

$$
\begin{aligned}
W_{\text{eff}}^{A;3d}(\sigma, m) =& \frac{1}{\beta_2}\sum_{j\neq k}^{N} \mathrm{Li}_2(e^{-2i(\sigma_j-\sigma_k)-2im_{\text{adj}}}) - \frac{1}{\beta_2}\sum_{j\neq k}^{N}(\sigma_j - \sigma_k + m_{\text{adj}}
\\
&+ \frac{1}{\beta_2}\sum_{j=1}^{N}\sum_{a=1}^{N_f}\mathrm{Li}_2(e^{-2i\sigma_j-2im_a}) - \frac{1}{\beta_2}\sum_{j=1}^{N}\sum_{a=1}^{N_f}(\sigma_j + m_a)^2
\\
&+ \frac{1}{\beta_2}\sum_{j=1}^{N}\sum_{a=1}^{N_f'}\mathrm{Li}_2(e^{2i\sigma_j-2im_a'}) - \frac{1}{\beta_2}\sum_{j=1}^{N}\sum_{a=1}^{N_f'}(\sigma_j - m_a')^2
\\
&- \frac{1}{\beta_2}\sum_{j\neq k}^{N}\mathrm{Li}_2(e^{2i(\sigma_j-\sigma_k)}) + \frac{1}{\beta_2}\sum_{j\neq k}^{N}(\sigma_j - \sigma_k)^2
\end{aligned}
$$

(II) B/G Corresp. for A_N

- The dictionary:

$$\pi u \longleftrightarrow \sigma, \quad \pi\eta \longleftrightarrow m_{\text{adj}}$$

$$M \longleftrightarrow N, \quad L \longleftrightarrow N_f = N_f^{'}$$

$$\{-\pi\eta s_a - \frac{\pi\eta}{2} + \pi\vartheta_a\} \longleftrightarrow m_a^{'}, \quad \{-\pi\eta s_a + \frac{\pi\eta}{2} - \pi\vartheta_a\} \longleftrightarrow m_a$$

- I Rep.: $\mathcal{R} = V \otimes V^* \oplus V \otimes \mathcal{F} \oplus V \otimes \mathcal{F} \oplus V^* \otimes \tilde{\mathcal{F}} \oplus V^* \otimes \tilde{\mathcal{F}}$
- II Rep.: $\mathcal{R} = V \otimes V^* \oplus V \otimes \mathcal{F} \oplus V \otimes \tilde{\mathcal{F}}$

Dual representation for BCDEF

- $\mathcal{R} = V \otimes V^* \oplus V \otimes \mathcal{F}$.
- Effective Pot.

$$
\begin{aligned}
W^{3d}_{\text{eff}}(\sigma, m) =& \frac{1}{\beta_2} \sum_{w \in \mathcal{R}} \sum_{a=1}^{N_f} \text{Li}_2(e^{2(-iw\cdot\sigma - im_a - i\beta_2 \tilde{c})}) \\
& - \frac{1}{4\beta_2} \sum_{w \in \mathcal{R}} \sum_{a=1}^{N_f} [2(w \cdot \sigma + m_a + \beta_2 \tilde{c})]^2 \\
& - \frac{1}{\beta^2} \sum_{\alpha \in \Delta} \text{Li}_2(e^{\frac{4}{\alpha_i^2} i\alpha\cdot\sigma}) + \frac{1}{4\beta_2} \sum_{\alpha \in \Delta} (\frac{4}{\alpha_i^2} \alpha \cdot \sigma)^2
\end{aligned}
$$

Param. Corresp. for B_N Case 1

- Case 1: if $s_1 = s_2 = -\frac{1}{2}$

$$\xi_+ = \xi_- = -\frac{\eta}{2} + \frac{1}{2}; \quad \vartheta_1 = \vartheta_2 = 0$$

$$or \quad \xi_+ = \xi_- = -\frac{\eta}{2}; \quad \vartheta_1 = \vartheta_2 = \frac{1}{2}$$

