ENSEMBLE AVERAGES, POISSON PROCESSES AND MICROSTATES

Cheng Peng （彭程）

KAVLI INSTITUTE FOR THEORETICAL SCIENCES (KITS)
UNIVERSITY OF CHINESE ACADEMY OF SCIENCES (UCAS)

Based on 2010.11192

第一届全国场论与弦论学术研讨会,彭桓武高能基础理论研究中心, 2020年11月30日

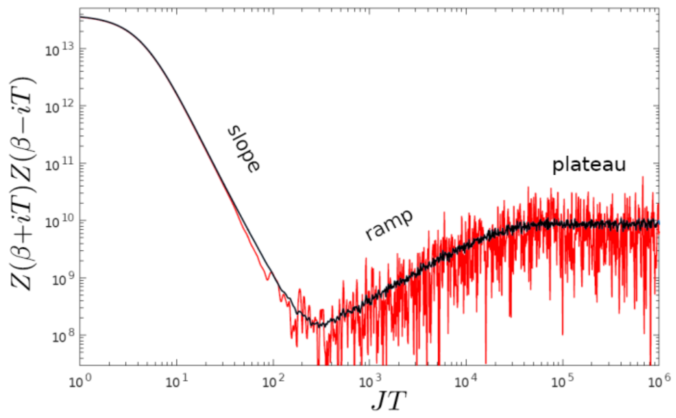
MOTIVATIONS - ENSEMBLE AVERAGES

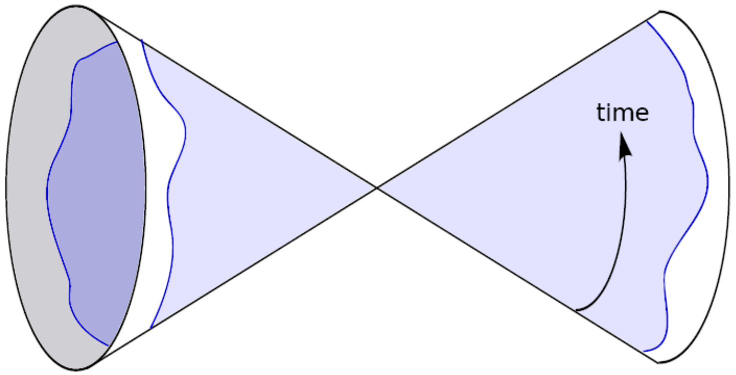
- Recent developments suggest

Gravitation Path Integrals (GPI) computes quantities in Ensemble Averaged Theories

(e.g. Jackiw-Teitelboim gravity is dual to random matrix theories)

- E.g. Spectral 2-point function $\langle |Z(\beta + iT)|^2 \rangle_J$ (Saad, Shenker, Stanford, 2018)





(Figures from SSS2018)

MOTIVATIONS - ENSEMBLE AVERAGES

- Recent developments suggest

Gravitation Path Integrals (GPI) computes quantities in Ensemble Averaged Theories

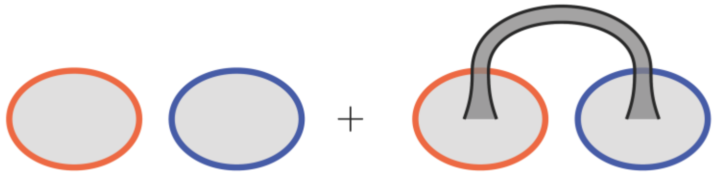
(e.g. Jackiw-Teitelboim gravity is dual to random matrix theories)

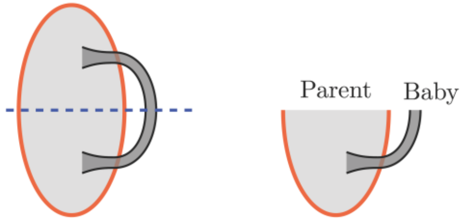
- E.g. A topological model of "Baby universes" (Marolf, Maxfield, 2020)

$$\Big\langle Z[J_1]\cdots Z[J_n]\Big\rangle := \int_{\Phi\sim J} \mathcal{D}\Phi\, e^{-S[\Phi]}$$

$$\mathfrak{Z}^{-1}\,\langle Z^n\rangle = \sum_{d=0}^{\infty} d^n p_d(\lambda), \qquad p_d(\lambda) = e^{-\lambda}\frac{\lambda^d}{d!}$$

$$\Big\langle Z[J_1]Z[J_2]\Big\rangle =$$





(Figures from MM2020)

