Refined Topological Strings on Compact Elliptic-fibered Calabi-Yau 3-folds

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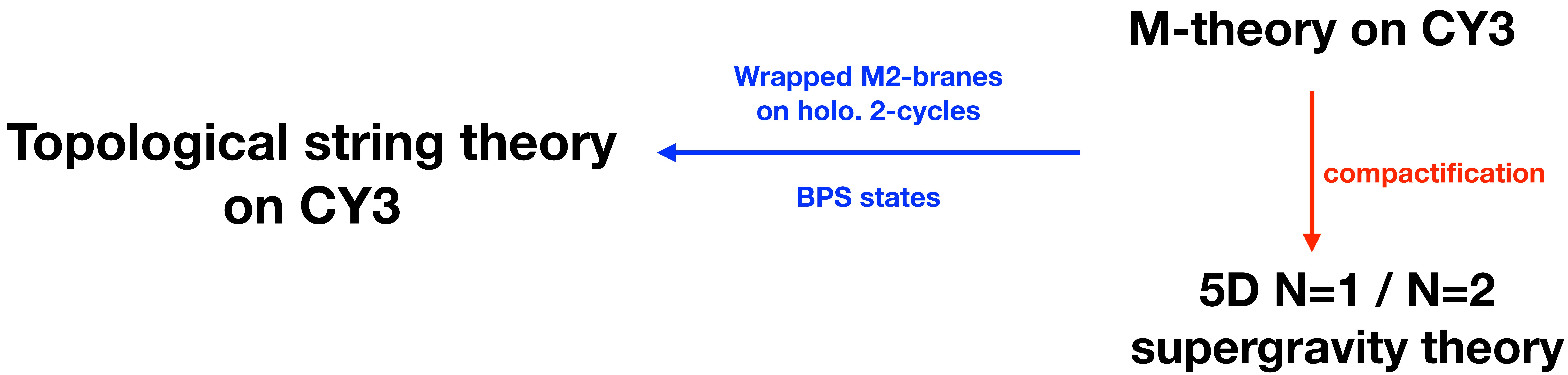
Motivation

For M-theory compactification on a generic Calabi-Yau 3-fold (CY3), the microscopic degeneracies N_{j_L, j_R} of spinning BPS states are captured *exactly* by topological string theory on CY3



Motivation

The BPS states in 5D are characterized by the spin (j_L, j_R) under the spatial rotation group

$$SO(4) \simeq SU(2)_L \times SU(2)_R$$

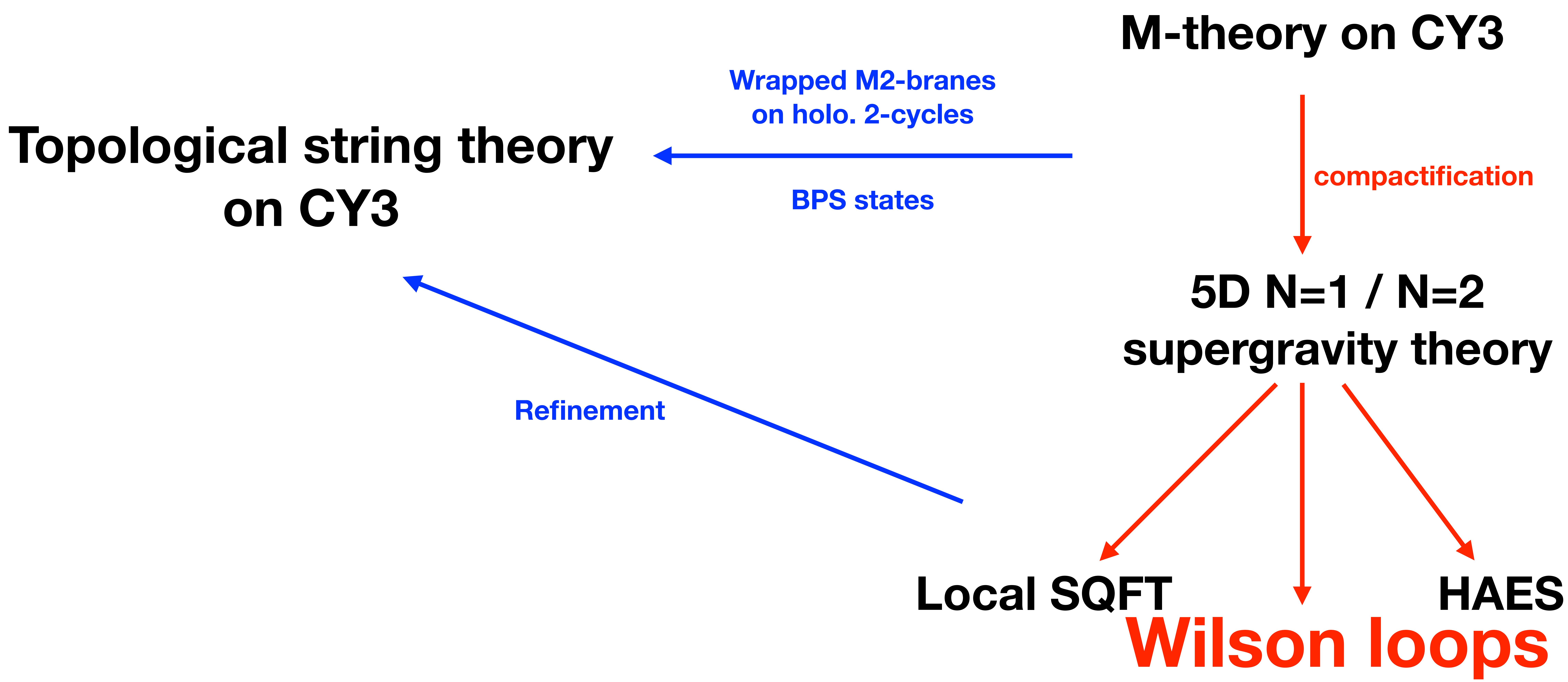
which can be captured by the topological A-model string amplitudes

However, the $SU(2)_R$ contents are irrelevant in the conventional topological string calculations.

Turning on $SU(2)_R$ gives a refined version of topological strings, known as refined topological strings —— Hard to define / calculate

The purpose of today's talk :

A proposal for REFINED topological strings on COMPACT Calabi-Yau 3-folds



Topological String Theory

- Topological String Theory is a simplified version of string theory focusing on the topological aspects of the string's worldsheet [Witten '89]

- It defines a map from a 2D $\mathcal{N}=(2,2)$ worldsheet theory to the target space

$$\phi_i : \Sigma \to X$$ = CY3 vanishing first Chern class

- We can define the A-model and B-model of the topological string theory by different types of topological twists.