- The map

$$\pi u \longleftrightarrow \sigma, \quad \pi\eta \longleftrightarrow m_{\text{adj}}$$

$$M \longleftrightarrow N, \quad 2(L-2) \longleftrightarrow N_f$$

$$\{-\pi\eta s_a - \frac{\pi\eta}{2} + \pi\vartheta_a, -\pi\eta s_a + \frac{\pi\eta}{2} - \pi\vartheta_a\} \longleftrightarrow m_a$$

Corresp. Param. for B_N Case 2

- Case 2: if $s_1 = s_2 = s_3 = s_4 = -\frac{1}{2}; \quad \xi_+ = \xi_- = -\frac{\eta}{2}$

$$\vartheta_1 = \vartheta_2 = 0$$

$$or \quad \vartheta_1 = \vartheta_2 = \frac{1}{2}$$

- The map

$$\pi u \longleftrightarrow \sigma, \quad \pi\eta \longleftrightarrow m_{\rm adj}$$

$$M \longleftrightarrow N, \quad 2(L-4) \longleftrightarrow N_f$$

Corresp. Param. for B_N Case 3

- Case 1: if $s_1 = s_2 = s_3 = -\frac{1}{2}$

$$\xi_+ = -\frac{\eta}{2} = -\xi_-; \quad \vartheta_1 = \frac{1}{2}, \quad \vartheta_2 = \vartheta_3 = 0$$

$$or \quad \xi_+ = -\frac{\eta}{2} + \frac{1}{2}, \ \xi_- = \frac{\eta}{2}; \quad \vartheta_1 = \frac{1}{2}, \quad \vartheta_2 = \vartheta_3 = 0$$

- The map

$$\pi u \longleftrightarrow \sigma, \quad \pi\eta \longleftrightarrow m_{\text{adj}}$$

$$M \longleftrightarrow N, \quad 2(L-3) \longleftrightarrow N_f$$

Bound. Cond. of C_N and D_N

- For C_N, Bound. Cond. of (II):

$$\xi_+ = \frac{\eta}{2}, \quad \xi_- = 0, \;\; \xi_+ = 0, \quad \xi_- = \frac{\eta}{2}$$

- Bound. Cond. Diff. from (I)

$$\xi_+ = \frac{1}{2}, \quad \xi_- = 0, \text{ or } \;\; \xi_+ = 0, \quad \xi_- = \frac{1}{2}$$

- For D_N, boundary conditions:
  $\xi_+ = \xi_- = i\infty$.
- Bound. Cond. intact
- Role of boundary condition and spin site are swapped.

Duality of Rep.

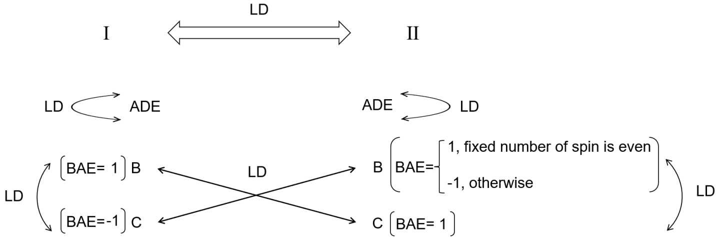

Figure: The Langlands dualities between the Rep. of (I) and (II).

Summary

- The Bethe equation for a given boundary parameter matches only one branch of the squared root of the vacuum equation.
- From Lie theory, ADE series are self-dual.
- B_n and C_n Langlands dual to each other, they belong to different branches.
- The Bethe/quiver gauge correspondence will give a concrete example of geometric representation for BCD Lie algebras.
- In a certain scene, we deal with a double covering theory of spin chain.
- The rep. I and II are Langlands dual; II is more concise, and no N_f' is needed (Except A_N).

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Thank You for Your Attention!