MOTIVATIONS - ENSEMBLE AVERAGES

- Recent developments suggest

Gravitation Path Integrals (GPI) computes quantities in Ensemble Averaged Theories

- Other discussions about ensemble averages
  - Pollack, Rozali, Sully and Wakeham 2002.02971
  - McNamara and Vafa 2004.06738
  - Afkhami-Jeddi, Cohn, Hartman and Tajdini 2006.04839
  - Maloney and Witten 2006.04855
  - Belin and de Boer 2006.05499
  - Cotler and Jensen 2006.08648
  - Bousso and Wildenhain 2006.16289
  - Stanford 2008.08570
  - …

MOTIVATIONS - RANDOMNESS

- Questions:

  How should we understand the ensemble average of random theories ?

  No such averages in familiar examples, don't know how to quantize

- Idea:

  - Models with true randomness
  - Microscopic model that display

    pseudo-randomness after coarse graining
  - The true randomness is an analogue

    of the pseudo-randomness
  - Emergent pseudo-randomness and emergent gravity

ENSEMBLE AVERAGES - DISCRETE DISTRIBUTIONS

- Previous analyses focus on Gaussian distributions
  - Simple
  - Well studied

- We consider discrete Poisson distributions
  - Quantum theories have discrete Hilbert space
  - Discrete distributions could appear in GPI (Marolf, Maxfield 2020)
  - Under control

ENSEMBLE AVERAGES - THE MODEL

- In practice, we consider $\mathcal{L}(\phi) = \partial_\mu \phi \partial^\mu \phi - J\phi$ where $J = J_0(x) + J_1(x)$
  - ➢ $J_0(x)$ a classical source
  - ➢ $J_1(x)$ a random source
- Integrate over the random source to get an effective action

$$e^{-S_{\text{eff}}} = \int \mathcal{D}J_1(x)\mathcal{P}(J_1(x))e^{-\int dV(x)\mathcal{L}(\phi)}$$

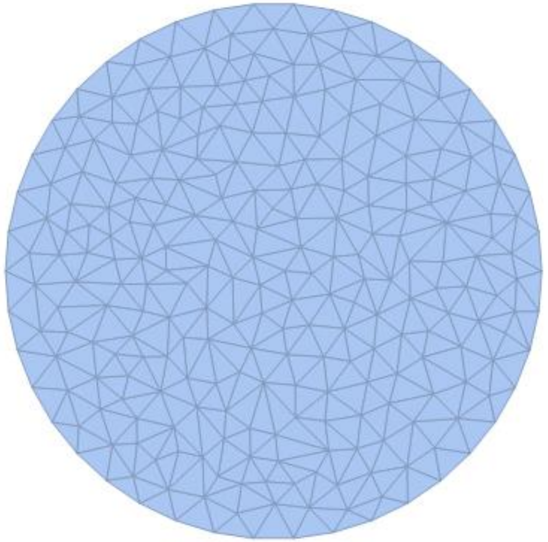
- Questions:
  - ➢ What set of theories/sources to be included ?
  - ➢ What is the measure for the average ?

ENSEMBLE AVERAGES - THE RANDOMNESS

- A sensible choice is (the physical description):

$$P(J_1(x)) = \prod_n \text{Pois}\left(J_1(x_n)dV(x_n), \lambda(x_n)dV(x_n)\right), \quad \forall dV(x_n) \text{ s.t. } \sum_n dV(x_n) = \mathcal{M}$$

where $\text{Pois}(m,\lambda) = e^{-\lambda}\frac{\lambda^m}{m!}, \quad m \in \mathbb{Z}_+$ and $\lambda(x)dV(x) = \langle J_1(x)dV(x)\rangle_{J_1}$



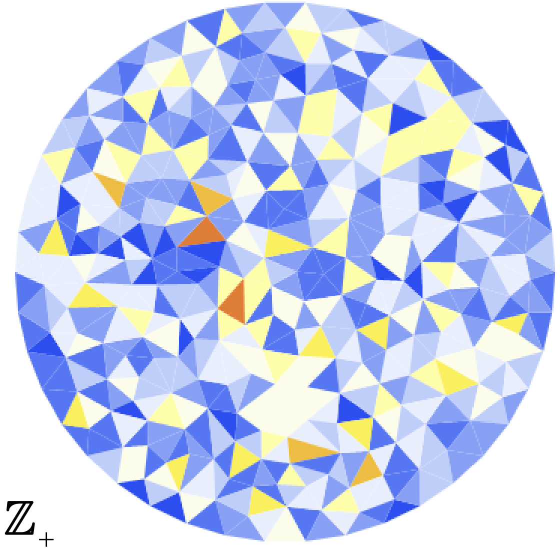
- Properties:
  - The distribution is local (x-dependent)
  - The discretization $dV(x)$ enters the probability distribution
  - The "fluxes" obey the discrete distribution
  - Shape of the distribution measured by $\lambda(x)$