Mirror Symmetry

A-model on X ⟷ **B-model on** $\widetilde{X}$

Topological String Theory

- The A-model is defined as Gromov-Witten theory in mathematics.

$$\mathcal{F}^{(g)}(t_i) = \sum_{\beta \in H_2(X;\mathbb{Z})} \mathrm{GW}_g^\beta e^{-\beta \cdot t}$$

- The generating function of these worldsheet instanton numbers is called the free energy of the topological strings. It only depends on the Kähler deformation parameters t_i — the masses of the BPS particles

$$\mathcal{F} = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}^{(g)}(t_i)$$

Topological String Theory

- The generating function can be expanded in terms of integral Gopakumar-Vafa (GV) invariants:

$$\mathcal{F} = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}^{(g)}(t_i) = \sum_{\beta,g} \sum_{k=1}^{\infty} \frac{n_g^\beta}{k} \left(2\sinh(\frac{k\lambda}{2})\right)^{2g-2} e^{-k\beta\cdot t}$$

which is derived by [Gopakumar-Vafa `98], from the Schwinger one-loop calculation for an electric particle in a background constant electromagnetic field

- The GV invariant is a topological invariant, it is related to the degeneracies N_{j_L,j_R}

$$n_g^\beta \sim \sum_{j_R} (2j_R+1) N_{j_L,j_R}^\beta$$

Topological String Theory

- The A-model GW or GV invariants are usually very hard to compute, but they can be relatively easy to compute from the B-model geometry via mirror symmetry.

- At genus 0, for a hypersurface CY, we can solve the genus 0 GV from the solutions of GKZ systems. E.g. Quintic CY3

d	n_0^d
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625

Topological String Theory

- Higher genus invariants can be computed from Holomorphic anomaly equations [Bershadsky, Cecotti, Ooguri, Vafa ’93]

  ➡ Quintic CY3

  - gap conditions: genus 51(+2) [Huang, Klemm, Quackenbush ’06]
  - Modularities on the DT invariants: genus 80

    [Alexandrov, Feyzbakhsh, Klemm, Pioline, Schimannek ’23]
    [Alexandrov, Feyzbakhsh, Klemm, Pioline ’23]

- Best playground of the resurgence in string theory

- Is there an all-genus expression? Black hole entropy [Vafa’s talk at String Math 2024]

Topological String Theory

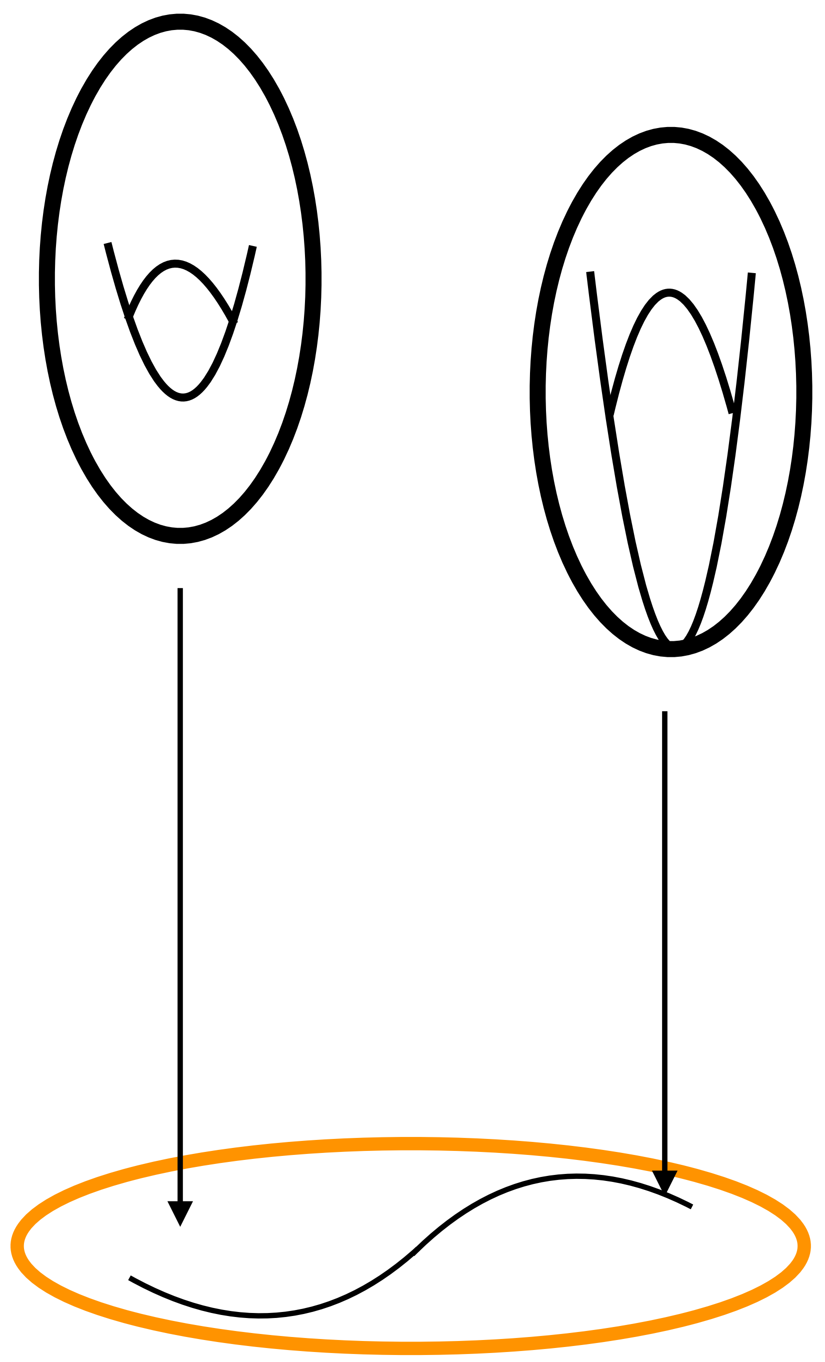
Elliptic fibered Calabi-Yau 3-folds:

E.g. elliptic fibration over $\mathbb{P}^2$

$$
\begin{array}{lcccccccccc}
l^{(1)} & = & (& -6 & ; & 2 & 3 & \boxed{1} & 0 & 0 & 0 &) \\
l^{(2)} & = & (& 0 & ; & 0 & 0 & \boxed{-3} & 1 & 1 & 1 &)
\end{array}
\qquad
\begin{array}{l}
\text{Fiber curve} \\
\text{Base curve}
\end{array}
$$

$$\downarrow$$
$$D$$

Compact divisor in local P2



Topological String Theory

Local Calabi-Yau 3-folds:

