Note on the Labelled tree graphs

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based on work with Yaobo Zhang, arXiv:2009.02394

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I: Motivation

- Scattering amplitude is one fundamental concept in the QFT. It connects the theoretical description and the experimental data.
- The well known standard method for the computation of scattering amplitudes is the Feynman diagrams. It has very nice physical picture, but when deal with theories with gauge symmetry, its practice faces a lot of challenge.
- In last twenty years, a big progress for its computation has been made, which is now called the On-shell program.

One of the on-shell frame is the CHY-frame by Cachazo, He and Yuan in 2013:

$$\mathcal{A}_n = \boxed{\int \frac{(\prod_{i=1}^{n} dz_i)}{d\omega}\Omega(\mathcal{E})}\mathcal{I}$$

[Freddy Cachazo, Song He, Ellis Ye Yuan , 2013, 2014]

In this frame:

- Each particle is represented by a puncture in Riemann sphere,i.e., a complex number z_i
- The expression holds for general D-dimension
- The box part is universal for all theories
- The CHY-integrand $\mathcal{I}$ determines the particular theory

For the universal part,

$$\Omega(\mathcal{E}) \equiv \prod_a{}' \delta(\mathcal{E}_a) = z_{ij} z_{jk} z_{ki} \prod_{a \neq i,j,k} \delta(\mathcal{E}_a)$$

provides the constraints:

- Scattering equations are defined

$$\mathcal{E}_a \equiv \sum_{b \neq a} \frac{2k_a \cdot k_b}{z_a - z_b} = 0, \qquad a = 1, 2, ..., n$$

- Only $(n-3)$ of them are independent by $SL(2, C)$ symmetry

$$\sum_a \mathcal{E}_a = 0, \quad \sum_a \mathcal{E}_a z_a = 0, \quad \sum_a \mathcal{E}_a z_a^2 = 0,$$

Universal part: $(n-3)$ integrations with $(n-3)$ delta-functions, so the integration becomes the sum over all solutions of scattering equations

$$\sum_{z\in \mathrm{Sol}} \frac{1}{\det'(\Phi)}\mathcal{I}(z)$$

where $\det'(\Phi)$ is the Jacobi coming from solving $\mathcal{E}_a$

$$\Phi_{ab}=\frac{\partial \mathcal{E}_a}{\partial z_b}=\begin{cases} \frac{s_{ab}}{z_{ab}^2} & a\neq b \\ -\sum_{c\neq a}\frac{s_{ac}}{z_{ac}^2} & a=b \end{cases},$$

The integrand $\mathcal{I}$ depends on the specific theory. So far, all physical theories have the form $\mathcal{I} = I_L \times I_R$. For example

$$\mathrm{PT}_n(\alpha)\mathrm{Pf}'\Psi_n, \quad \mathrm{Pf}'\Psi_n(\epsilon)\mathrm{Pf}'\Psi_n(\tilde{\epsilon}), \quad \mathrm{PT}_r(\alpha)\mathrm{Pf}\Psi_{n-r}(\epsilon)\mathrm{Pf}'\Psi_n(\tilde{\epsilon})$$

for YM, gravity, and single trace EYM.

- The Parke-Taylor factor $\mathrm{PT}_n(\alpha)$ is defined by

$$\mathrm{PT}_n(\alpha) = \frac{1}{\sigma_{\alpha_1\alpha_2}\sigma_{\alpha_2\alpha_3}\cdots\sigma_{\alpha_n\alpha_1}}.$$

- The reduced Pfaffian $\mathrm{Pf}'\Psi_n$ is

$$\mathrm{Pf}'\Psi_n = 2\frac{(-1)^{i+j}}{\sigma_{ij}}\mathrm{Pf}(\Psi^{ij}_{ij}).$$

The $2n \times 2n$ anti-symmetric matrix is

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix},$$

where

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}} & a \neq b, \\ 0 & a = b, \end{cases} \qquad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} & a \neq b, \\ 0 & a = b, \end{cases}$$

and

$$C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}} & a \neq b, \\ -\sum_{c=1, c\neq a} \frac{\epsilon_a \cdot k_c}{\sigma_{ac}} & a = b. \end{cases}$$

If we view the frame in a more abstracted way, CHY-frame defines a mapping from rational functions of z_α's to another rational function of k_i, ϵ_i's. Such a mapping can be summarized by the **"integration rule"**:

- For each subset, we define the **pole index** of each subset $A_i \subset \{1, 2, ..., n\}$ is defined as

$$\chi(A_i) \equiv L[A_i] - 2(|A_i| - 1) \ , \tag{1}$$

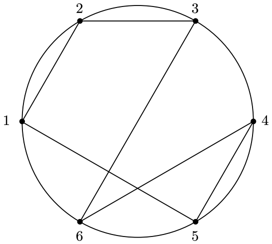
- For a given subset A with the pole index $\chi[A] \geq 0$, the amplitude could have terms with poles like $\frac{1}{s_{A_i}^{\chi[A_i]+1}}$, where

$$s_{A_i} = (\sum_{a \in A_i} k_a)^2 = (\sum_{b \in \overline{A}_i} k_b)^2 \ . \tag{2}$$

- By considering compatible conditions with $(n-3)$ poles , we get a possible term in the Feynman diagrams.

[Baadsgaard, Bjerrum-Bohr, Bourjaily and Damgaard, 2015] [Cardona, Feng, Gomez, Huang, 2016]

Example of 6-point



- Simple poles s_A with with $\chi[A] = 0$,

$$s_{12} \ s_{23} \ s_{45} \ s_{123}$$

- Maximum compatible combinations with $m = n - 3 = 3$

$$\{s_{12}\ , s_{45}\ , s_{123}\} \quad \{s_{23}\ , s_{45}\ , s_{123}\}$$

- The scalar propagator

$$\frac{1}{s_{12}s_{45}s_{123}} + \frac{1}{s_{23}s_{45}s_{123}}$$

Among them, one of theory, the Bi-Adjoint scalar theory is defined as

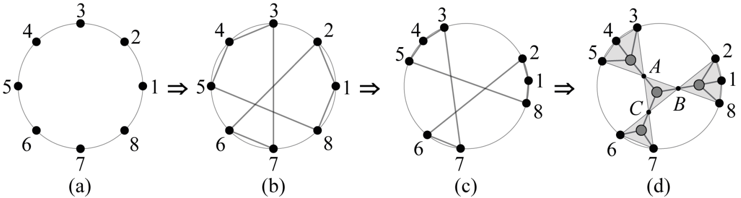
$$\mathcal{I} = \mathrm{PT}(\alpha)\mathrm{PT}(\beta)$$

has played very important role:

- It is the simplest theory in CHY frame.
- It is also the basis for all other theories, since it provides the skeleton of Feynman diagrams. Any other theories, can be written as the linear combination of bi-adjoint scalars.

One way to deal with it is given by CHY:

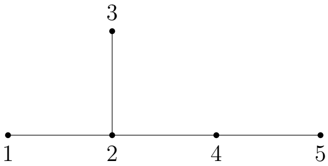
[Freddy Cachazo, Song He, Ellis Ye Yuan , 2013, 2014]



The cubic Feynman diagrams of $\mathcal{I}[(12345678)|(12673458)]$

$$\frac{1}{s_{128}}\frac{1}{s_{345}}\frac{1}{s_{67}}\left(\frac{1}{s_{12}}+\frac{1}{s_{18}}\right)\left(\frac{1}{s_{34}}+\frac{1}{s_{45}}\right)$$

Recently, in 1708.08701 Gao, He and Zhang have given a new class of weight two CHY-integrands (the so called "Cayley functions"), which largely generalized the Parke-Taylor factor.