ENSEMBLE AVERAGES - THE RANDOMNESS

- A sensible choice is (the physical description):

$$P(J_1(x)) = \prod_n \mathrm{Pois}\left(J_1(x_n)dV(x_n), \lambda(x_n)dV(x_n)\right), \quad \forall dV(x_n) \text{ s.t. } \sum_n dV(x_n) = \mathcal{M}$$

where $\mathrm{Pois}(m,\lambda) = e^{-\lambda}\frac{\lambda^m}{m!},\ m \in \mathbb{Z}_+$ and $\lambda(x)dV(x) = \langle J_1(x)dV(x)\rangle_{J_1}$



- Properties:
  - The distribution is local (x-dependent)
  - The discretization $dV(x)$ enters the probability distribution
  - The "fluxes" obey the discrete distribution $\int J_1(x)dV(x) \cong J_1(x)dV(x) \in \mathbb{Z}_+$
  - "Shape" of the distribution measured by $\lambda(x)$

ENSEMBLE AVERAGES - THE EFFECTIVE ACTION I

- Averaging over the random source with this measure leads to

$$
\begin{aligned}
&\langle e^{\int dV(x) J_1(x)\phi(x)} \rangle_{J_1} \\
&= \left(\prod_n \sum_{k=0}^{\infty} \mathrm{Pois}\left(J_1(x_n) dV(x_n) = k, dV(x_n)\lambda(x_n) \right) \right) e^{\sum_n dV(x_n)\left(J_1(x_n)\left(\phi(x_n)+i\pi\right)+2\lambda(x)\right)} \\
&= \exp\left(\int dV(x)\lambda(x)(e^{\phi(x)} - 1) \right)
\end{aligned}
$$

- Adding back the other terms gives the effective action

$$
S_{\mathrm{eff}} = \int dV(x)\left(\partial_\mu \phi \partial^\mu \phi - J_0(x)\phi - \lambda(x)(e^{\phi(x)} - 1) \right)
$$

Generalized Liouville theory

ENSEMBLE AVERAGES - THE SIGN

- The sign of the potential term is "wrong"

- To cure this we consider instead the integration measure

$$\mathcal{P}(J_1(x)) = \prod_n \text{Pois}\left(J_1(x_n)dV(x_n), \lambda(x_n)dV(x_n)\right)(-1)^{\mathcal{F}}, \qquad (-1)^{\mathcal{F}} \equiv (-1)^{\int J_1(x)dV(x)} \underbrace{e^{2\int dV(x)\lambda(x)}}_{\text{normalization}}$$

- This leads to the averaged action

$$S_{\text{eff}} = \int dV(x)\left(\partial_\mu\phi\partial^\mu\phi - J_0(x)\phi + \lambda(x)(e^{\phi(x)} - 1)\right)$$

- Is there a more accurate description of what we did?

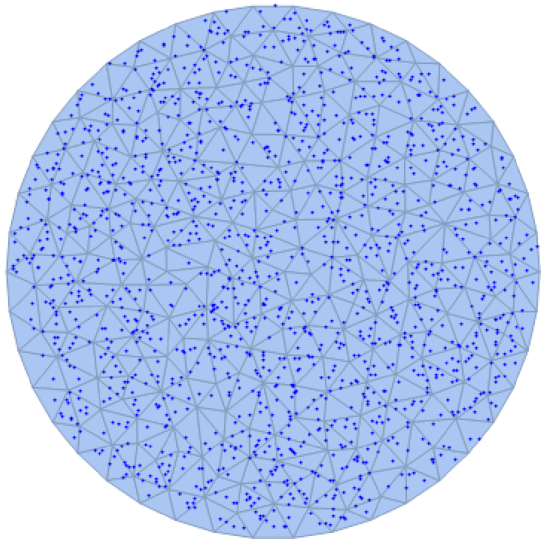
POISSON PROCESS - DEFINITION

- Consider the same theory, now reconsider it in terms of the Poisson Process.