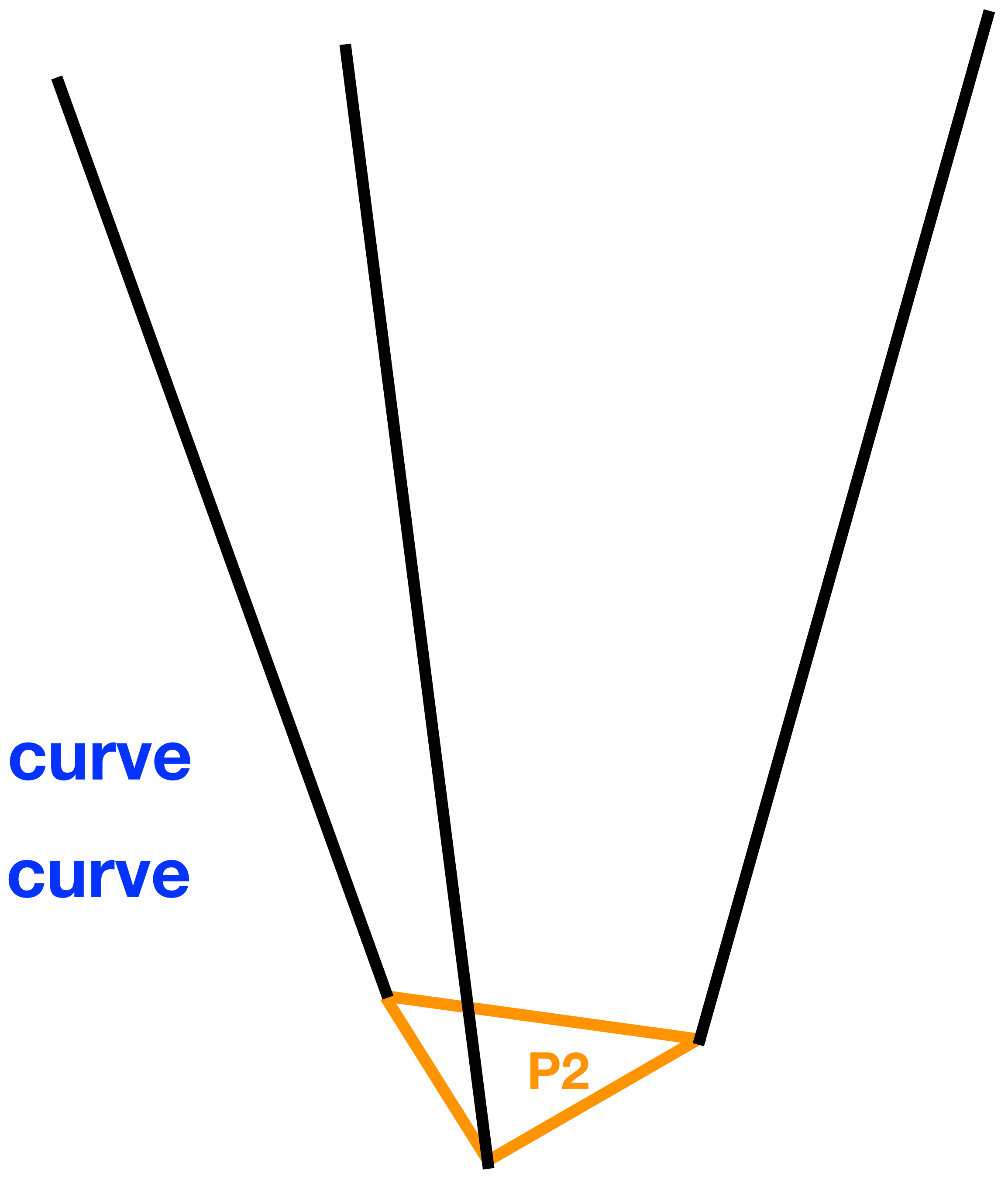
E.g. local $\mathbb{P}^2$, large fiber limit of elliptic $\mathbb{P}^2$

$$\begin{array}{ccccccccccc} l^{(1)} & = & (& -6 & ; & 2 & 3 & \boxed{1} & 0 & 0 & 0 &) \\ l^{(2)} & = & (& 0 & ; & 0 & 0 & \boxed{-3} & 1 & 1 & 1 &) \\ & & & & & & & \downarrow & & & & \\ & & & & & & & D & & & & \end{array}$$

Fiber curve

Base curve

Compact divisor in local P2



Local P2

- Holomorphic anomaly equation
- Topological vertex
- Topological recursion / Remodeling conjecture

Refined Topological String Theory

Refinement of Local Calabi-Yau 3-folds:

- 4D or 5D supersymmetric gauge theories (on $\mathbb{R}^4$ or $\mathbb{R}^4 \times S^1$) can be geometric engineered from local Calabi-Yau 3-folds X, from IIA/M-theory compactification on X

- The instanton partition function of the gauge theory in the (extended) Coulomb branch can be computed via supersymmetric localization, it is equal to the topological string partition function

$$Z_{\text{gauge}}(t, \lambda) = Z_{\text{top}}(t, \lambda)$$

	Gauge theory	Topological string theory
t	VEV of scalars in the vector multiplets, mass parameters,...	Kähler parameters
λ	Chemical potential for the rotation symmetry	Tological string coupling

Refined Topological String Theory

Refinement of Local Calabi-Yau 3-folds:

- Nekrasov proposed the gauge theory on the Omega-deformed background, by turning on the chemical potentials $\epsilon_\pm = \frac{1}{2}(\epsilon_1 \pm \epsilon_2)$ for the 4D rotation symmetry

$$SO(4) \simeq SU(2)_L \times SU(2)_R$$

$$Z_{\text{gauge}}(t, \lambda) \xrightarrow{\text{refinement}} Z_{\text{gauge}}(t, \epsilon_1, \epsilon_2)$$

- The BPS particles are characterized by the spins (j_L, j_R)

Refined Topological String Theory

Refinement of Local Calabi-Yau 3-folds:

- Refined topological strings are proposed from the refinement of gauge theory

$$Z_{\text{ref}}(t,\epsilon_1,\epsilon_2)=Z_{\text{gauge}}(t,\epsilon_1,\epsilon_2)$$
$$=\exp\left[\sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{k=1}^{\infty}\sum_{j_L,j_R}(-1)^{2j_L+2j_R}\frac{1}{k}N^{\beta}_{j_L,j_R}\frac{\chi_{j_L}(k\epsilon_-)\chi_{j_R}(k\epsilon_+)}{2\sinh(\frac{k\epsilon_1}{2})2\sinh(\frac{k\epsilon_2}{2})}e^{-k\beta\cdot t}\right]$$

