第一届“场论与弦论” 及相关数学物理研讨会

Form factors of high-dimensional operators in non-planar N=4 SYM and QCD

Gang Yang

Based on recent work:

- arXiv:2011.02494 with Qingjun Jin (靳庆军) and Ke Ren (任可)
- arXiv:2011.06540 with Guanda Lin (林冠达)

Content

Motivation

Form factors

QCD

Non-planar N=4 SYM

Outlook

Background and Motivation

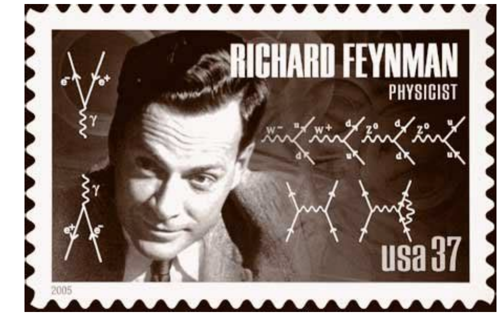
Progress in amplitudes

Significant progress has been made in the study of amplitudes in past years.

New Structures

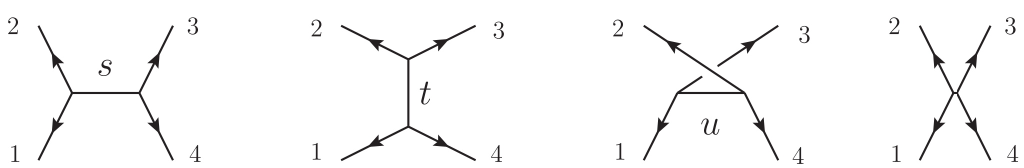
New Methods

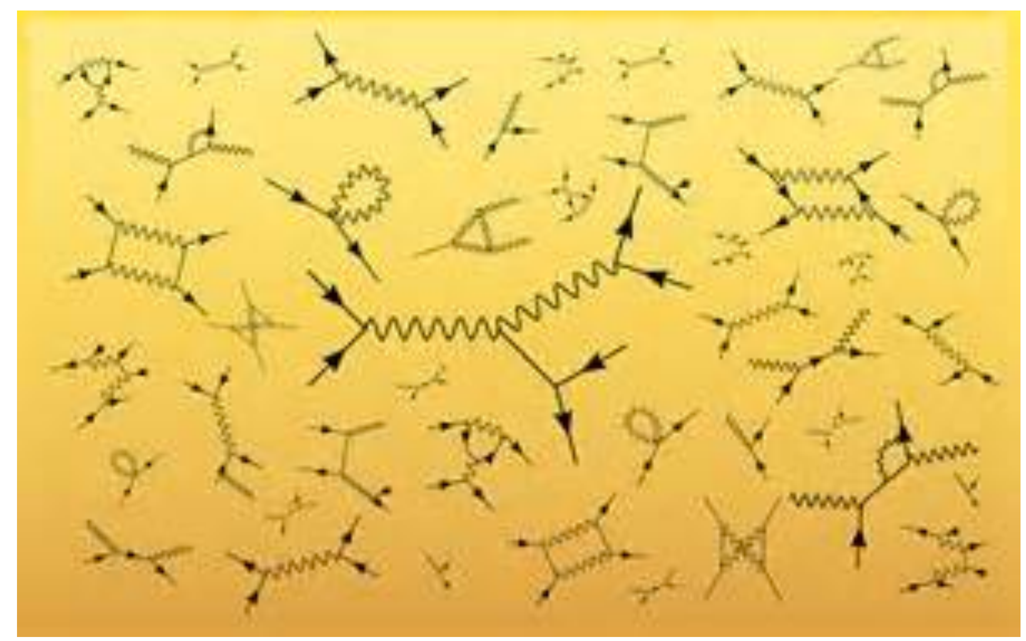
Feynman diagram



Feynman diagram is a universal tool, but in practice it can be very complicated.

4-gluon tree:





n-gluon tree amplitudes:

n	4	5	6	7	8	9	10
# graphs	4	25	220	2485	34300	559405	10525900

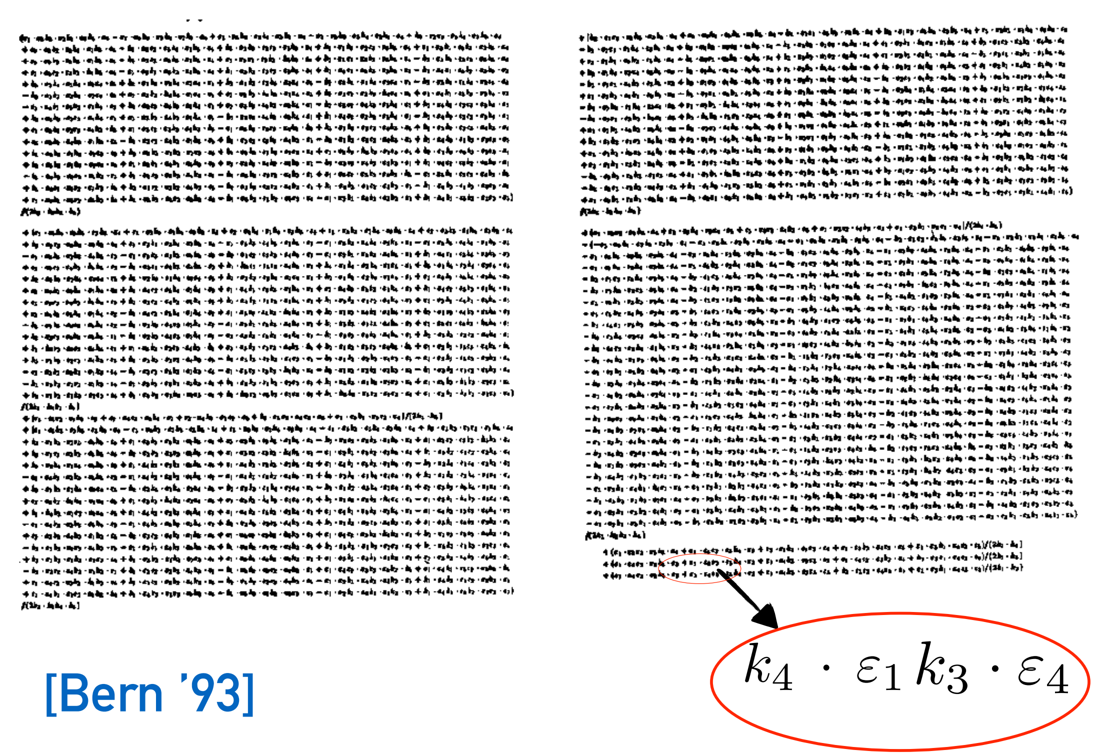
Parke-Taylor MHV formula

[Parke, Taylor, 1986]

Any n-gluon tree MHV amplitudes:

$$A_n^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}$$

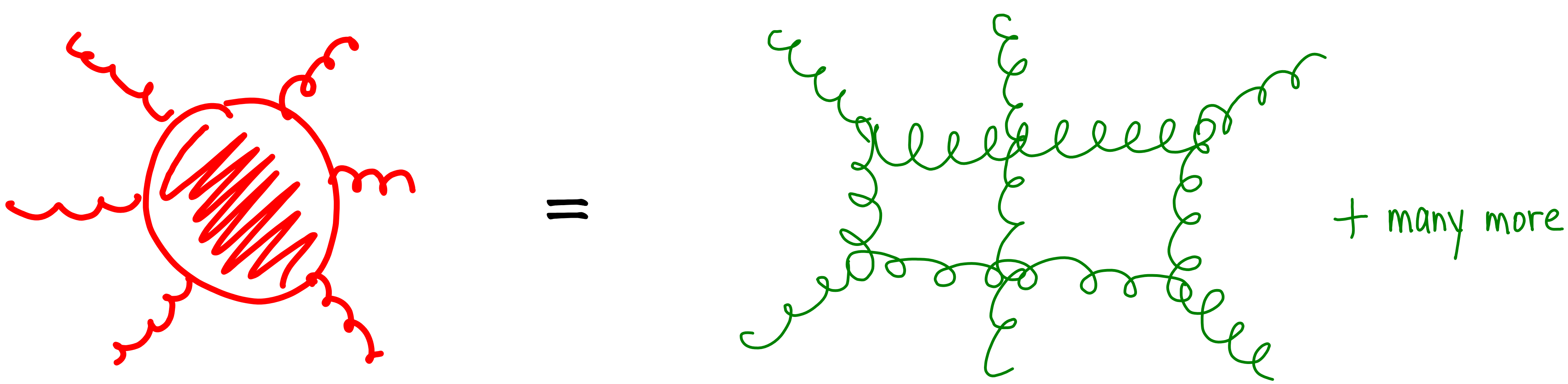
compare with results obtained by Feynman diagrams:



[Bern '93]