Definition: Poisson process

A random countable subset Π on a given carrier space $\mathcal{M}$, s.t.

- For disjoint subsets $A_i \subset \mathcal{M}$, the random variable $N(A_i) \equiv \#(\Pi \cap A_i)$ are mutually independent
- $N(A_i)$ satisfies the Poisson distribution $\text{Pois}(N(A_i), \Lambda(A_i)) = e^{-\Lambda(A_i)} \dfrac{\Lambda(A_i)^{N(A_i)}}{N(A_i)!}$ with $\Lambda(A_i) = \mathbb{E}(N(A_i))$



- Mean measure $\Lambda(A)$, determined by the intensity function $\lambda(x)$

$$\Lambda(A) = \int_A \lambda(x) dV(x)$$

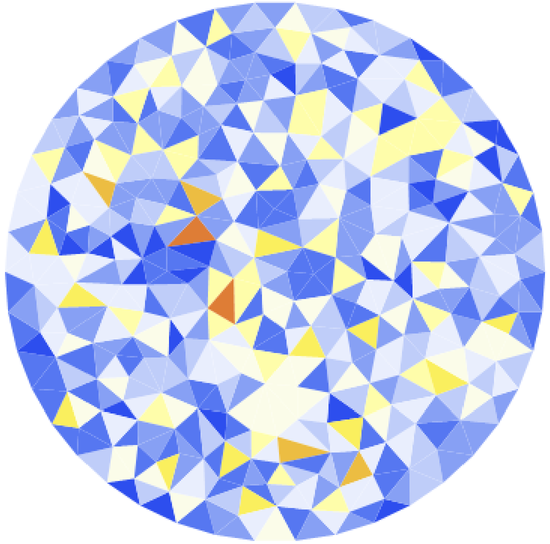
POISSON PROCESS - DEFINITION

- Consider the same theory, now reconsider it in terms of the Poisson Process.

Definition: Poisson process

A random countable subset Π on a given carrier space $\mathcal{M}$, s.t.

- For disjoint subsets $A_i \subset \mathcal{M}$, the random variable $N(A_i) \equiv \#(\Pi \cap A_i)$ are mutually independent
- $N(A_i)$ satisfies the Poisson distribution Pois$(N(A_i), \Lambda(A_i)) = e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^{N(A_i)}}{N(A_i)!}$ with $\Lambda(A_i) = \mathbb{E}(N(A_i))$



- Mean measure $\Lambda(A)$, determined by the intensity function $\lambda(x)$

$$\Lambda(A) = \int_A \lambda(x) dV(x)$$

POISSON PROCESS - INTERPRETATION

- In terms of the Poisson processes, we reinterpret our computation as follows:
  - The fluxes of the random source is identified with the random counting measure

$$\int_B dV(x) J_1(x) = N(B)$$

For infinitesimal subset $B \sim dV(x)$, we get

$$N(dx) = J_1(x) dV(x) \longrightarrow \int dV(x) J_1(x) \phi(x) = \int N(dx) \phi(x)$$

  - Average over the random source is identified with the Laplace functional

$$\mathbb{E}[e^{\int dV(x) J_1(x) \phi(x)}] = \mathbb{E}[e^{\int N(dx) \phi(x)}]$$

POISSON PROCESS - LAPLACE FUNCTIONAL

- Generically, the Laplace functional of a test function $f(x)$ is

$$\mathbb{E}\left[e^{\alpha\int_{\mathcal{M}} N(dx) f(x)}\right] = e^{\int_{\mathcal{M}} \Lambda(x)(e^{\alpha f(x)}-1)}$$

- In our computation, we thus have

$$\mathbb{E}[e^{\int dV(x) J_1(x)\phi(x)}] = \mathbb{E}[e^{\int \mathcal{J}(dx)\phi(x)}] = e^{\int_{\mathcal{M}} \Lambda(x)(e^{\phi}(x)-1)}$$

- Same as the result we got previously

- Sign flip can be accomplished by an $(-1)^{N(B)}$ factor

$$P(N(B)=n) = \frac{\left(-\Lambda(B)\right)^n}{n!} e^{\Lambda(B)}$$

ENSEMBLE AVERAGES – A SHORT SUMMARY

- What we have done is similar to the known example

SYK —(Average over random couplings)→ effective action —(Low energy limit)→ Schwarzian/JT gravity