[X. Gao, S. He, and Y. Zhang]

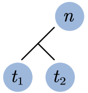
- The Cayley functions T_{n-1} could be understood better in the gauge fixed form $z_n \to \infty$. With this gauge fixing, each Cayley function is mapped to a **labelled tree graph** with $(n-1)$-nodes $1, 2, ..., n-1$ and $(n-2)$ edges connecting these nodes.
- From a Cayley tree, we can read out the corresponding weight two CHY-integrand as

$$C_n(T_{n-1}) := \frac{1}{\prod_{\{i,j\} \in Edges(T_{n-1})} (z_i - z_j)} \tag{3}$$

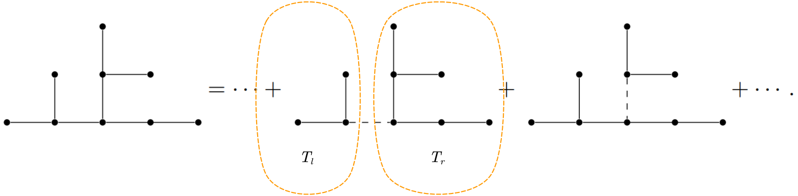
or the $SL(2, C)$ covariant form

$$C_n(T_{n-1}) := \frac{\prod_{k \in Vertexes(T_{n-1})} (z_k - z_n)^{v_k - 2}}{\prod_{\{i,j\} \in Edges(T_{n-1})} (z_i - z_j)} \tag{4}$$

In 1708.08701 the iterative construction of cubic Feynman diagrams of CHY-integrands $(C_n(T_{n-1}))^2$ with $C_n(T_{n-1})$ defined by Cayley tree's has been given by a recursive method.

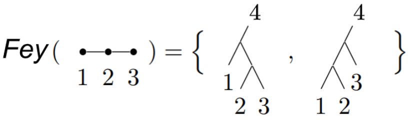
$$Fey(T_{n-1}) = \bigsqcup_{T_l \bullet\!-\!-\!-\!\bullet T_r \,=\, T_{n-1}} \left\{ \right.$$


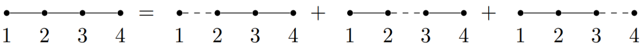
$$\left. \middle| \; t_1 \in Fey(T_l), t_2 \in Fey(T_r) \right\}$$

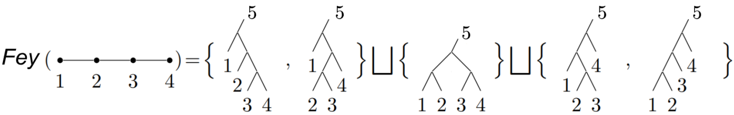


[X. Gao, S. He, and Y. Zhang]

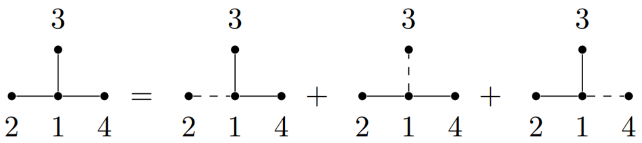
Examples

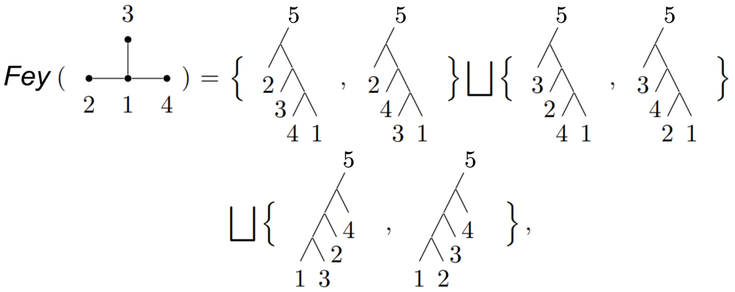




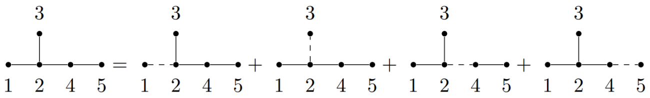


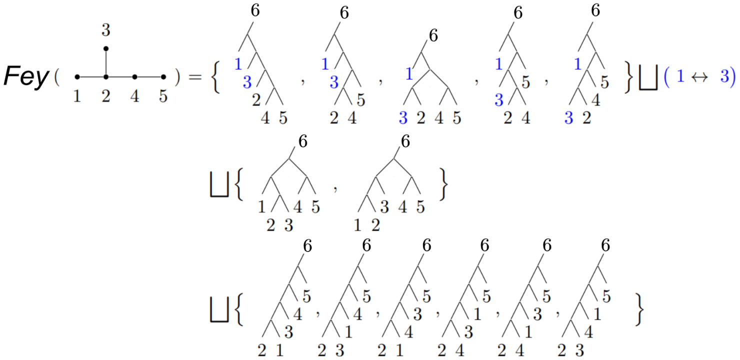
Examples





Examples





[X. Gao, S. He, and Y. Zhang]

- For arbitrary Cayley tree's, the number of Feynman diagrams depends on all its subdivisions.

$$\|Fey(T_{n-1})\| = \sum_{T_l \bullet\!-\!-\!-\!\bullet T_r = T_{n-1}} \|Fey(T_l)\| \, \|Fey(T_r)\|$$

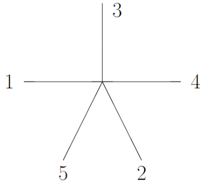
- When the topology of Cayley tree is simple (such as a line or star), the corresponding Feynman diagrams has a symmetry feature.
- For complicated Cayley tree's, there are a lot of Feynman diagrams, thus a good way to organize these diagrams will be very useful for our further understanding of various questions related to these theories.

- We will show that they can be re-organized to some much simpler effective Feynman diagrams, for complicated cubic tree Feynman diagrams produced by "Labelled tree graphs".
- Using these effective Feynman diagrams, it is much easier to capture the theory, since the pole structure will be much more clear and organized and the connection to geometric picture (i.e., the combinatoric polytope) will be more transparent.
- For generalization of the bi-adjoint theory, we have suggested an algorithm to pick out terms with a given pole structure.

II:Effective Feynman diagrams

Among all Cayley tree's, there are two special types, for which the pole structures are clear.

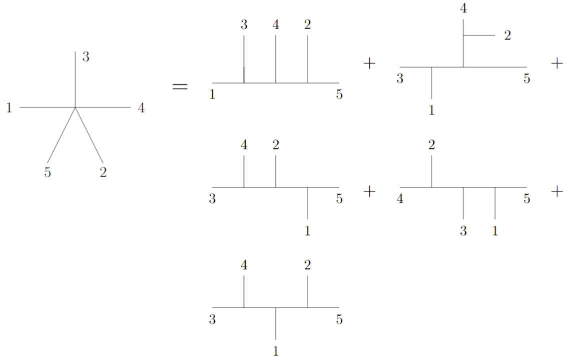
- The first one is just a line, for example, the Cayley tree T_4 with edge-list $\{\{1,3\},\{3,4\},\{4,2\}\}$.



When rewriting into the $SL(2,C)$ covariant form, it is nothing but the familiar Parke-Taylor graph.

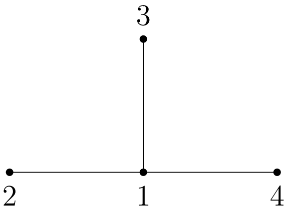
Since we know the pole structure of these Feynman diagrams, we can compactly represent them by "an effective Feynman vertex" V_C, which is defined as

$$V_C(\alpha) \equiv \{\text{the sum of all } \alpha \text{ color ordered cubic Feynman diagrams}\} \quad (5)$$



(The expansion of the effective vertex $V_C(1,3,4,2,n=5)$)

- The second type is is the star graph (node), where among $(n-1)$-points, $(n-2)$ of them connect to the remaining point.



(Star graph of $n=5$)

The "shuffle" algebra.