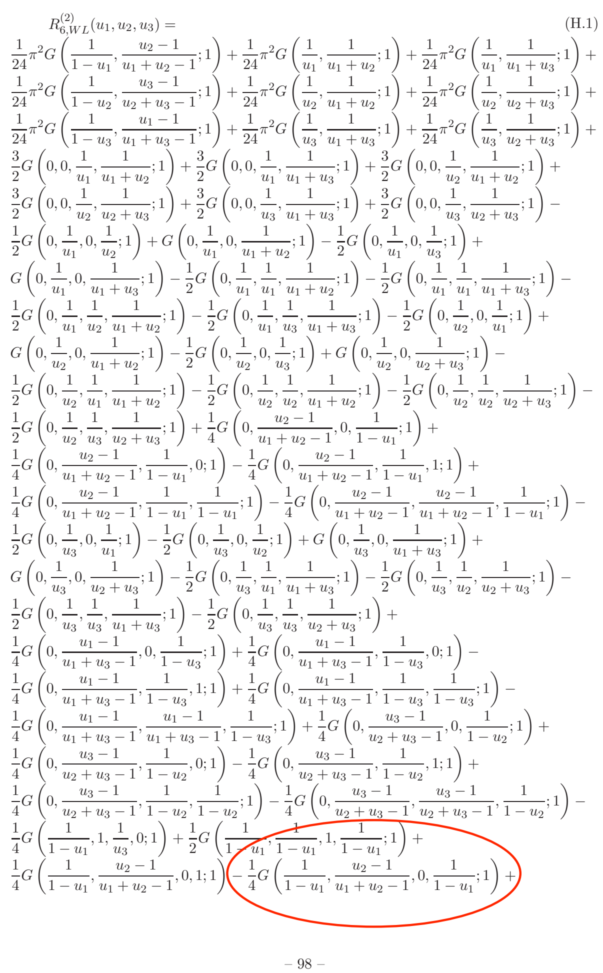
Another two-loop example

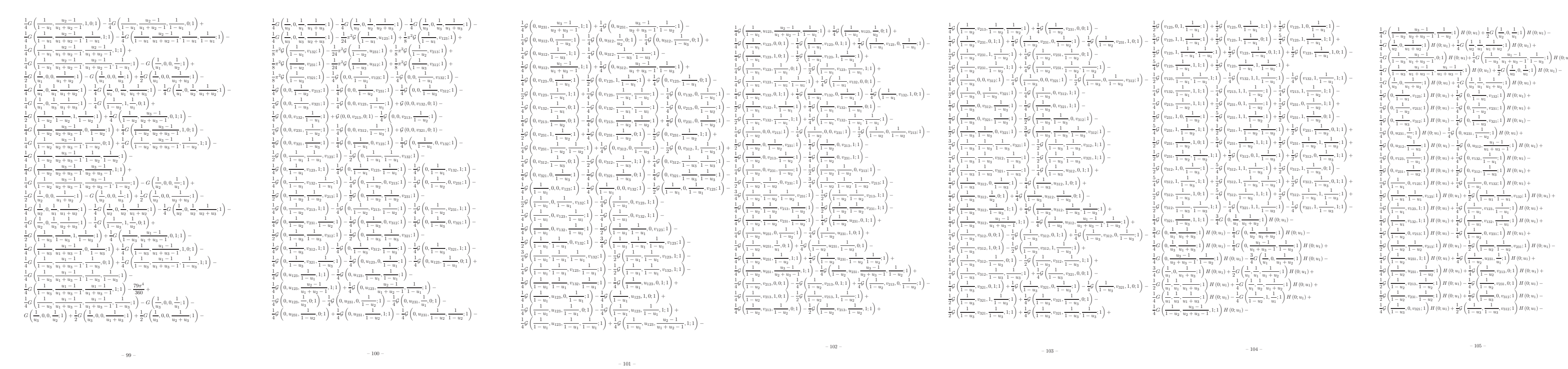
Six-gluon MHV amplitudes in N=4 SYM



[Del Duca, Duhr, Smirnov 2010] (heroic computation)

Complicated results of 17 pages

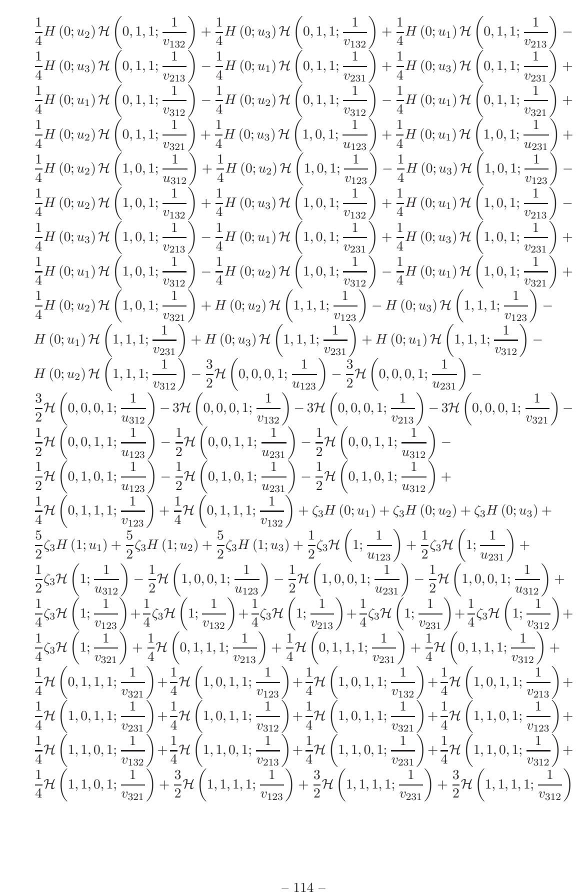




[Del Duca, Duhr, Smirnov 2010]

"multiple(Goncharov)-polylogrithm function"

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)$$



Result can be remarkably simple

17 pages =

[Goncharov, Spradlin, Vergu, Volovich 2010]

$$\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{1}{24}J^4+\frac{\pi^2}{12}J^2+\frac{\pi^4}{72}$$

$$L_4(x^+,x^-)=\frac{1}{8!!}\log(x^+x^-)^4+\sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-))$$

$$\ell_n(x)=\frac{1}{2}\left(\mathrm{Li}_n(x)-(-1)^n\,\mathrm{Li}_n(1/x)\right)$$

$$J=\sum_{i=1}^{3}(\ell_1(x_i^+)-\ell_1(x_i^-)).$$

a line result in terms of classical polylogarithms!

→ require advanced mathematical tools: “**Symbol**”

Alexander Goncharov

Progress in amplitudes

Significant progress has been made in the study of amplitudes in past years.

New Structures

New Methods

Such simplicity is unexpected and also hard to understand using traditional Feynman diagrams.

Lessons from modern amplitudes

New structures and new formulations

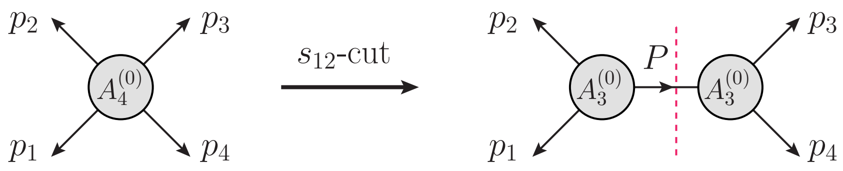
Witten's twistor theory
Double-copy
CHY formalism
New mathematical structure

c.f. talks by Song He, Bo Feng, Junjie Rao, Zhihao Fu

New computational methods

Spinor helicity variables

BCFW recursion relation



Unitarity cuts

New algebraic reduction and integration methods

c.f. talk by Yang Zhang

Spinor helicity formalism

Massless momentum:

$$p_\mu \quad \to \quad p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}$$

$$p_\mu p^\mu = 0 \quad \to \quad p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \,, \qquad \alpha, \dot{\alpha} = 1, 2$$