- The non-negative number $N^{\beta}_{j_L,j_R}$ counts the degeneracy of the BPS particle with spin (j_L,j_R) and mass $\beta\cdot t$, it comes from M2-branes wrapping over the curve class β

- It is called refined BPS invariants (non-negative integers)

Refined Topological String Theory

Refined topological strings on non-compact CY3

- Refined holomorphic anomaly equations

$$F = \sum_{g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} F^{(n,g)}(t_i)$$

genus $g \to$ genus (n, g)

- Refined topological vertex

- Blowup equations

- Elliptic fibration over non-compact toric surface (6d SCFTs, 6d LSTs)

  - Modular bootstrap: can be refined

Refined Topological String Theory

Known results for refined topological strings on compact CY3's

- $K3 \times T2$ [Katz, Klemm, Pandharipande, `14]
- Elliptic CY3, $(n \leq 1, g)$ done by [Huang Katz, Klemm, `20]

Our result: refined topological strings on compact (elliptic-fibered) CY3 for any (n, g)

$$F = \sum_{g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} F^{(n,g)}(t_i)$$

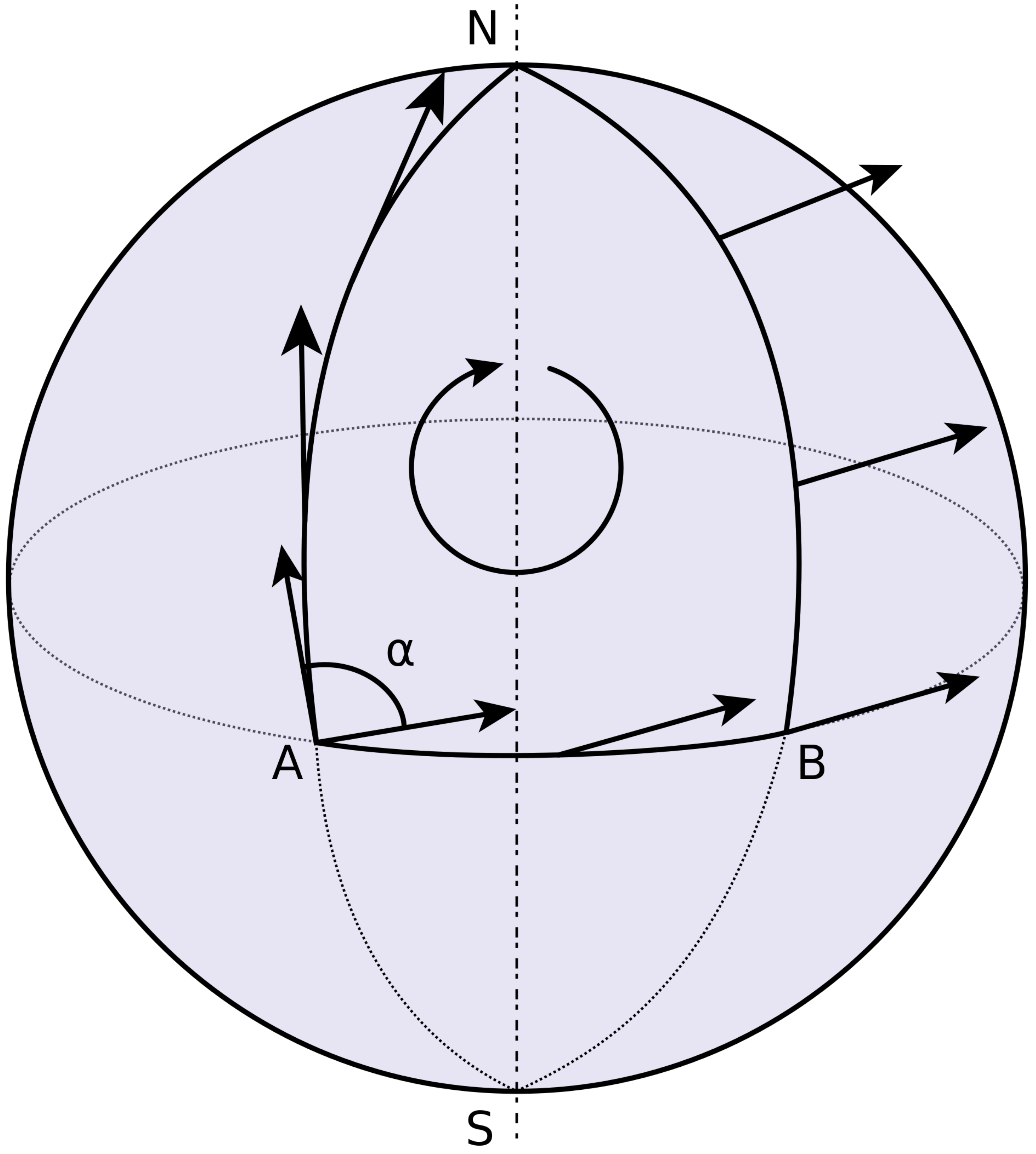
$$= \exp\left[\sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L, j_R} (-1)^{2j_L + 2j_R} \frac{1}{k} N^{\beta}_{j_L, j_R} \frac{\chi_{j_L}(k\epsilon_-)\chi_{j_R}(k\epsilon_+)}{2\sinh(\frac{k\epsilon_1}{2}) 2\sinh(\frac{k\epsilon_2}{2})} e^{-k\beta \cdot t} \right]$$

Remark: The number $N^{\beta}_{j_L, j_R}$ may not be an invariant but depends on the complex deformation of the CY3

Wilson loop

- In gauge theory, Wilson lines/loops are gauge invariant operators
- They arise from the parallel transport of gauge variables around closed loops
- They can be generated from the worldline C of static infinitely massive quarks in rep.$\mathbf{r}$ — Polyakov loop

$$W_{\mathbf{r}}(C) = \mathrm{Tr}_{\mathbf{r}}\left[\mathcal{P}\exp\left(i\oint_C A_\mu dx^\mu\right)\right]$$



Wilson loop

- Supersymmetric version

$$W_{\mathbf{r}}[C] = \mathrm{Tr}_{\mathbf{r}}\, \mathcal{P} \exp \int_C \left(i A_\mu \dot{x}^\mu + |\dot{x}| \phi \right) ds$$

- a scalar is added to preserve some of the supersymmetries.