For a two ordered sets, their shuffle is defined as

$$\alpha \sqcup\!\sqcup \emptyset = \alpha, \quad \emptyset \sqcup\!\sqcup \beta = \beta,$$
$$\alpha \sqcup\!\sqcup \beta = \{\alpha_1, \{\alpha_2, ..., \alpha_m\} \sqcup\!\sqcup \beta\} + \{\beta_1, \alpha \sqcup\!\sqcup \{\beta_2, ..., \beta_k\}\} \quad (6)$$

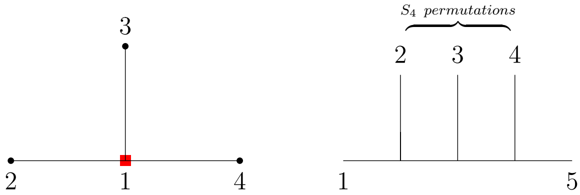
Using this notation, all Feynman diagrams coming from the star graph can be summarized as $(a_1; \{a_2\} \sqcup\!\sqcup \{a_3\} ... \sqcup\!\sqcup \{a_{n-1}\}; a_n)$, which will be the sum of sequences of the form

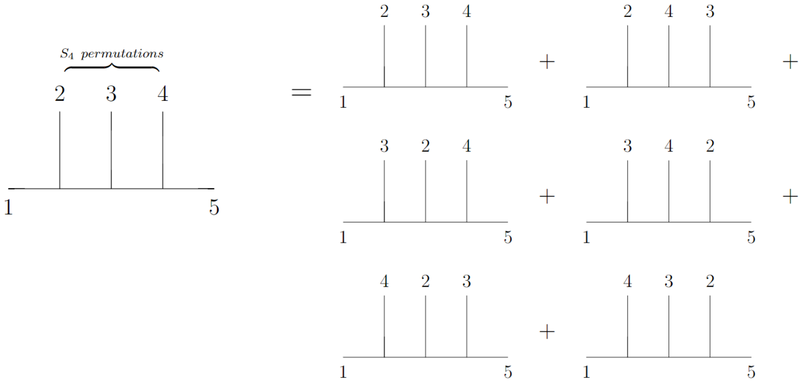
$$(a_1; a_2, a_3, ..., a_k; a_n) \to \frac{1}{\prod_{t=2}^{n-2} s_{a_1 a_2 ... a_t}} \quad (7)$$

Because the pattern for the star graph, we can compactly represent them by another **"effective Feynman vertex"** V_P ,where the subscript P means the P-type vertex (the permutation type vertex).

Example: $V_P(1;\{2\}\sqcup\{3\}\sqcup\{4\};5)$

The $V_P(1;\{2\}\sqcup\{3\}\sqcup\{4\};5)$ vertex contains three parts: the starting external leg 1, the ending external leg $n=5$ and the middle sequence coming from shuffle algebra.

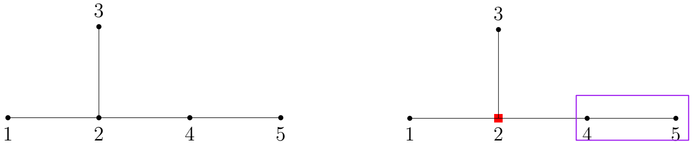



S_4 permutations
2 3 4
1 5
=
2 3 4
1 5
+
2 4 3
1 5
+
3 2 4
1 5
+
3 4 2
1 5
+
4 2 3
1 5
+
4 3 2
1 5

These two special Cayley tree's correspond to the line and vertex (with multiple branches) structures in the general tree graphs respectively. Thus it is very natural to guess that there should be able to properly combine about two special structures of Feynman diagrams to compactly represent all Feynman diagrams of a given arbitrary Cayley tree.

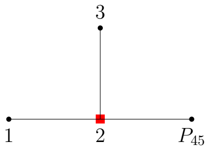
Now, we use next-to-Star graph of $n = 6$ as examples to demonstrate the idea, especially the algorithmic way to read out all effective Feynman diagrams.

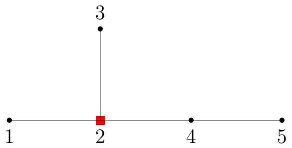
- (A) At the first step, choose a marked node, for example, k.



- (B) Then remove all edges connecting to the marked node k. Now the Cayley tree is separated to node k and several subgraphs, which we can denote as $K_1, K_2, .., K_t$.

- (C) For each subgraph K_i, we need to consider its all substructures by all possible contractions of edges in the subgraph. For a given contraction, we will generate following data:
  - (1) First we shrink all contracted edges in the subgraph K_i (so nodes at the two ends of the edge will be merged to a single node) to generate a new graph $\widetilde{K}_i$;
  - (2) Secondly, all edges having been contracted will become several disconnected sub-Cayley tree, which we will denote as $K_{i,j}$.

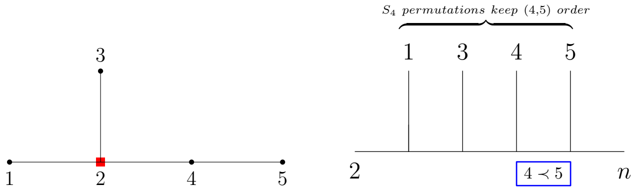




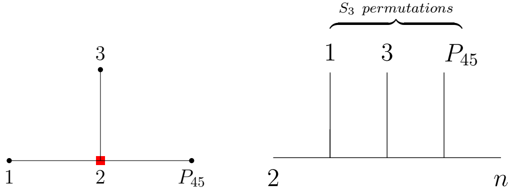
- (D) Having data from the step (C), now we have an effective vertex V_P, which is given by reconnecting the node k with all $\widetilde{K}_i$. The roughly expression will be

$$V_P(k; \widetilde{K}_1 \sqcup\!\sqcup \widetilde{K}_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \widetilde{K}_t; n) \tag{8}$$

where for each $\widetilde{K}_i$ in the shuffle algebra.

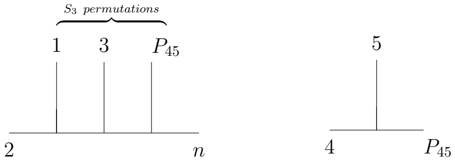


$V_P(2; \{1\} \sqcup\!\sqcup \{3\} \sqcup\!\sqcup \{4, 5\}; n = 6)$



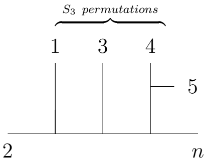
$V_P(2; \{1\} \sqcup \{3\} \sqcup \{P_{45}\}; n = 6)$

- (E) For each sub-Carley tree $K_{i,j}$, we can repeat the steps from (A) to (D) to get the corresponding effective vertex $V_{K_{i,j}}$.



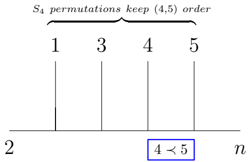
$V_P(2; \{1\} \sqcup \{3\} \sqcup \{P_{45}\}; n = 6)$ and $V_C(\{4, 5, P_{45}\})$

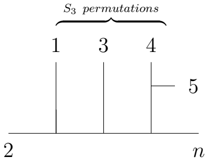
- (F) Then we connect vertex $V_{K_{i,j}}$ to the merged node in $\widetilde{K}_i$ by corresponding propagator to construct the effective Feynman diagram.



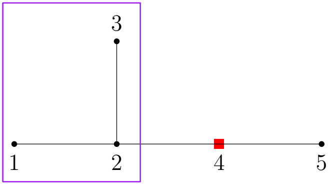
$$V_P\left(2;\{1\}\sqcup\{3\}\sqcup\{P_{45}\};n=6\right)\frac{1}{P_{45}^2}V_C\left(\{4,5,P_{45}\}\right)$$

- (G) Iterating above steps, we will arrive effective Feynman diagrams for any Cayley tree.

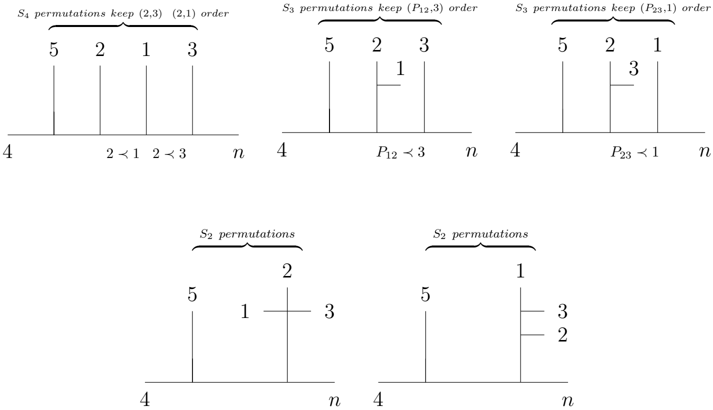




In above construction of effective Feynman diagrams, we have used the node 2 as the starting (marked) point. In general, we can choose any point to start with the whole construction.