Polarisation vector:

$$\varepsilon^{(-)}_{i,\alpha\dot{\alpha}} = \frac{\lambda_i \, \tilde{\xi}}{[\tilde{\lambda}_i \, \tilde{\xi}]} \,, \qquad \varepsilon^{(+)}_{i,\alpha\dot{\alpha}} = \frac{\xi \, \tilde{\lambda}_i}{\langle \xi \lambda_i \rangle}$$

"Chinese Magic" [Xu, Zhang, Zhang, 84]

Use good on-shell variables

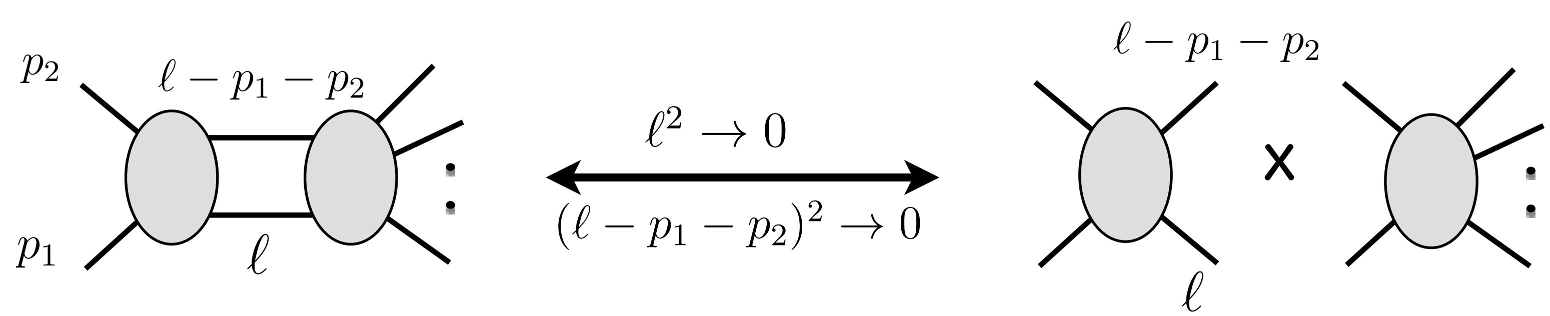
Unitarity cut method (幺正性)

"The S-matrix is an analytic function of all momentum variables with only those singularities required by unitarity."

"singularities": physical poles and branch cuts.

Unitarity-cut method provides an efficient method to compute loop integrand.

Cutkosky cutting rule: $\frac{1}{\ell^2} \to 2\pi\delta^{(+)}(\ell^2)$



Progress in planar N=4 SYM

In the planar limit, N=4 SYM is believed to be exactly solvable.

Thanks to: **Integrability and AdS/CFT correspondence**

(Dual conformal and Yangian symmetries)

See also other theories, Fishnet theory, 3D ABJM, c.f. talk by Junbao Wu

Many exact solutions were found:

- anomalous dimensions
- scattering amplitudes/Wilson loops   Pentagon OPE program, [Basso, Pedro, Sever; …]
- correlation functions…   Hexagon form factor program, [Basso, Komatsu, Pedro; …]

Operator mixing and spectrum

Different operators can mixing with each other at quantum level via renormalization:

$$\mathcal{O}_{R,i} = Z_i^{\ j} \mathcal{O}_{B,j}$$

From the renormalization constant matrix, one can obtain the dilatation operator:

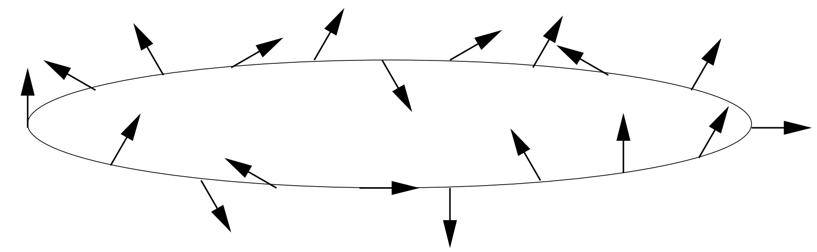
$$\mathscr{D} = -\frac{d \log Z}{d \log \mu}$$

The anomalous dimension is given by the eigenvalue of the dilatation operator.

$$\mathscr{D} \cdot \mathcal{O}_{\text{eigen}} = \gamma \cdot \mathcal{O}_{\text{eigen}}$$

Integrability

$$\mathrm{tr}(F_{\alpha\beta}F^{\beta\gamma}\psi_{\gamma A}\phi_{BC}\bar{\psi}_{D\dot{\alpha}}\ldots)$$



Dilatation operator ⟷ **Spin chain Hamiltonian**

c.f. talks by Guangliang Li, Junpeng Cao, Wenli Yang, Xiwen Guan, Yuzhu Jiang…

Example: the scalar operators at 1-loop:

$$\mathfrak{D}^{(1)} = \mathbb{H}_{\mathrm{SO}(6)} = \sum_i 2(\mathbb{1} - \mathbb{P})_{i\,i+1} + \mathbb{T}_{i\,i+1}$$ [Minahan, Zarembo 02]

Direct evidence that N=4 SYM is integrable.