Our model —(Average over random potentials)→ Liouville gravity

- We have demonstrated the importance of choosing
  - which set of theories to be averaged over (fluxes but not sources)
  - what is an appropriate measure for the average (the $(-1)^{\mathcal{F}}$ factor)

MICROSTATES - RATIONALE

- Recap:

the ensemble average of random potentials could be an analogue of some pseudo-randomness originated from the ignorance of some microscopic structure

- We will
  - Construct a microscopic model
  - Go to a special double scaled, low energy limit
  - Illustrate how pseudo-randomness appears
  - Demonstrate its equivalence to the type of true randomness we have discussed previously

MICROSTATES - A REALIZATION

- A lattice, sites labelled by a position vector x
- On each site: A complex fermion : $\bar{\psi}_x |0\rangle = |1\rangle, \quad N_x = \bar{\psi}_x \psi_x, \quad N_x |i\rangle = i|i\rangle, \quad i = 0,1$

  A real boson: ϕ_x
- The Hamiltonian of the system:

$$H = \sum_x H_{x,0} + H_{x,1}, \quad H_{x,0} = \Pi_x^2 + \frac{M}{2}\phi_x^2 + \sum_y t_{xy}\phi_x\phi_y + J_0(x)\phi_x, \quad H_{x,1} = m\bar{\psi}_x\psi_x - \bar{\psi}_x\psi_x\phi_x$$

- Prepare the system in the state

$$\rho = \rho_\phi \otimes \rho_\psi, \quad \rho_\psi = \bigotimes_x \rho_x, \quad \rho_x = (1 - p(x))|0\rangle_x\langle 0| + p(x)|1\rangle_x\langle 1|$$

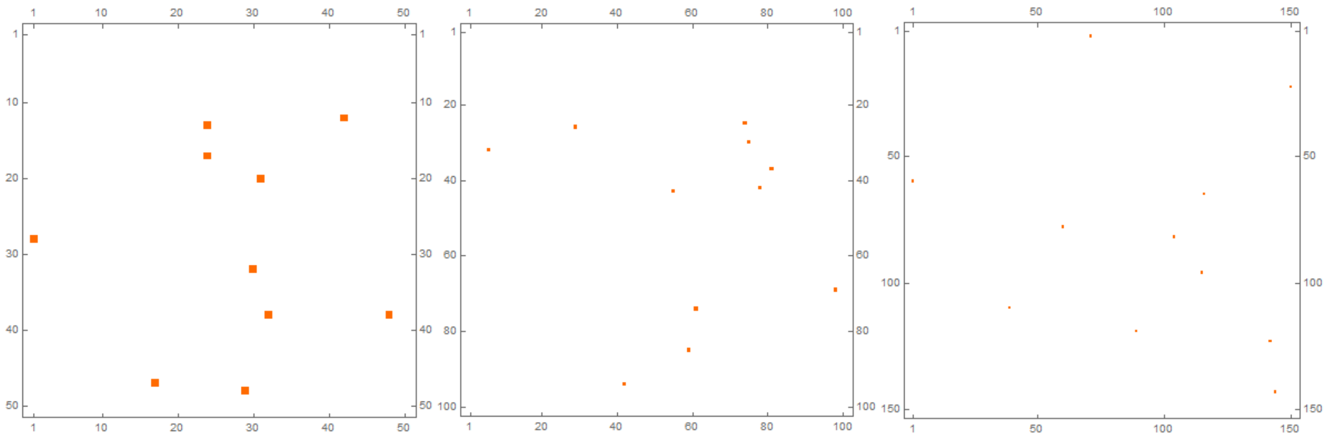
where $p(x)$ is the probability of fermionic excited state on site x

MICROSTATES - A DOUBLE SCALED LIMIT

- Consider the continuous limit: $n = \frac{1}{a^d} \to \infty$

  where a is the lattice spacing, n is the number of site per unit volume
- In this limit, countable infinite lattice sites in each open set.
- Further a double scaling limit: the number of sites (per unit volume) where fermionic d.o.f. is excited remains finite

$$\lim_{n\to\infty} np(x) \equiv \lambda(x), \qquad \lambda(x) \sim \mathcal{O}(1)$$