The node 4 as the marked point



The Effective Feynman diagram representation from node 4

Enumerate Feynman diagrams from effective Feynman diagram

Each effective Feynman diagram will code several cubic Feynman diagrams, thus the counting of these cubic Feynman diagrams is a very important check for this algorithm.

- For an effective Feynman diagram, its counting N_F is given by

$$N_F = \prod_{i=1}^{t} n_i \tag{9}$$

where t is the number of effective vertexes in the diagram and for each effective vertex, n_i is the number of its expansion to cubic Feynman sub-diagrams.

The effective vertex can only be two types, i.e., either the P-type or the C-type.

- For the C-type effective vertex, the $|V_C\{(l_1, l_2, ..., l_n)\}|$ enumerates all the cubic Feynman trees respecting the colour order of the list $\{(l_1, l_2, ..., l_n)\}$. It is the n-th Catalan number Cat_n, which is given directly in terms of binomial coefficients by

$$\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^{n}\frac{n+k}{k} \qquad \text{for } n \geq 0 \quad (10)$$

$$\mathrm{Cat}_2 = 1, \ \ \mathrm{Cat}_3 = 2, \ \ \mathrm{Cat}_4 = 5, \ \ \mathrm{Cat}_5 = 14, \qquad (11)$$

$$\mathrm{Cat}_6 = 42, \ \ \mathrm{Cat}_7 = 132, \ \ \mathrm{Cat}_8 = 429 \qquad (12)$$

- For the P-type vertex, two arbitrary ordering list $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_n\}$. The shuffle $\alpha \sqcup\!\sqcup \beta$ contains all possible permutations of the list $\alpha \cup \beta$, which preserve the relative ordering in α and β respectively. Thus the counting is given by

$$|\alpha \sqcup\!\sqcup \beta| = \frac{(m+n)!}{m!n!}, \tag{13}$$

This counting can be easily generalized to multiple lists, thus we have

$$|V_P(o; \alpha_1 \sqcup\!\sqcup \alpha_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \alpha_k; n)| = \frac{(\sum_{i=1}^{k} |\alpha_i|)!}{\prod_{j=1}^{k} |\alpha_j|!} \tag{14}$$

when all list α_i's do not have substructure.

When the V_P has some substructures, we could recursively count the number. For example

$$V_P(1;\{2,3 \shuffle 4\} \shuffle \{5,6 \shuffle 7\};8)$$

there are two levels of shuffle algebra.

- The first layer $\{2,3 \shuffle 4\} \shuffle \{5,6 \shuffle 7\}$, using the formula, we get the counting

$$\frac{(3+3)!}{3!3!} = 20$$

.

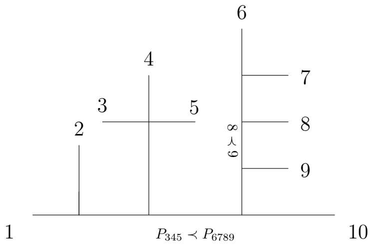
- The second shuffle layer, i.e., $\{3 \shuffle 4\}$ and $\{6 \shuffle 7\}$, each of them gives the number

$$\frac{(1+1)!}{1!1!} = 2$$

.

Putting them together, we get the counting $20 \times (2 \times 2) = 80$.

For the example with mixing V_C and V_P types, its expression is given by



$$V_P(1;\{2\} \sqcup \{P_{345}, P_{6789}\}; 10)\frac{1}{P_{345}} V_C(\{3,4,5,P_{345}\}) \quad (15)$$

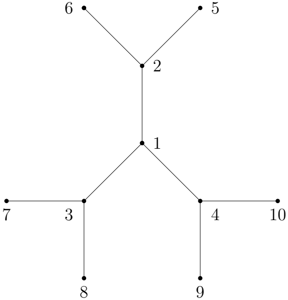
$$\frac{1}{P_{6789}} V_P(6;\{7\} \sqcup \{8,9\}; P_{6789}) \quad (16)$$

and the counting is given by

$$\frac{(1+2)!}{2} \times \frac{1}{2+1} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \times \frac{(1+2)!}{2} = 18 \quad (17)$$

More examples

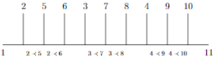
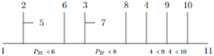
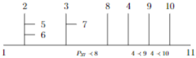
The effective Feynman diagrams can very compactly encode all cubic Feynman diagrams coming from a given Cayley tree, especially this Cayley tree has symmetric structure.

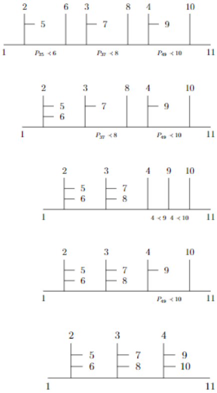


$C_{11}\{\{1,2\},\{1,3\},\{1,4\},\{3,7\},\{3,8\},\{2,5\},\{2,6\},\{4,9\},\{4,10\}\}$

These 32 diagrams can be divided into ten types:

- (1) one without any contraction;
- (2) six of one contraction;
- (3) twelve of two contractions at the different branches;
- (4) three of two contractions at the same branch;
- (5) twelve of three contractions at the two different branches;
- (6) eight of three contractions at the three different branches;
- (7) twelve of four contractions at the three different branches;
- (8) three of four contractions at the two different branches;
- (9) six of five contractions at the three different branches;
- (10) one of six contractions.

EFD	counting	# of EFD
	$\frac{(3+3+3)!}{3!3!3!} \times 2! \times 2! \times 2! = 13440$	1
	$\frac{(2+3+3)!}{2!3!3!} \times 2! \times 2! = 2240$	6
	$\frac{(2+2+3)!}{2!2!3!} \times 2! = 420$	12
	$\frac{(1+3+3)!}{1!3!3!} \times 2! \times 2! \times 2! = 1120$	3
	$\frac{(1+2+3)!}{1!2!3!} \times 2! \times 2! = 240$	12



$\frac{(2+2+2)!}{2!2!2!} = 90$	8
$\frac{(1+2+2)!}{1!2!2!} \times 2! = 60$	12
$\frac{(1+1+3)!}{1!1!3!} \times 2! \times 2! \times 2! = 160$	3
$\frac{(1+1+2)!}{1!1!2!} \times 2! \times 2! = 48$	6
$\frac{(1+1+1)!}{1!1!1!} \times 2! \times 2! \times 2! = 48$	1

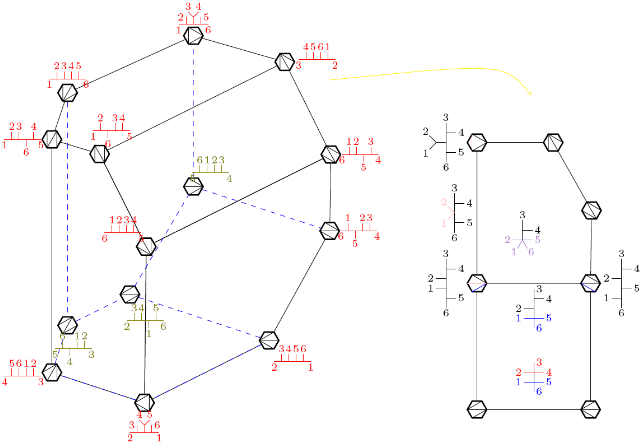
These effective diagrams have coded 40416 cubic Feynman diagrams

Application of effective Feynman diagrams

The first application is to a geometric object, the so called "Polytope of Feynman diagrams", which is defined for a collection of cubic Feynman diagrams of n points as following

- (1) Each vertex of this polytope corresponds to a cubic Feynman diagrams (so there are $(n-3)$ poles).
- (2) Two vertexes will be connected by an edge when and only when they share same $(n-4)$ poles.
- (3) All vertexes on a surface share same $(n-5)$ poles.
- (4) In general, vertexes of a dimension r surface share same $(n-3-r)$ poles.