Cusp anomalous dimension

Non-perturbative result via integrability method:

[Beisert, Eden, Staudacher '06]

$$K_{ij} = j(-1)^{i(j+1)} \int_0^\infty \frac{dt}{t} \frac{J_i(2gt)J_j(2gt)}{e^t - 1}$$

$$\Gamma_{\text{cusp}} = 4g^2 \left(\frac{1}{1+K}\right)_{11}$$

Weak coupling expansion:

$$\Gamma_{\text{cusp}} = 4g^2 - \frac{4\pi^2}{3} g^4 + \frac{44\pi^4}{45} g^6 - 8\left(4\zeta_3^2 + \frac{73}{630}\pi^6\right) + \mathcal{O}(g^{10})$$

[Belitsky, Gorsky, Korchemsky'03],
[Kotikov,Lipatov,Onishchenko,Velizhanin'04]
[Bern,Czakon,Dixon,Kosower,Smirnov'06]
[Cachazo,Spradlin,Volovich'06] ……

Strong coupling expansion (AdS/CFT):

$$\Gamma_{\text{cusp}} = 2g - \frac{3\log 2}{2\pi} + \mathcal{O}(1/g)$$

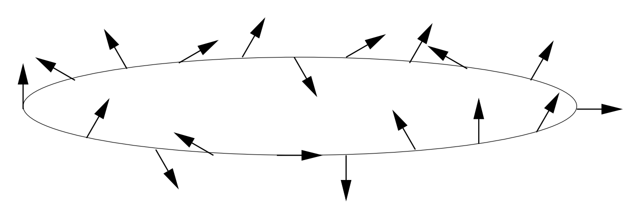
[Gubser, Klebanov,Polyakov'02],
[Frolov,Tseytlin'02] [Kruczenski'02],
[Makeenko'02] ……

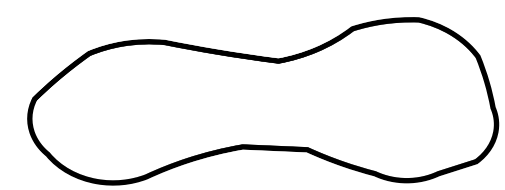
Underlying picture

Single trace operators can be viewed as states of a dynamic, cyclic, quantum spin chain. The latter can be related to a string picture.

single trace operators	⟷	spin chain model	⟷	string sigma-model

$\mathrm{tr}(F_{\alpha\beta}F^{\beta\gamma}\psi_{\gamma A}\phi_{BC}\bar{\psi}_{D\dot{\alpha}}\ldots)$

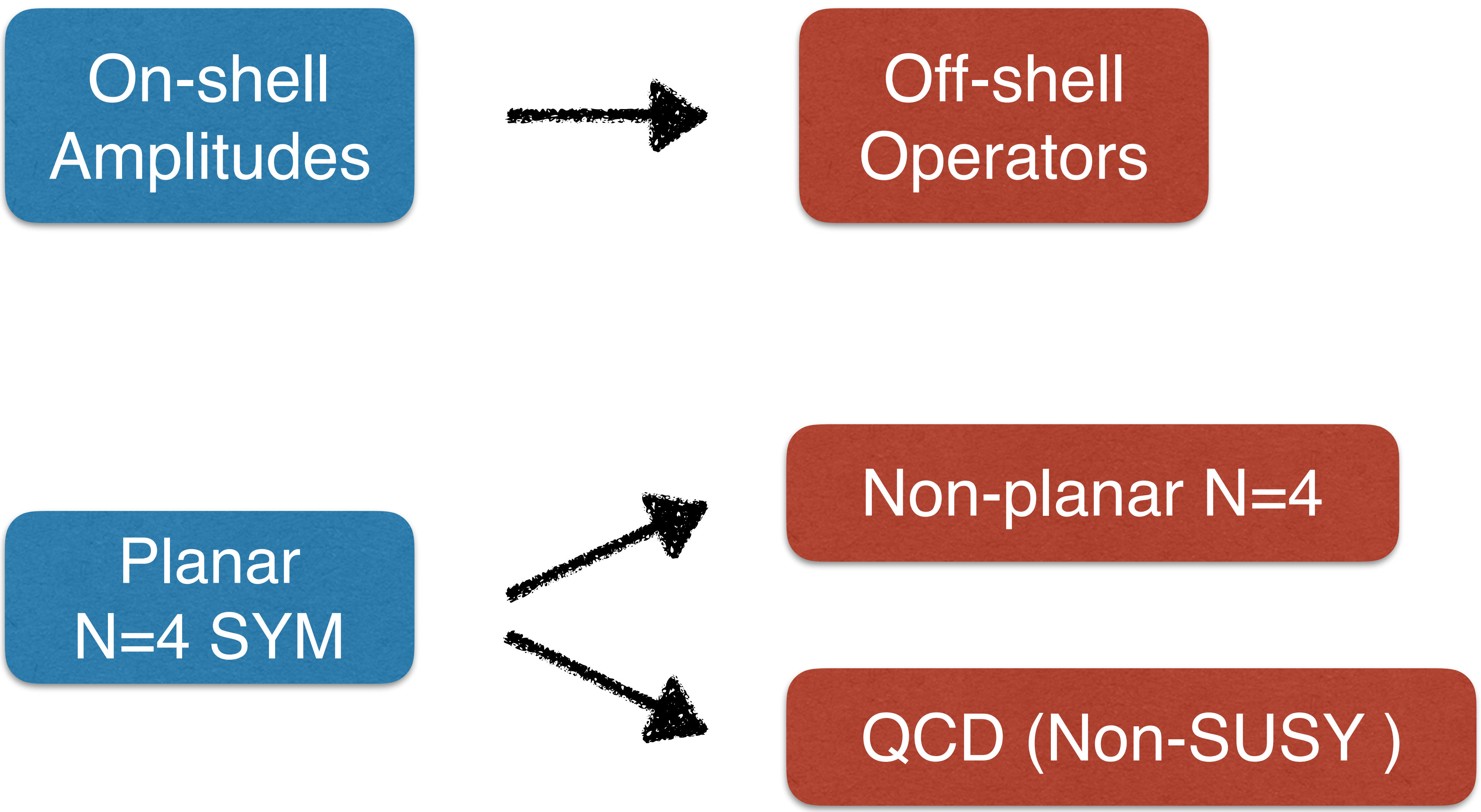




Scaling operator / anomalous dimension	Spin chain Hamiltonian/Energy	String Hamiltonian/ Energy
Weak coupling	Non-perturbative	Strong coupling