MICROSTATES - A DOUBLE SCALED LIMIT

- The fermionic factor of the density matric in a small enough subset $dV(x)$ is

$$\bigotimes_{x\in dV(x)} \rho_x = \bigotimes_{x\in dV(x)} (1-p(x))|0\rangle_x\langle 0| + p(x)|1\rangle_x\langle 1|$$

$$= \sum_k P(n_x = k)\left(|0\rangle_x\langle 0|\right)^{\otimes(n'-k)} \otimes \left(|1\rangle_x\langle 1|\right)^{\otimes k} \qquad P(n_x = k) = \binom{n'}{k} p(x)^k (1-p(x))^{n'-k}$$

- In the above limit, $P(n_x = k)$ becomes

$$\lim_{n'\to\infty} P_{dx}(n_x = k) = \frac{n'!}{k!(n'-k)!n'^k}(n'p)^k(1-\frac{n'p}{n'})^{n'-k} = \frac{\Lambda_{dx}(x)^k}{k!}e^{-\Lambda_{dx}(x)} = P_{\text{Pois}}(k, \Lambda_{dx}(x))$$

a Poisson distribution with $\Lambda_{dx}(x) = \lambda(x)dV(x)$.

MICROSTATES - TRACING OUT FERMIONS

- Next we get an effective action for the bosonic field ϕ_x

- Integrating over the fermionic degrees of freedom

$$e^{-\beta H_{\text{eff}}} = \mathrm{Tr}_{\mathcal{H}_\psi}\left((-1)^F \rho e^{-\beta H}\right) = \mathrm{Tr}_{\mathcal{H}_\psi}\left(\rho e^{-\beta H + i\pi F}\right) := \mathrm{STr}_{\mathcal{H}_\psi}\left(\rho e^{-\beta H}\right)$$

- Such a trace is chosen so that it is base free

MICROSTATES - TRACING OUT FERMIONS

- Tracing over the fermions leads to

$$e^{-\beta H_{\text{eff}}} = \mathrm{Tr}_{\mathcal{H}_\psi}\left(\bigotimes_{dV(x)} \rho_x e^{-\beta\sum_{dV(x)}\left(\sum_{x\in dV(x)}\left(H_x - \frac{i\pi}{\beta}F_{dx}\right)\right)}\right) = \prod_{dV(x)}\left(\sum_k \mathrm{Pois}(k,\Lambda_{dx}(x))e^{-\beta\left(k(m-\phi_x-\frac{i\pi}{\beta})\right)}\right) = e^{-\beta\int dV(x)\frac{\lambda(x)}{\beta}\left(e^{-\beta(m-\phi_x)}+1\right)}$$

- Redefining $b = \frac{\beta}{2}, \qquad \mu = \frac{\lambda}{2\pi\beta}e^{-\beta m} = \frac{\lambda}{4\pi b}e^{-2bm}$,

and adding back the pure bosonic terms, we get

$$\mathcal{H}_{\text{eff}}(x) = \pi_x^2 + \frac{M}{2}\phi_x^2 + \sum_y t_{xy}\phi_x\phi_y + J_0(x)\phi(x) + 2\pi\mu(x)e^{2b\phi_x} + \frac{\lambda(x)}{2b}$$

MICROSTATES – THE LOW ENERGY LIMIT

- Take the low energy limit by focusing on the lowest few Fourier modes

- For simplicity, we choose $t_{xy} = t_{yx} = \frac{1}{2} t \delta\left(\| x - y \| - 1\right)$

- The low energy effective theory is

$$\mathcal{H}_{\text{eff}}(x) = \pi_x^2 + \frac{1}{c^2}\left(\partial_x \phi(x)\right)^2 + m_\phi^2 \phi(x)^2 - J_0(x)\phi(x) + 2\pi\mu(x) e^{2b\phi(x)} + \frac{\lambda(x)}{2b}$$

where

$$m_\phi^2 = \frac{M}{2} - t, \qquad c^2 = \frac{2}{a^2 t} \geq 0, \qquad m_\phi, a, c \in \mathbb{R}$$

MICROSTATES - THE DUAL CHANNEL

- Can also integrate out the ϕ_x field to get a quantum mechanical model of the fermions

$$e^{-\int dtL_{\text{eff}}} = \int \mathcal{D}\phi_x e^{-\int dtL} \sim e^{\frac{1}{2}\log(g)+\int dt\left(i\sum_x \bar{\psi}_x\dot{\psi}_x + m\sum_x \bar{\psi}_x\psi_x + \frac{1}{2}g_{xy}\sum_{x,y}\bar{\psi}_x\psi_x\bar{\psi}_y\psi_y\right)}$$

where $g^{-1}{}_{xy} = -\partial_\tau^2\delta_{x,y} + M\delta_{x,y} + t_{x-1,x}\delta_{y,x-1} + t_{x,x+1}\delta_{y,x+1}$ and leads to a nearest neighbor coupling.