Above construction of polytope has used the **bottom-up** approach. Our definition of effective vertexes has used an opposite approach, i.e., the **top-down** method. For example, for the CHY-integrand $(\mathrm{PT}(\{1, 2, ..., n\}))^2$, all cubic Feynman diagrams are represented by a single effective vertex $V_C(\{1, 2, ..., n\})$. This single vertex corresponds the $(n-3)$-dimension polytope, the so called "associahedron".



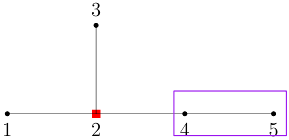
Associahedron of $(\mathrm{PT}(\{1,2,3,4,5,6\}))^2$

Its codimension one boundary correspond to fix a givenpole, which corresponds to split the single V_C-type effective vertex to two V_C-type effective vertexes connected by this given pole, i.e,

$$V_C(\{1,2,3,4,5,6\}) \rightarrow \begin{cases} V_C(\{1,2,P_{12}\})\frac{1}{s_{12}}V_C(\{P_{12},3,4,5,6\}) \\ 6 \text{ cases }: s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61} \\ \\ V_C(\{1,2,3,P_{123}\})\frac{1}{s_{123}}V_C(\{P_{123},4,5,6\}) \\ 3 \text{ cases }: s_{123}, s_{234}, s_{345} \end{cases} \quad (18)$$

Thus there are 9 faces. By counting each effective Feynman diagram, we see that 6 faces have five edges and five vertexes while 3 faces have four edges and four vertexes. We can split effective vertex further to get the representation of edges, such a picture has been discussed in arXiv:1801.08965.

The same splitting picture holds for the V_P-type effective vertex. For next-to-Star graph,



the whole polytope is given by two effective Feynman diagrams:

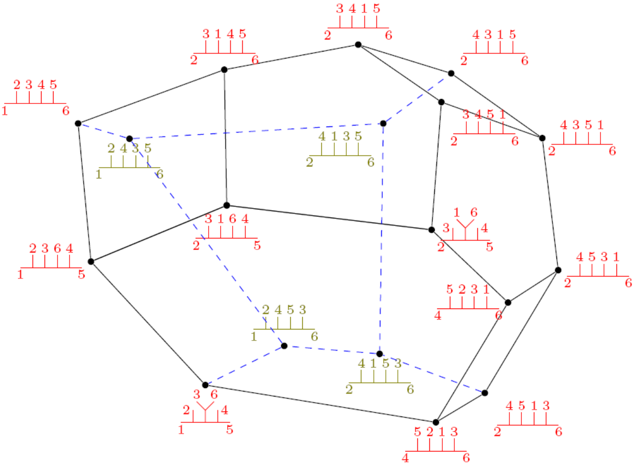
$$
\begin{aligned}
F_A &= V_P(2;\{1\}\sqcup\{3\}\sqcup\{P_{45}\};n)\frac{1}{P_{45}^2}V_C(\{4,5,P_{45}\}) \\
F_B &= V_P(2;\{1\}\sqcup\{3\}\sqcup\{4,5\};n) \qquad (19)
\end{aligned}
$$

- the effective diagram F_A has the fixed pole s_{45}, it defines a two-dimension surface.
- For the effective diagram F_B, when considering the relative orderings we have following types of splitting

$$F_B = V_P(2; \{1\} \sqcup \{3\} \sqcup \{4,5\}; n)$$

$$\to \begin{cases} V_P(2;\{1\};P_{12})\frac{1}{s_{12}}V_P(P_{12};\{3\}\sqcup\{4,5\};n\} \\ \text{3 cases}: s_{21}, s_{23}, s_{24} \\ \\ V_P(2;\{1\}\sqcup\{3\};P_{123})\frac{1}{s_{123}}V_P(P_{123};\{4,5\};n\} \\ \text{4 cases}: s_{213}, s_{214}, s_{234}, s_{245} \\ \\ V_P(2;\{1\}\sqcup\{3\}\sqcup\{4\};P_{1234})\frac{1}{s_{1234}}V_P(P_{1234};\{5\};n) \\ \text{3 cases}: s_{2134}, s_{2145}, s_{2345} \end{cases} \quad (20)$$

Adding together, we find the polytope has 11 two-dimension surfaces. Each surface is defined by an effective Feynman diagram. Counting the effective Feynman diagrams, we can find that there are 4 surfaces with four edges, 5 surfaces with five edges and 2 surfaces with six edges. Which two surfaces share an edge can also be easily identified.



The polytope of the Next-to-Star graph with n=6

III: Pick up Poles

For a bi-adjoint scalar theory defined by two PT-factors, there is a way to extract a subset of all Feynman diagrams containing a particular pole structure. It is given by using following cross-ratio factor

$$\mathcal{P}^{ac}_{bd} := \frac{[ac][bd]}{[ad][bc]} \quad , \quad [ab] := z_{ab} \tag{21}$$

[B. Feng,, 2016]

For example, with the CHY-integrand

$$\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{56}^2 z_{61}^2}$$

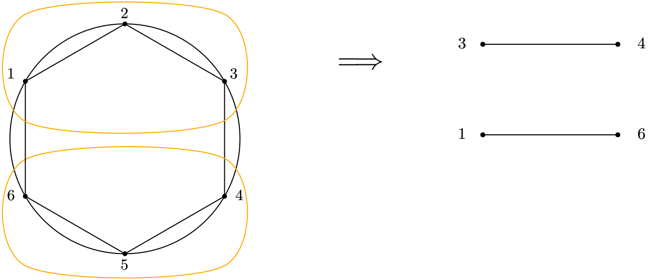
we produce following fourteen Feynmann diagrams up to a sign

$$\frac{1}{s_{12}s_{34}s_{56}}+\frac{1}{s_{12}s_{56}s_{123}}+\frac{1}{s_{23}s_{56}s_{123}}+\frac{1}{s_{12}s_{34}s_{126}}+\frac{1}{s_{16}s_{34}s_{126}}+\frac{1}{s_{16}s_{23}s_{156}}+\frac{1}{s_{16}s_{34}s_{156}}+\frac{1}{s_{23}s_{56}s_{156}}+\frac{1}{s_{34}s_{56}s_{156}}+\frac{1}{s_{16}s_{23}s_{45}}+\frac{1}{s_{12}s_{123}s_{45}}+\frac{1}{s_{23}s_{123}s_{45}}+\frac{1}{s_{12}s_{126}s_{45}}+\frac{1}{s_{16}s_{126}s_{45}}\,. \tag{22}$$

To pick out all items containing s_{123}.

- First we split the whole set into the pole set $A = \{1,2,3\}$ and its complement $\overline{A} = \{4,5,6\}$
- Define the set $\mathrm{Links}[A, \overline{A}]$ as the collections of lines connecting the set A and $\overline{A}$. Each line will be represented by two nodes: one is in A and another one, in $\overline{A}$
- Furthermore, we should distinguish the solid line(corresponding the factor $[ab]$ in the denominator) and dashed line(corresponding the factor $[ab]$ in the numerator) by $\underline{\{a,b\}}$ and $\overline{\{a,b\}}$ respectively.

Now using the pair in the Links set, we can construct a single cross-ratio $\mathcal{P}_{34}^{16} = \frac{z_{16}z_{34}}{z_{14}z_{36}}$.



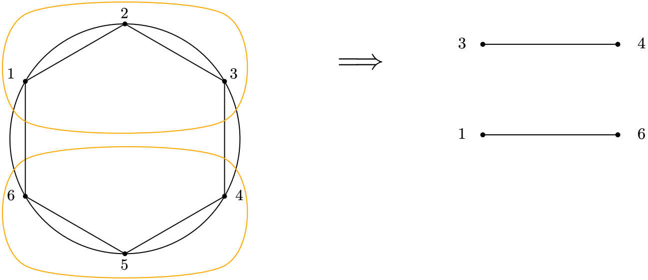
Links[{1, 2, 3}, {4, 5, 6}] of integrand $\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{56}^2 z_{61}^2}$

One can easily check that by multiplying such cross-ratio factor to the original CHY-integrand

$$\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{56}^2 z_{61}^2} \mathcal{P}_{34}^{16} = \frac{1}{z_{12}^2 z_{14} z_{16} z_{23}^2 z_{34} z_{36} z_{45}^2 z_{56}^2} \quad (23)$$

produce only four terms all containing the pole s_{123}

$$\frac{1}{s_{12} s_{45} s_{123}} + \frac{1}{s_{23} s_{45} s_{123}} + \frac{1}{s_{12} s_{56} s_{123}} + \frac{1}{s_{23} s_{56} s_{123}} \quad (24)$$



$Links[\{1,2,3\},\{4,5,6\}]$ of integrand $\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{56}^2 z_{61}^2}$

In many situations, such as the soft limitand collinear limit, we want to know the singular behavior of a given amplitude. These singular behaviors are connected with particular poles, thus how to isolating contributionsfrom these poles becomes important to many studies.