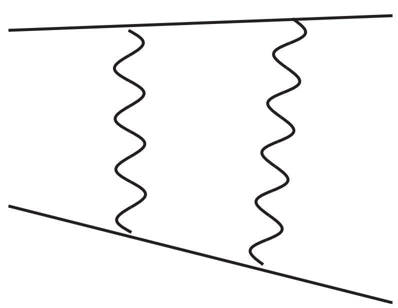
See the review by Beisert et.al, 2010

Motivation


On-shell Amplitudes
Off-shell Operators
Planar N=4 SYM
Non-planar N=4
QCD (Non-SUSY)

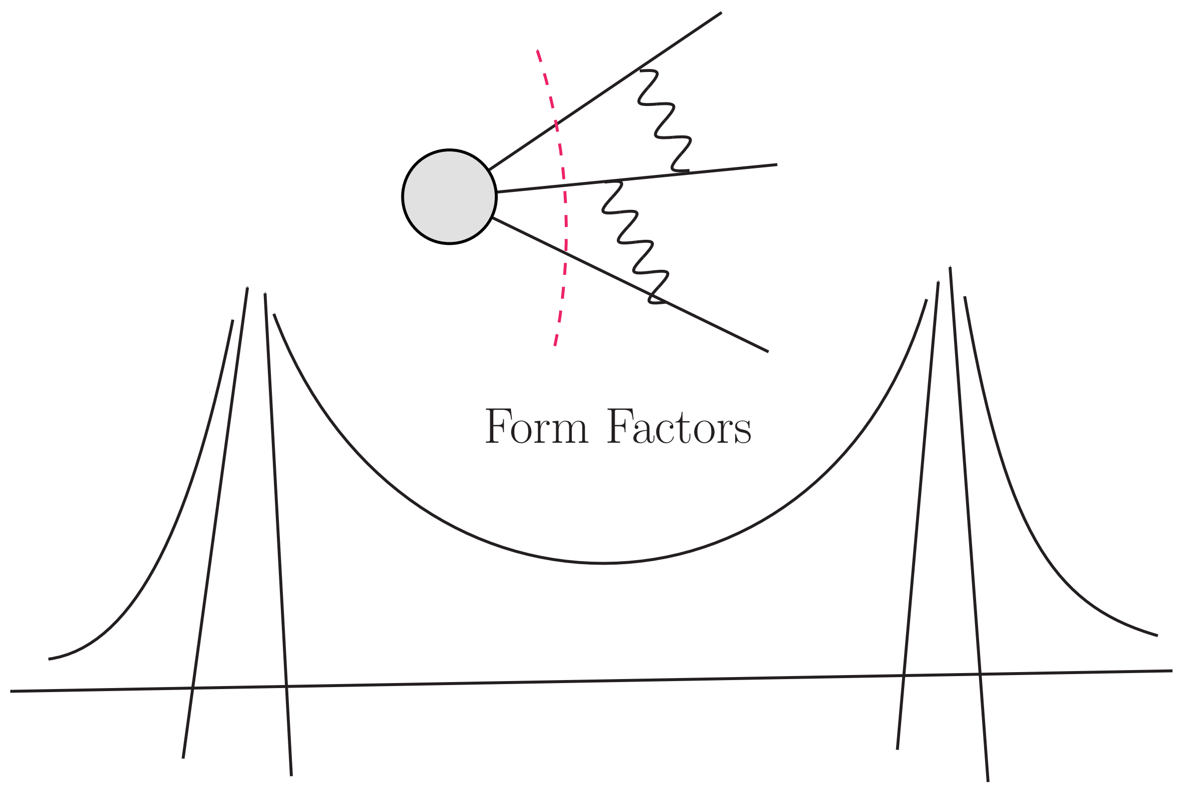
Form factor:
A probe to the off-shell World

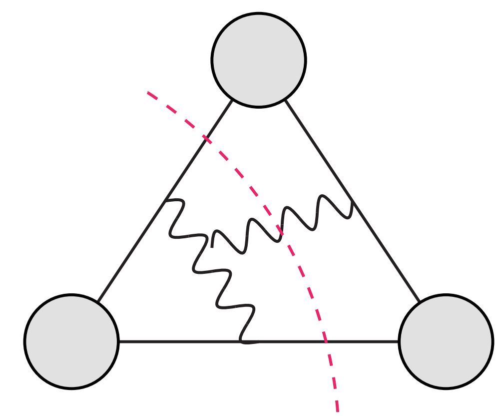
Form factors



Scattering Amplitudes

$\langle p_1 p_2 \ldots p_n \rangle$





Correlation Functions

$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle$

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x} \langle p_1 \ldots p_n | \mathcal{O}(x) | 0 \rangle$$

$$= \delta^{(4)}\Big(\sum_{i=1}^{n} p_i - q\Big) \langle p_1 \ldots p_n | \mathcal{O}(0) | 0 \rangle$$

Gauge invariant operators

Local gauge invariant operators are constructed as traces of covariant fields.

$$\mathcal{O}(x) = \mathrm{Tr}(\mathcal{W}_1^{(m_1)}\mathcal{W}_2^{(m_2)}\ldots\mathcal{W}_n^{(m_n)})(x)$$

gauge transformation

$$\mathcal{W} \to U\mathcal{W}U^\dagger$$

$$\mathcal{W}^{(m)} := D^m\mathcal{W}\,, \qquad D_{\alpha\dot{\alpha}}\mathcal{W} = \partial_{\alpha\dot{\alpha}}\mathcal{W} - ig_{\mathrm{YM}}[A_{\alpha\dot{\alpha}},\, \mathcal{W}]$$