- Expanding by number of derivatives, the only relevant piece of the interaction is

$$g_{xy} = \left(M\delta_{x,y} + t_{x-1,x}\delta_{y,x-1} + t_{x,x+1}\delta_{y,x+1}\right)^{-1} = \frac{1}{M\sqrt{1-\frac{4t^2}{M^2}}}\sum_{p=-\infty}^{\infty}\frac{\left(\frac{2t}{M}\right)^{2|p|}}{\left(\sqrt{1-\frac{4t^2}{M^2}}+1\right)^{2|p|}}\delta_{y,x+2p} - \frac{\left(\frac{2t}{M}\right)^{2|p|+1}}{\left(\sqrt{1-\frac{4t^2}{M^2}}+1\right)^{2|p|+1}}\delta_{y,x+2p+1}$$

- The range of parameter is $\frac{M}{t} = 2(1+\frac{m_\phi^2}{t}) \geq 2$
- A branch cut at $m_\phi = 0$, corresponding to integrating out a massless mode.

A GRAVITY INTERPRETATION ?

- In previous analyses, probabilistic measures emerge. Interpret it as a geometric volume measure in gravity ?
- This helps understand Gravitational Path Integral = Ensemble Average of Theories

- Recall our effective action

$$S_{\text{eff}} = \int dV(x)\left(\partial_\mu^\mu \phi \partial \phi(x) - J_0(x)\phi + \lambda(x)(e^{\phi(x)} - 1)\right)$$

and the Liouville gravity action

$$S_{\text{L}} = \frac{1}{4\pi}\int \mathrm{d}^2 x\sqrt{|h|}\left(Q\Phi(x)R_h(x) + (\nabla\Phi)^2 + 4\pi\mu e^{2b\Phi(x)}\right)$$

A GRAVITY INTERPRETATION ?

- Comparing the actions, we find they are identical once we identify

$$Q\sqrt{|h|}R_h(x) = -J_0(x), \qquad 4\pi\mu\sqrt{|h|} = \frac{\lambda(x)}{2b}e^{-2mb}, \qquad \sqrt{|h|}h^{\mu\nu}\partial_\mu\partial_\nu = \delta^{\mu\nu}\partial_\mu\partial_\nu$$

- The last relation trivializes in the conformal gauge $h_{\mu\nu} = e^{\rho(x)}\delta_{\mu\nu}$, and the remaining two relations become

$$Q\delta^{\mu\nu}\partial_\mu\partial_\nu\rho(x) = J_0(x), \qquad 4\pi\mu e^{\rho(x)} = \frac{\lambda(x)}{2b}e^{-2mb}$$

- This gives the relation between the probabilistic measure $\lambda(x)$

  and the geometric measure $\rho(x)$

A GRAVITY INTERPRETATION ?

- Comments

1. This connection is only true if $J_0(x)$ correlates with $\lambda(x)$ according to

$$J_0(x) = Q\delta^{\mu\nu}\partial_\mu\partial_\nu \log\left(\lambda(x)\right)$$

   i.e. not all average of random theories have gravity descriptions

2. Curiously $J_0(x)$ was introduced as a source of $\phi(x)$: $J_0(x) = \delta^{\mu\nu}\partial_\mu\partial_\nu\phi(x)$. Recall $e^{\phi(x)}$ is originally the Weyl factor in getting Liouville; this put $\lambda(x)$ and $e^{\phi(x)}$ on the same footing, and confirms the geometric interpretation of $\lambda(x)$

3. The parameter Q sets up a scale.

4. The gravity description only captures the "mean" probability measure $\lambda(x)$, but not the details of the microscopic model. They could encode the information of the quantum aspects of gravity?

SUMMARY

- Quantum theories have discrete Hilbert spaces, so we consider averaging over theories with discrete random variables.
- Suitable ensemble average of these discrete theories, with a mathematically rigorous description in terms of Poisson processes.
- Averaged theories of this type have an equivalent description of tracing over parts of the microstates in a single theory.
- The results from both approaches mirror Liouville gravity.

THANK YOU!