- By removing these singularcontributions, the two-loop CHY-integrand of the planar bi-adjoing scalar theory has beenconstructed.

  [B. Feng , 2016,2014]

- The technique of picking poles has been used to study the symmetryproperties of different PT-integrands.

  [R. Huang, F. Teng, and B. Feng, 2018]

- The same technique has been used to thecontraction of the one-loop CHY-integrand for general bi-adjoing scalar theory

  [B. Feng and C. Hu, 2019]

- Another interesting point of picking pole technique is following. In the previous part,we have talked about the effective Feynman diagrams and the corresponding geometric picture, i.e., polytope. For the square Cayley integrand, all Feynman diagrams constitute a high-dimensional polytope. The process of picking out a particular pole corresponds precisely to the operation of projecting from a high-dimensional volume onto a specific face.

In these mentioned applications, the pole picking is constraint to the CHY-integrandsof two PT-factors. We will give an algorithm to pick up aparticular pole for the most general CHY-integrands, which do not contain any higher order poles.

The General algorithm

- Unlike the bi-adjoint scalar theory, the general CHY-integrand of weight four contain both denominators and numerators and in general, amplitudes will depend on both, so the constructed cross-ratio factor of picking pole should depend on both too.
- Another important fact is that the index of a given pole is determined by links inside these nodes, thus the cross-ratio factor should not affect the index, which means lines used in the construction should come from these lines connecting the set A and its complement $\overline{A}$.

The General algorithm

In general case, there are two types of linking lines. Thus when we construct the cross-ratio factor, we should put their role into count.

- In the first situation where both linking lines are solid lines, for example, $\underline{\{a, c\}}$ and $\underline{\{b, d\}}$, we will call them **pure primary cross-ratio factor**.

$$\mathcal{P}_{bd}^{ac} := \frac{[ac][bd]}{[ad][bc]} \quad , \qquad [ab] := z_{ab} \tag{25}$$

- In the second situation where both linking lines are dashed lines, for example, $\overline{\{a, c\}}$ and $\overline{\{b, d\}}$, which we will write as a inverse

$$\bar{\mathcal{P}}_{bd}^{ac} \equiv \frac{z_{ad} z_{bc}}{z_{ac} z_{bd}} = (\mathcal{P}_{bd}^{ac})^{-1} \tag{26}$$

- The third situation is most tricky one, where one linking line is solid and another one, dashed, for example, $\overline{\{a,b\}}$ and $\underline{\{e,f\}}$. However, using this pair, it is impossible to reach the goal. A way to solve the difficulty is to involve another linking line, for example, $\overline{\{c,d\}}$ and define following combination

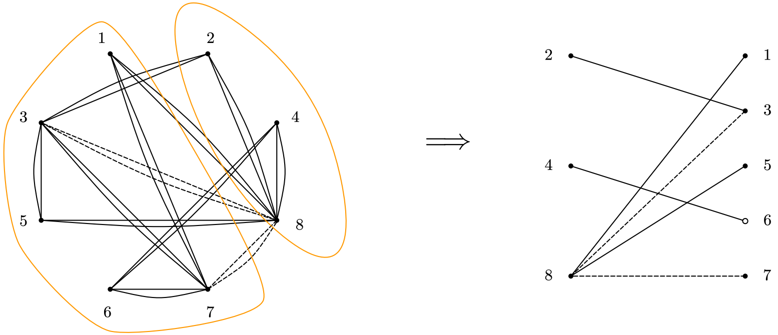
$$\mathcal{P}^{\overline{ab}}_{\underline{ef};\overline{cd}} \equiv \mathcal{P}^{ab}_{cd}\bar{\mathcal{P}}^{cd}_{ef} = \frac{z_{ab}z_{cd}}{z_{ad}z_{cb}}\frac{z_{ed}z_{cf}}{z_{ef}z_{cd}} = \frac{z_{ab}z_{ed}z_{cf}}{z_{ef}z_{ad}z_{cb}} \qquad (27)$$

We will call $\mathcal{P}^{\overline{ab}}_{\underline{ef};\overline{cd}}$ the **mixed rimary cross-ratio factor**. There is another possibility, i.e., involving $\underline{\{c,d\}}$. We must point out, although the line $\overline{\{c,d\}}$ acts as an intermediate variable, we do need to impose the condition $a \neq c,\ e \neq c$ and $b \neq d,\ f \neq d$.

The General algorithm

Based on examples, now we present our algorithm:

- (1) Given a CHY-integrand, draw the corresponding 4-regular graph, where the factor in the denominator is represented by solid line while the factor in the numerator is represented by dashed line.
- (2) To pick up the pole s_A, we divide all nodes of the graph into two subsets, i.e., the A and its complement $\overline{A}$. Now there are lines connecting the subset A and $\overline{A}$. Collecting them (by removing duplications) we form the linking set $\text{Links}[A, \overline{A}]$.



The partition of the CHY-integrand $\frac{z_{38}^2 z_{78}^2}{z_{17}^2 z_{18}^2 z_{23}^2 z_{28}^2 z_{35}^2 z_{37}^2 z_{46}^2 z_{48}^2 z_{58}^2 z_{67}^2}$

- (3) From the linking set $\mathrm{Links}[A, \overline{A}]$, we generate four collections of primary cross-ratio factors:
  - (I) Using two solid lines according to the formula;

$$\mathcal{P}_{23}^{18} = \frac{z_{18}z_{23}}{z_{12}z_{38}} \quad \mathcal{P}_{46}^{18} = \frac{z_{18}z_{46}}{z_{14}z_{68}} \quad \mathcal{P}_{46}^{23} = \frac{z_{23}z_{46}}{z_{26}z_{34}}$$
$$\mathcal{P}_{58}^{23} = \frac{z_{23}z_{58}}{z_{25}z_{38}} \quad \mathcal{P}_{58}^{46} = \frac{z_{46}z_{58}}{z_{45}z_{68}}$$

  - (II) Using two dashed lines according to the formula;
  - (III) Using two solid lines and one dashed line according to the formula.

$$\mathcal{P}_{78;23}^{18} = \frac{z_{18}z_{27}}{z_{12}z_{78}} \quad \mathcal{P}_{38;46}^{23} = \frac{z_{23}z_{68}}{z_{26}z_{38}} \quad \mathcal{P}_{78;23}^{46} = \frac{z_{27}z_{38}z_{46}}{z_{26}z_{34}z_{78}}$$
$$\mathcal{P}_{78;46}^{23} = \frac{z_{23}z_{47}z_{68}}{z_{26}z_{34}z_{78}} \quad \mathcal{P}_{78;23}^{58} = \frac{z_{27}z_{58}}{z_{25}z_{78}} \quad \mathcal{P}_{38;46}^{18} = \frac{z_{18}z_{34}}{z_{14}z_{38}}$$
$$\mathcal{P}_{78;46}^{18} = \frac{z_{18}z_{47}}{z_{14}z_{78}} \quad \mathcal{P}_{38;46}^{58} = \frac{z_{34}z_{58}}{z_{38}z_{45}} \quad \mathcal{P}_{78;46}^{58} = \frac{z_{47}z_{58}}{z_{45}z_{78}}$$

  - (IV) Using two dashed lines and one solid line according to the inverse of the formula.

- (4)We need to have some criterions to see if it is the right answer. The first criterion is that
  - **Criterion I:** The new CHY-integrand, i.e, the multiplication of original CHY-integrand and the pick-factor, should not contain any new poles or higher order poles comparing with the original CHY-integrand.