In N=4 SYM, there are following ‘letters’:

$$\mathcal{W}_i \in \{\phi_{AB}\,,\ F_{\alpha\beta}\,,\ \bar{F}_{\dot{\alpha}\dot{\beta}}\,,\ \bar{\psi}_{\dot{\alpha}A}\,,\ \psi_{\alpha ABC}\}$$

$$\alpha, \dot{\alpha} = 1, 2$$

$$A = 1, 2, 3, 4$$

Operators and on-shell kinematics

In terms of spinor helicity variables: [Beisert 10] [Zwiebel 11] [Wilhelm 14]

$$
\begin{array}{ccc}
\bar{F}_{\dot{\alpha}\dot{\beta}} & \xrightarrow{g_+} & \tilde{\lambda}^{\dot{\alpha}}\tilde{\lambda}^{\dot{\beta}} \\
\bar{\psi}_{\dot{\alpha}A} & \xrightarrow{\bar{\psi}_{\dot{\alpha}A}} & \tilde{\lambda}^{\dot{\alpha}}\eta^{A} \\
\phi_{AB} & \xrightarrow{\phi_{AB}} & \eta^{A}\eta^{B} \\
\psi_{\alpha ABC} & \xrightarrow{\psi_{\alpha ABC}} & \lambda^{\alpha}\eta^{A}\eta^{B}\eta^{C} \\
F_{\alpha\beta} & \xrightarrow{g_-} & \lambda^{\alpha}\lambda^{\beta}\eta^{1}\eta^{2}\eta^{3}\eta^{4} \\
D_{\alpha\dot{\alpha}} & \longrightarrow & \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}
\end{array}
$$

Operators and form factors

Applying the rules:

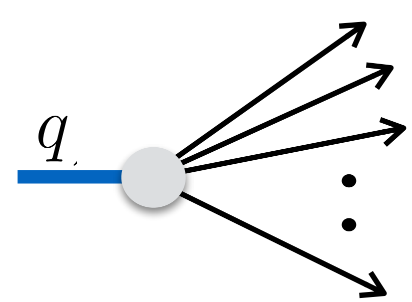
$\bar{F}_{\dot\alpha\dot\beta}$	$\xrightarrow{g_+}$	$\tilde\lambda^{\dot\alpha}\tilde\lambda^{\dot\beta}$
$\bar\psi_{\dot\alpha A}$	$\xrightarrow{\bar\psi_{\dot\alpha A}}$	$\tilde\lambda^{\dot\alpha}\eta^A$
ϕ_{AB}	$\xrightarrow{\phi_{AB}}$	$\eta^A\eta^B$
$\psi_{\alpha ABC}$	$\xrightarrow{\psi_{\alpha ABC}}$	$\lambda^\alpha\eta^A\eta^B\eta^C$
$F_{\alpha\beta}$	$\xrightarrow{g_-}$	$\lambda^\alpha\lambda^\beta\eta^1\eta^2\eta^3\eta^4$
$D_{\alpha\dot\alpha}$	$\longrightarrow$	$\lambda^\alpha\tilde\lambda^{\dot\alpha}$

$$\mathrm{tr}(\bar F_{\alpha\beta}F^{\alpha\beta}) \to \lambda_1^\alpha\lambda_1^\beta\lambda_{2\alpha}\lambda_{2\beta}(\eta_1)^4(\eta_2)^4 = \langle 1\,2\rangle^2(\eta_1)^4(\eta_2)^4$$

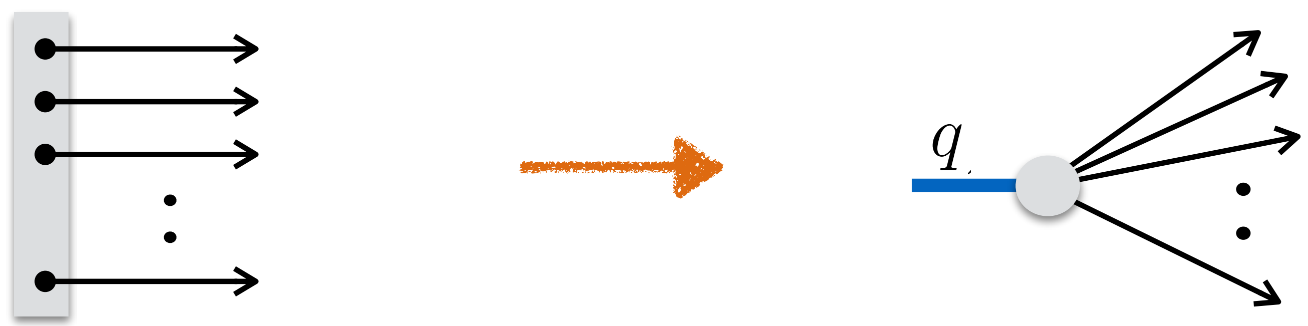
$$\mathrm{tr}(\bar F_{\dot\alpha}{}^{\dot\beta}\bar F_{\dot\beta}{}^{\dot\gamma}\bar F_{\dot\gamma}{}^{\dot\alpha}) \to \tilde\lambda_1^{\dot\alpha}\tilde\lambda_{1\dot\beta}\tilde\lambda_2^{\dot\beta}\tilde\lambda_{2\dot\gamma}\tilde\lambda_3^{\dot\gamma}\tilde\lambda_{3\dot\alpha} = [1\,2][2\,3][3\,1]$$

The RHS exactly reproduce the minimal form factor results:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x}\,\langle p_1\ldots p_n|\mathcal{O}(x)|0\rangle = \delta^{(4)}(\sum_{i=1}^{n} p_i - q)\,\langle p_1\ldots p_n|\mathcal{O}(0)|0\rangle$$



Operators and form factors



$\mathcal{O}(x) = \mathrm{Tr}(\mathcal{W}_1^{(m_1)}\mathcal{W}_2^{(m_2)}\ldots\mathcal{W}_n^{(m_n)})(x)$

$F_{n,\mathcal{O}}(1,\ldots,n)$

One can translate any local operator into the "on-shell" language!