- Half-BPS operators in 5D $\mathcal{N} = 1$ gauge theories on S^1
  - The rotation symmetry $SO(4)$ is preserved
  - GV-like expansion [Huang, Lee, XW, '22][Kim, Kim, Kim, '21]

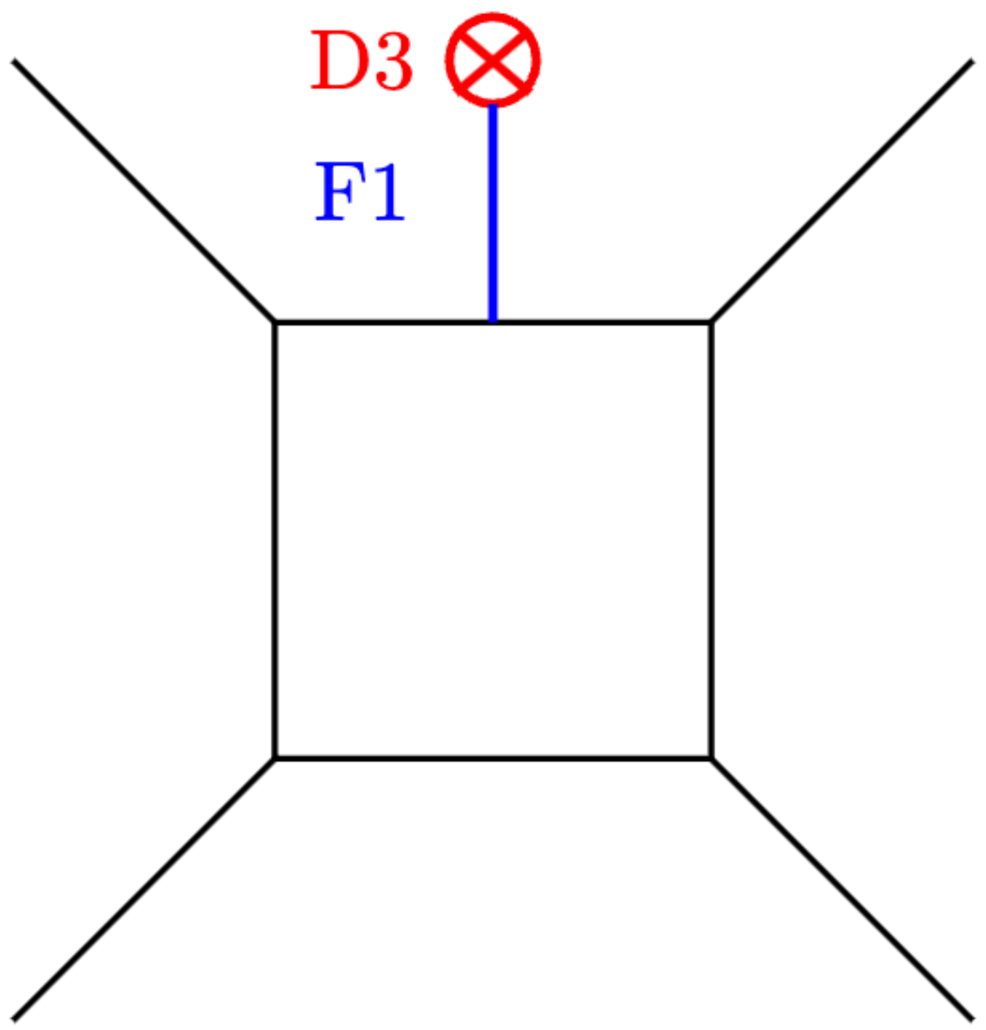
Wilson loop

- **Coulomb branch**: the scalar field ϕ gets the expectation value in the Cartan subalgebra of the gauge group, which breaks the gauge group to $U(1)^r$.

- The scalar expectation values $\phi_i, i = 1, \cdots, r$ parametrize the moduli space on the Coulomb branch.

- The representation of the Wilson loop becomes the electric charge of the Wilson loop particle —— a **heavy, stationery electric particle** located at the origin of the space $\mathbb{R}^4$

Half-BPS Wilson loop operators — IIB realization [David Tong, '14] …

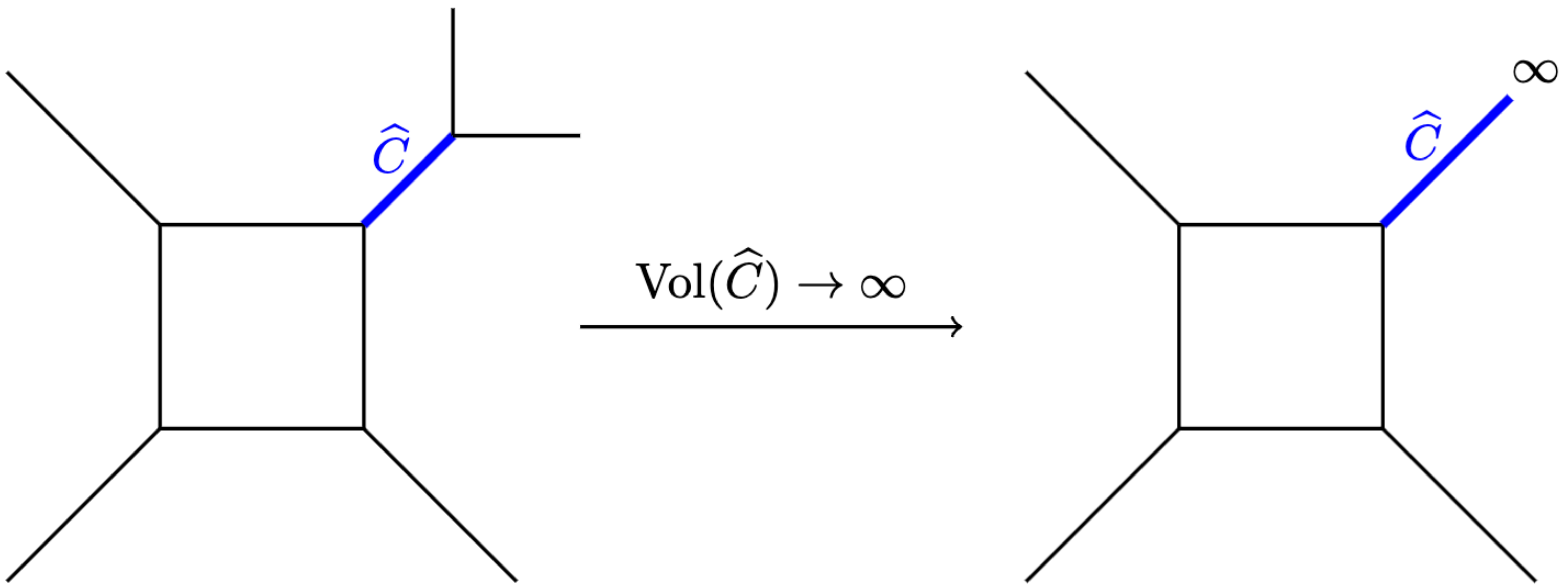


	0	1	2	3	4	5	6	7	8	9
D5	•	•	•	•	•	•				
NS5	•	•	•	•	•		•			
$5_{(p,q)}$	•	•	•	•	•	θ	θ			
F1	•						•			
D3	•							•	•	•

The half-BPS Wilson loop operators are realized by adding semi-infinite F1 strings with charge 1, stretched between D3 branes and D5 branes.