  It is worth to emphasize that when we say it does not create new poles or higher poles, we are just calculate the **pole index** of a subset A_i as $\chi(A_i) \equiv L[A_i] - 2(|A_i| - 1)$,
  - **Criterion II:** The pick-factor should remove all incompatible poles of the original integrand.

  More explicitly, for each pick-factor in the list, we calculate the remaining poles (by just calculating the pole index) after multiplying it to the original CHY-integrand.

- (4)We need to have some criterions to see if it is the right answer. The first criterion is that
  - **Criterion I:** The new CHY-integrand, i.e, the multiplication of original CHY-integrand and the pick-factor, should not contain any new poles or higher order poles comparing with the original CHY-integrand.

  It is worth to emphasize that when we say it does not create new poles or higher poles, we are just calculate the **pole index** of a subset A_i as $\chi(A_i) \equiv L[A_i] - 2(|A_i| - 1)$,

Now we use above four collections to construct the wanted cross-ratio factor,which picks up a particular pole.

- (7b) Secondly, we denote elements in the collection (III) by $a_i, i = 1, ..., M_{III}$.
- (7c) Now we consider following combination

$$\mathcal{T}_{i;k} \prod_{1 \leq i_1 < ... < i_m \leq M_{III}} a_{i_1} ... a_{i_m} \tag{28}$$

When searching through the combination, there are four variables. The first one is the changing of the index i. The second one is the changing of the index k. The third one is the number m and the fourth one is the different choices of a given number m. For the four nested loops. The outmost loop is $m = 1$ to $m = M_{III}$. The next loop is from largest i to smallest i. The third loop is the different choice of k and the innermost loop is different combinations of $a_{i_1} ... a_{i_m}$ with given m.

Now we use above four collections to construct the wanted cross-ratio factor,which picks up a particular pole.

- (5) First we collect all cross-ratio factors obtained from the multiplication of the primary cross-ratio factors in the collection (I), which satisfy the Criterion I. We denote them as $\mathcal{T}_{i;k}$ where i is the number of primary cross-ratio factors in the multiplication and k distinguishes different combinations with same number i. The allowed choices of k will be denoted as N_i.

$\mathcal{I}$	# of kept incompatible poles
$\mathcal{I}_8\mathcal{T}_{1;1} \equiv \mathcal{I}_8\mathcal{P}_{23}^{18} = \frac{z_{38}z_{78}^2}{z_{12}z_{17}^2z_{18}z_{23}z_{28}^2z_{35}^2z_{37}^2z_{46}^2z_{48}^2z_{58}^2z_{67}^2}$	10
$\mathcal{I}_8\mathcal{T}_{1;2} \equiv \mathcal{I}_8\mathcal{P}_{46}^{23} = \frac{z_{38}^2z_{78}^2}{z_{17}^2z_{18}^2z_{23}z_{26}z_{28}^2z_{34}z_{35}^2z_{37}^2z_{46}z_{48}^2z_{58}^2z_{67}^2}$	9
$\mathcal{I}_8\mathcal{T}_{1;3} \equiv \mathcal{I}_8\mathcal{P}_{58}^{23} = \frac{z_{38}z_{78}^2}{z_{17}^2z_{18}^2z_{23}z_{25}z_{28}^2z_{35}^2z_{37}^2z_{46}^2z_{48}^2z_{58}z_{67}^2}$	11
$\mathcal{I}_8\mathcal{T}_{1;4} \equiv \mathcal{I}_8\mathcal{P}_{46}^{18} = \frac{z_{38}^2z_{78}^2}{z_{14}z_{17}^2z_{18}z_{23}^2z_{28}^2z_{35}^2z_{37}^2z_{46}z_{48}^2z_{58}^2z_{67}^2z_{68}}$	12
$\mathcal{I}_8\mathcal{T}_{1;5} \equiv \mathcal{I}_8\mathcal{P}_{58}^{46} = \frac{z_{38}^2z_{78}^2}{z_{17}^2z_{18}^2z_{23}^2z_{28}^2z_{35}^2z_{37}^2z_{45}z_{46}z_{48}^2z_{58}z_{67}^2z_{68}}$	11
$\mathcal{I}_8\mathcal{T}_{2;1} \equiv \mathcal{I}_8\mathcal{P}_{23}^{18}\mathcal{P}_{58}^{23} = \frac{z_{78}^2}{z_{12}z_{17}^2z_{18}z_{25}z_{28}^2z_{35}^2z_{37}^2z_{46}^2z_{48}^2z_{58}z_{67}^2}$	7
$\mathcal{I}_8\mathcal{T}_{2;2} \equiv \mathcal{I}_8\mathcal{P}_{23}^{18}\mathcal{P}_{46}^{18} = \frac{z_{38}z_{78}^2}{z_{12}z_{14}z_{17}^2z_{23}z_{28}^2z_{35}^2z_{37}^2z_{46}z_{48}^2z_{58}^2z_{67}^2z_{68}}$	6
$\mathcal{I}_8\mathcal{T}_{2;3} \equiv \mathcal{I}_8\mathcal{P}_{23}^{18}\mathcal{P}_{58}^{46} = \frac{z_{38}z_{78}^2}{z_{12}z_{17}^2z_{18}z_{23}z_{28}^2z_{35}^2z_{37}^2z_{45}z_{46}z_{48}^2z_{58}z_{67}^2z_{68}}$	4
$\mathcal{I}_8\mathcal{T}_{2;4} \equiv \mathcal{I}_8\mathcal{P}_{46}^{23}\mathcal{P}_{58}^{23} = \frac{z_{38}z_{78}^2}{z_{17}^2z_{18}^2z_{25}z_{26}z_{28}^2z_{34}z_{35}^2z_{37}^2z_{46}z_{48}^2z_{58}z_{67}^2}$	5
$\mathcal{I}_8\mathcal{T}_{2;5} \equiv \mathcal{I}_8\mathcal{P}_{46}^{18}\mathcal{P}_{58}^{23} = \frac{z_{38}z_{78}^2}{z_{14}z_{17}^2z_{18}z_{23}z_{25}z_{28}^2z_{35}^2z_{37}^2z_{46}z_{48}^2z_{58}z_{67}^2z_{68}}$	6
$\mathcal{I}_8\mathcal{T}_{2;6} \equiv \mathcal{I}_8\mathcal{P}_{58}^{23}\mathcal{P}_{58}^{46} = \frac{z_{38}z_{78}^2}{z_{17}^2z_{18}^2z_{23}z_{25}z_{28}^2z_{35}^2z_{37}^2z_{45}z_{46}z_{48}^2z_{67}^2z_{68}}$	6
$\mathcal{I}_8\mathcal{T}_{3;1} \equiv \mathcal{I}_8\mathcal{P}_{23}^{18}\mathcal{P}_{46}^{18}\mathcal{P}_{58}^{23} = \frac{z_{78}^2}{z_{12}z_{14}z_{17}^2z_{25}z_{28}^2z_{35}^2z_{37}^2z_{46}z_{48}^2z_{58}z_{67}^2z_{68}}$	3
$\mathcal{I}_8\mathcal{T}_{3;2} \equiv \mathcal{I}_8\mathcal{P}_{23}^{18}\mathcal{P}_{58}^{23}\mathcal{P}_{58}^{46} = \frac{z_{78}^2}{z_{12}z_{17}^2z_{18}z_{25}z_{28}^2z_{35}^2z_{37}^2z_{45}z_{46}z_{48}^2z_{67}^2z_{68}}$	2

All the combinations without add new poles of $\frac{z_{38}^2z_{78}^2}{z_{17}^2z_{18}^2z_{23}^2z_{28}^2z_{35}^2z_{37}^2z_{46}^2z_{48}^2z_{58}^2z_{67}^2}$

Now we use above four collections to construct the wanted cross-ratio factor,which picks up a particular pole.