The lowest energy modes on such F1 strings are fermionic, so there can be at most one F1 string stretched between a D3 brane and a D5 brane.

Half-BPS Wilson loop operators — IIB realization



To have a geometric realization, we can also add a flavor matter by blowing up the geometry at one point.

In the large volume limit of the exceptional curve, only single F1 states contribute

Half-BPS Wilson loop operators — IIB realization

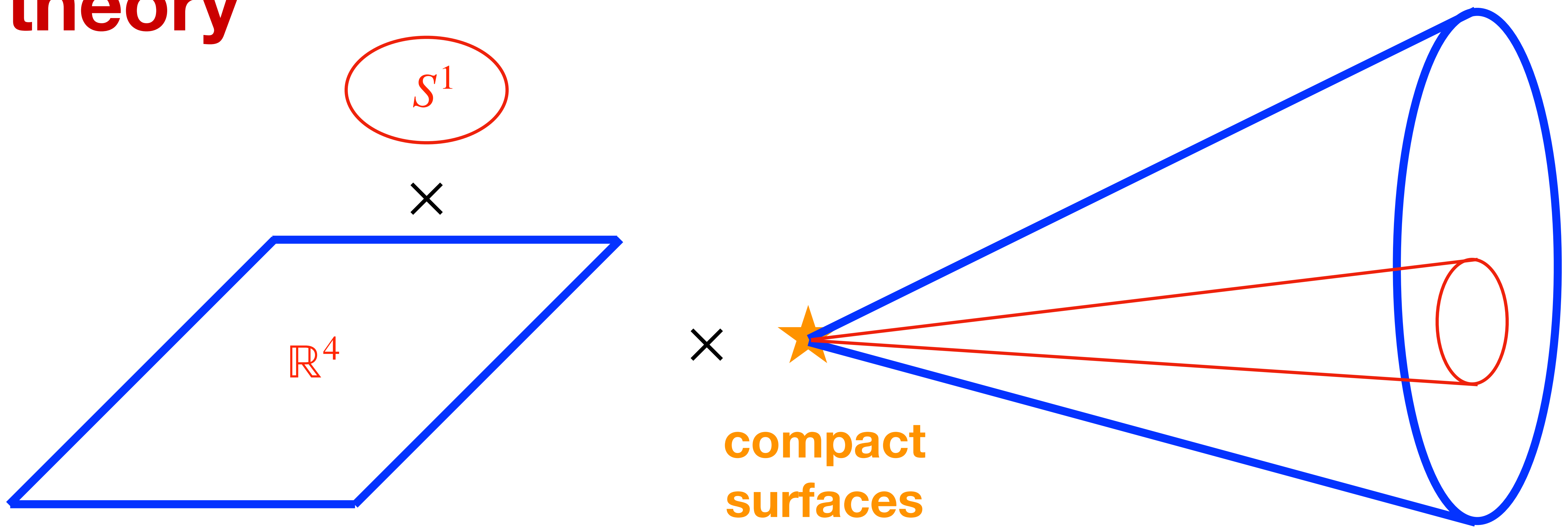
- The geometric definition can be extended to non-gauge theory, e.g. local P2

d_h/d_C	0	1	2
0	0	1	0
1	3	-2	1
2	-6	5	-4
3	27	-32	35
4	-192	286	-400
5	1695	-3038	5187

Genus 0 invariants for Wilson loops of local P2

- And theories with broken one-form symmetry

Realization in M theory



Wilson loop | **Geometry X**

heavy, stationery electric particle | a non-compact curve C in CY3 extended to infinity
Heavy

BPS particles | M2-branes wrapping around $C + \mathsf{C}$, $C \in H_2(X; \mathbb{Z})$

Representation | Charges of C $q_i = D_i \cdot \mathsf{C}$

Compact divisor

How the Wilson loops of a local CY3 are connected to the compact CY3

Genus 0 GV invariants:

d_h/d_C	0	1	2
0	0	1	0
1	3	-2	1
2	-6	5	-4
3	27	-32	35
4	-192	286	-400
5	1695	-3038	5187

Wilson loops for local P2

d_h/d_C	0	1	2
0	0	1	0
1	3	-2	0
2	-6	5	0
3	27	-32	7
4	-192	286	-110
5	1695	-3038	1651

one point blowup of local P2

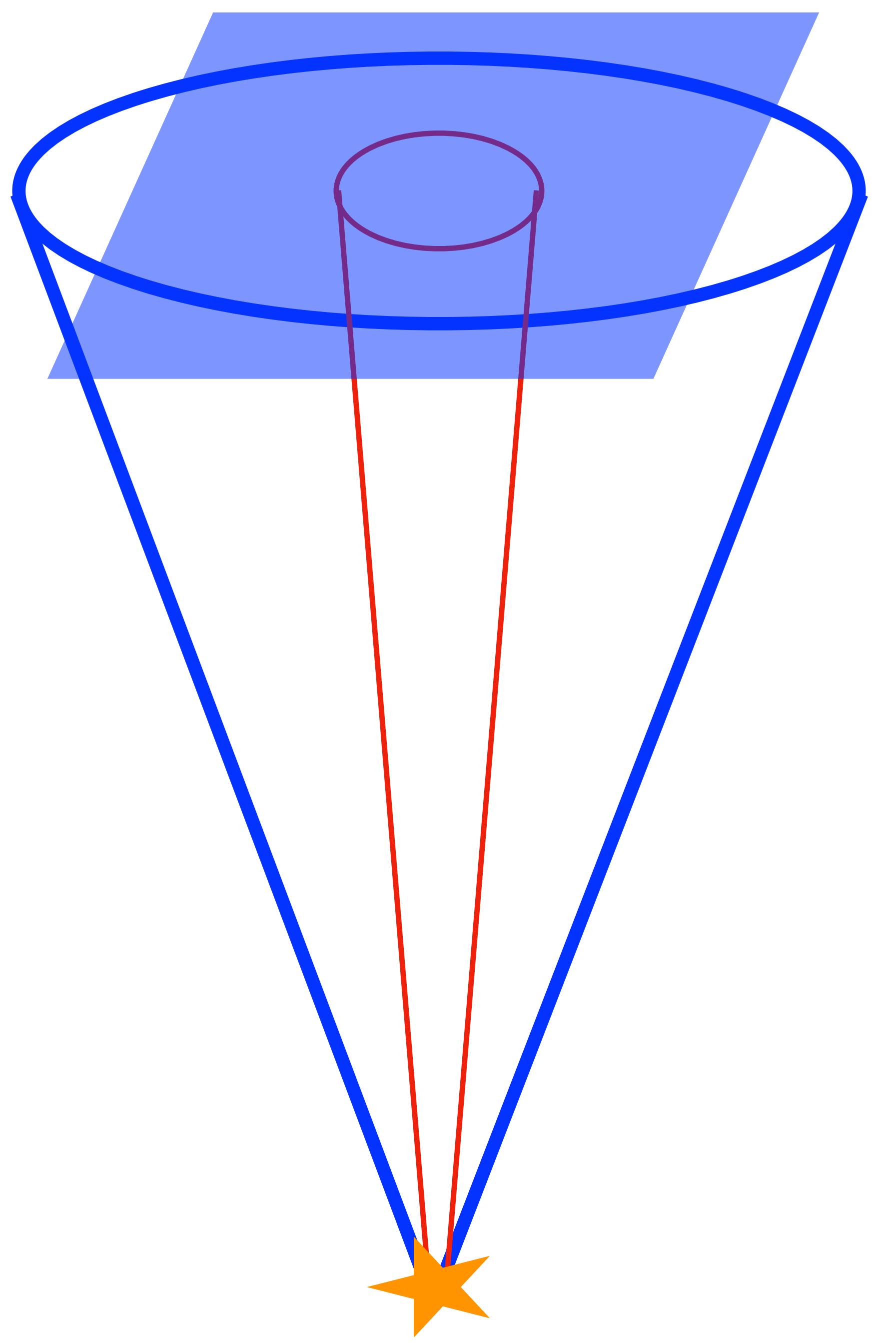
d_h/d_C	0	1	2
0	0	$540\times(1)$	540
1	3	$540\times(-2)$	143370
2	-6	$540\times(5)$	-574560
3	27	$540\times(-32)$	5051970
4	-192	$540\times(286)$	-57879900
5	1695	$540\times(-3038)$	751684050

elliptic P2

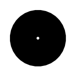
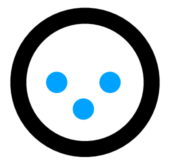
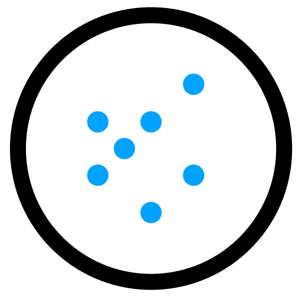
The Euler number for elliptic P2 is -540

This is not a coincidence, the red numbers are all related to Wilson loops of local P2

How the Wilson loops of a local CY3 are connected to the compact CY3



Ending of the 2-cycle at infinity

	Wilson loop of local P2	dP1	elliptic P2
Wilson loop particle			
	1	$\frac{1}{2\sinh(\frac{\epsilon_1}{2})\cdot 2\sinh(\frac{\epsilon_2}{2})}$	$\frac{546[0,0]\oplus[1,\frac{1}{2}]}{2\sinh(\frac{\epsilon_1}{2})\cdot 2\sinh(\frac{\epsilon_2}{2})}$

If a half-BPS particle in the 5D supergravity theory is heavy enough

It becomes the half-BPS Wilson loop particle in the local 5D quantum field theory

We can refine the BPS spectrum of the 5D supergravity theory via refinement of the 5D Wilson loops

Elliptic P2

$$\begin{array}{ccccccccccccc} l^{(1)} & = & (& -6 & ; & 2 & 3 & \boxed{1} & 0 & 0 & 0 &) \\ l^{(2)} & = & (& 0 & ; & 0 & 0 & \boxed{-3} & 1 & 1 & 1 &) \end{array}$$

Fiber curve q

Base curve Q

$$\downarrow$$
$$D$$

Compact divisor in local P2

Define the expansion

$$\mathcal{F}(Q, q; \epsilon_1, \epsilon_2) = \sum_{d_1=0}^{\infty} \mathcal{F}_{d_1}(Q; \epsilon_1, \epsilon_2)(qQ^{1/3})^{d_1}$$

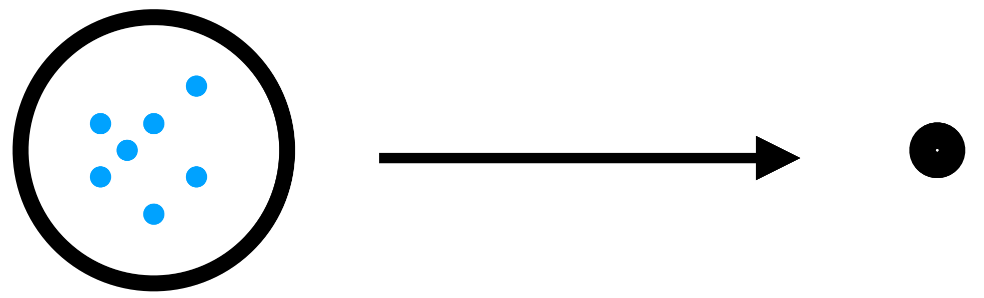
We have the conjecture :

$$\mathcal{F}_1(Q; \epsilon_1, \epsilon_2) = f(\epsilon_1, \epsilon_2)\mathcal{F}^{\mathbb{P}^2}_{\mathrm{W},[1]}$$