- (6) Secondly, we denote elements in the collection (III) by $a_i, i = 1, ..., M_{III}$.
- (7) Now we consider following combination

$$\mathcal{T}_{i;k} \prod_{1 \leq i_1 < ... < i_m \leq M_{III}} a_{i_1} ... a_{i_m} \qquad (29)$$

When searching through the combination, there are four variables. The first one is the changing of the index i. The second one is the changing of the index k. The third one is the number m and the fourth one is the different choices of a given number m. For the four nested loops. The outmost loop is $m = 1$ to $m = M_{III}$. The next loop is from largest i to smallest i. The third loop is the different choice of k and the innermost loop is different combinations of $a_{i_1} ... a_{i_m}$ with given m.

Since we can not find the right pick-factor from just the collection (I) of $\frac{z_{38}^2 z_{78}^2}{z_{17}^2 z_{18}^2 z_{23}^2 z_{28}^2 z_{35}^2 z_{37}^2 z_{46}^2 z_{48}^2 z_{58}^2 z_{67}^2}$. According to our algorithm we need to included the collection (III) with following 9 primary cross-ratio factors

$$
\begin{aligned}
a_1 &\equiv \mathcal{P}^{18}_{78;23} = \frac{z_{18}z_{27}}{z_{12}z_{78}} & a_2 &\equiv \mathcal{P}^{23}_{38;46} = \frac{z_{23}z_{68}}{z_{26}z_{38}} \\
a_3 &\equiv \mathcal{P}^{46}_{78;23} = \frac{z_{27}z_{38}z_{46}}{z_{26}z_{34}z_{78}} & a_4 &\equiv \mathcal{P}^{23}_{78;46} = \frac{z_{23}z_{47}z_{68}}{z_{26}z_{34}z_{78}} \\
a_5 &\equiv \mathcal{P}^{58}_{78;23} = \frac{z_{27}z_{58}}{z_{25}z_{78}} & a_6 &\equiv \mathcal{P}^{18}_{38;46} = \frac{z_{18}z_{34}}{z_{14}z_{38}} \\
a_7 &\equiv \mathcal{P}^{18}_{78;46} = \frac{z_{18}z_{47}}{z_{14}z_{78}} & a_8 &\equiv \mathcal{P}^{58}_{38;46} = \frac{z_{34}z_{58}}{z_{38}z_{45}} \\
a_9 &\equiv \mathcal{P}^{58}_{78;46} = \frac{z_{47}z_{58}}{z_{45}z_{78}}
\end{aligned}
\qquad (30)
$$

There are 13's $\mathcal{T}_{i;k}$ with $i = 3, 2, 1$ and $N_3 = 2, N_2 = 6, N_1 = 5$ respectively. Searching along the nested loops, we find that

- (1) With $m = 1$, we find all 13×9 can not satisfy both Criterion I and II at the same time.
- (2) With $m = 2$, we find all 13×36 can not satisfy both Criterion I and II at the same time.

Now we use above four collections to construct the wanted cross-ratio factor,which picks up a particular pole.

- (3) With $m = 3$, we tested all 13×84 items.Twelve combinations satisfy both Criterion I and II:

$$
\begin{array}{lll}
\mathcal{T}_{2;1}: & \mathcal{P}^{18}_{23}\mathcal{P}^{23}_{58}\mathcal{P}^{18}_{38;46}\mathcal{P}^{46}_{78;23}\mathcal{P}^{58}_{78;46} & \mathcal{P}^{18}_{23}\mathcal{P}^{23}_{58}\mathcal{P}^{18}_{78;46}\mathcal{P}^{46}_{78;23}\mathcal{P}^{58}_{38;46} \\
\mathcal{T}_{2;2}: & \mathcal{P}^{18}_{23}\mathcal{P}^{18}_{46}\mathcal{P}^{23}_{38;46}\mathcal{P}^{58}_{78;23}\mathcal{P}^{58}_{78;46} & \mathcal{P}^{18}_{23}\mathcal{P}^{18}_{46}\mathcal{P}^{23}_{78;46}\mathcal{P}^{58}_{38;46}\mathcal{P}^{58}_{78;23} \\
\mathcal{T}_{2;3}: & \mathcal{P}^{18}_{23}\mathcal{P}^{46}_{58}\mathcal{P}^{18}_{78;46}\mathcal{P}^{23}_{38;46}\mathcal{P}^{58}_{78;23} & \mathcal{P}^{18}_{23}\mathcal{P}^{46}_{58}\mathcal{P}^{18}_{38;46}\mathcal{P}^{23}_{78;46}\mathcal{P}^{58}_{78;23} \\
\mathcal{T}_{2;4}: & \mathcal{P}^{23}_{46}\mathcal{P}^{23}_{58}\mathcal{P}^{18}_{38;46}\mathcal{P}^{18}_{78;23}\mathcal{P}^{58}_{78;46} & \mathcal{P}^{23}_{46}\mathcal{P}^{23}_{58}\mathcal{P}^{18}_{78;23}\mathcal{P}^{18}_{78;46}\mathcal{P}^{58}_{38;46} \\
\mathcal{T}_{2;5}: & \mathcal{P}^{18}_{46}\mathcal{P}^{23}_{58}\mathcal{P}^{18}_{78;23}\mathcal{P}^{23}_{38;46}\mathcal{P}^{58}_{78;46} & \mathcal{P}^{18}_{46}\mathcal{P}^{23}_{58}\mathcal{P}^{18}_{78;23}\mathcal{P}^{23}_{78;46}\mathcal{P}^{58}_{38;46} \\
\mathcal{T}_{2;6}: & \mathcal{P}^{23}_{58}\mathcal{P}^{46}_{58}\mathcal{P}^{18}_{78;23}\mathcal{P}^{18}_{78;46}\mathcal{P}^{23}_{38;46} & \mathcal{P}^{23}_{58}\mathcal{P}^{46}_{58}\mathcal{P}^{18}_{38;46}\mathcal{P}^{18}_{78;23}\mathcal{P}^{23}_{78;46}
\end{array}
$$

- All of above twelve combinations of primary cross-ratio factors give the same pick-factor:

$$\frac{z_{18}^2 z_{23}^2 z_{27} z_{46} z_{47} z_{58}^2}{z_{12} z_{14} z_{25} z_{26} z_{38}^2 z_{45} z_{78}^2} \qquad (31)$$

It gives a new integrand.

$$\mathcal{I}_8' = \frac{z_{27} z_{47}}{z_{12} z_{14} z_{17}^2 z_{25} z_{26} z_{28}^2 z_{35}^2 z_{37}^2 z_{45} z_{46} z_{48}^2 z_{67}^2} \qquad (32)$$

The Feynman diagrams corresponding to this new integrand contains the only s_{248} terms.

IV: Final Remark

In this talk, we have presented two results

- The first result, base on the topological structure of the Cayley tree graph, there are two types of primary effective Feynman vertex: the colour ordered type V_C corresponding the line subgraph of Cayley tree and the permutation type V_P which corresponding to node with multiple legs.
    - For the V_C-type effective vertex, the $|V_C\{(l_1, l_2, ..., l_n)\}|$ represents all the cubic Feynman diagrams respecting the colour order of the list $\{(l_1, l_2, ..., l_n)\}$.
    - For the P-type vertex $V_P(k; \widetilde{K}_1 \sqcup\!\sqcup \widetilde{K}_2 \sqcup\!\sqcup ... \sqcup\!\sqcup \widetilde{K}_t; n)$, it represents the DDM type of cubic Feynman graph diagrams with ordering from the shuffle algebra.

- Second result is the construction of picking up factor for general CHY-integrandscontaining only simple poles. Unlike the situation of bi-adjoint theory where one needs only a cross-ratio factor, for general CHY-integrand, we need to construct all possible cross-ratio factors from the linking set $\mathrm{Links}[A, \overline{A}]$,including the denominators and numerators. When multiplying them together, we need to introduce two criteria to select the right combination.

Based on results in this note, there are several interesting directions one can investigate.

- Since one can represent these diagrams using the polytope, it is naturalto ask, could we using the geometric picture, i.e., projecting from an dimension higherobject to its specific face, to understand and construct the same picking up factor?
- For general theory, such as Gravity and Yang-Mills, the CHY-integrands aregeneral. Thus one can try if it is possible to construct the one-loop integrands for thesegeneral theory by removing singular contribution using the picking up factor developed inthis note.
- If this picking pole factor provides another way to reduce higher polesto lower poles?

Thanks a lot for the attension !!!