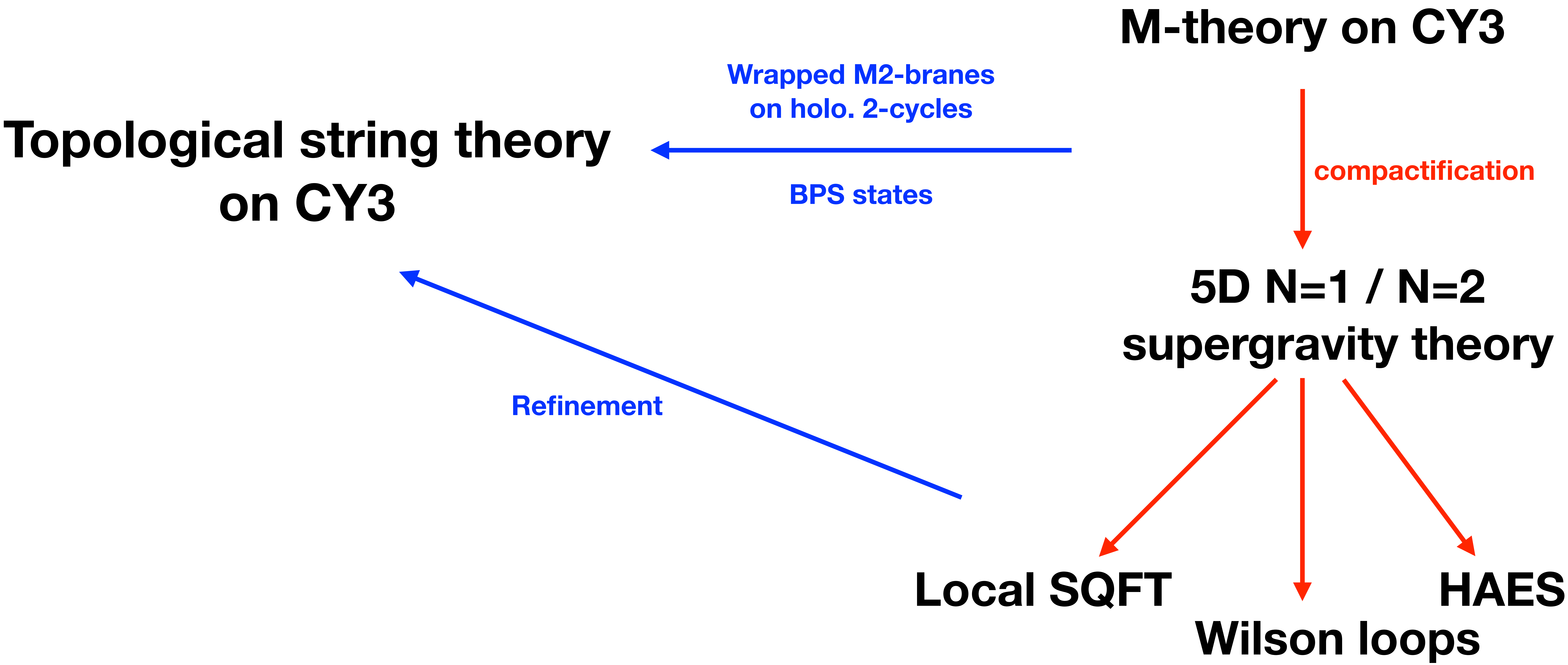
BPS numbers in the fiber direction

$$f(\epsilon_1, \epsilon_2) = \frac{546 - (q_-^{-1} + q_-)(q_+^{-2} + 1 + q_+^2)}{2\sinh(\epsilon_1/2) \cdot 2\sinh(\epsilon_2/2)}$$

Frozen contribution



Conclusion


M-theory on CY3
Wrapped M2-branes
on holo. 2-cycles
BPS states
Topological string theory
on CY3
compactification
5D N=1 / N=2
supergravity theory
Refinement
Local SQFT
Wilson loops
HAES

Conclusion

We can also consider the gluing of multiple local theories

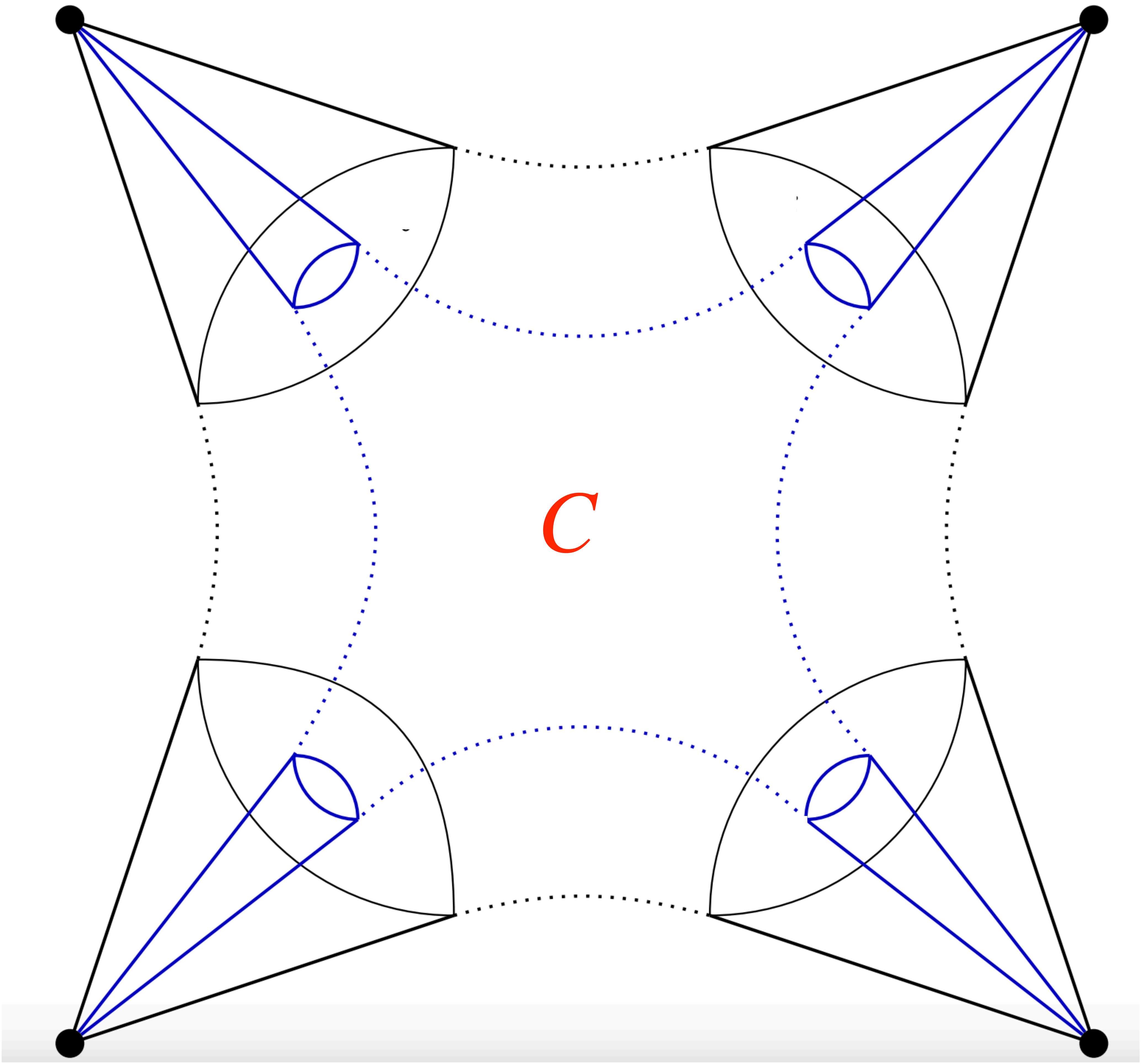


Figure 1 in 2307.13027

Thank you!