Starting from tree minimal form factors, one can construct non-minimal form factors and loop form factors.

Simplicity of MHV Form factors

Parke-Taylor structure of form factors: [Brandhuber, Spence, Travaglini, GY 2011]

$$F_n^{\mathrm{MHV}}(1^+, .., i_\phi, .., j_\phi, .., n^+; \mathrm{tr}(\phi^2)) = \delta^4(\sum_{i=1}^{n} p_i - q) \frac{\langle ij \rangle^2}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

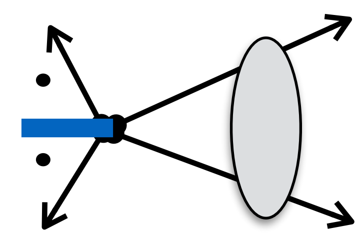
Recall the Parke-Taylor formula for amplitudes:

$$A_n^{\mathrm{MHV}}(1^+, .., i^-, .., j^-, .., n^+) = \delta^4(\sum_{i=1}^{n} p_i) \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

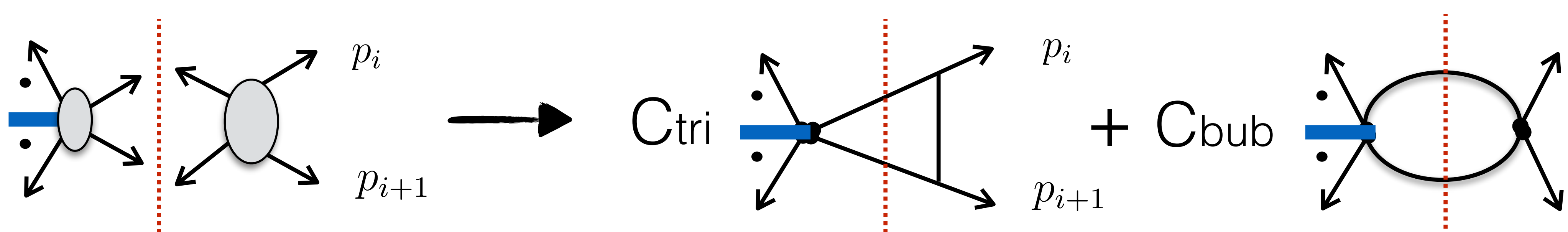
Form factor via Unitarity

At one-loop, there are only ‘range-2’ interactions:

$$F_n^{(1)} = \sum_{i=1}^{n}$$



The basis is very simple:



One-loop anomalous dimension is given by the bubble coefficients:

$$Z^{(1)} = -\frac{C_{\text{bub}}}{\epsilon}, \quad \mathscr{D}^{(1)} = 2\epsilon Z^{(1)} = -2C_{\text{bub}}$$

$$(\mathbb{D}^{(1)})_{\mathrm{SO}(6)} = \sum_i 2(\mathbb{1} - \mathbb{P})_{i\,i+1} + \mathbb{T}_{i\,i+1}$$

Loop structure of form factors

At higher loops, the IR and UV are mixed:

General structure of (bare) form factors:

$$\log F_{\text{bare}} = \sum_{\ell=1}^{\infty} g^{2\ell}\Big(-\frac{\gamma_{\text{cusp}}^{(\ell)}}{(2\ell\epsilon)^2} - \frac{\mathcal{G}_0^{(\ell)}}{2\ell\epsilon}\Big) \sum_{i=1}^{n} \Big(\frac{\mu^2}{-s_{ii+1}}\Big)^{\ell\epsilon} - (\log Z) + \text{Fin} + \mathcal{O}(\epsilon)$$

Universal infrared divergences

wanted UV divergences

QCD Spectrum and Higgs Amplitudes

- arXiv:2011.02494 with Qingjun Jin (靳庆军) and Ke Ren (任可)

QCD operators

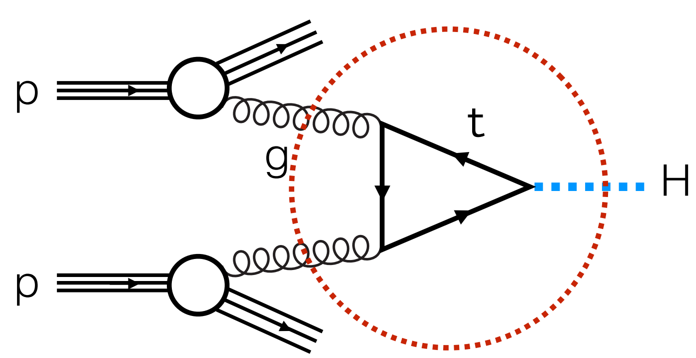
We consider scalar gauge invariant local operators:

$$\mathcal{O} \sim \; c(a_1, ..., a_n)\big(D_{\mu_{11}}...D_{\mu_{1m_1}}F_{\nu_1\rho_1}\big)^{a_1}\cdots\big(D_{\mu_{n1}}...D_{\mu_{nm_n}}F_{\nu_n\rho_n}\big)^{a_n}X(\eta,\epsilon)$$

$$D_\mu \star = \partial_\mu + ig[A_\mu,\star]\,, \qquad [D_\mu, D_\nu]\star = ig[F_{\mu\nu},\star] \qquad F_{\mu\nu} = F^a_{\mu\nu}T^a, \qquad [T^a,T^b] = if^{abc}T^c$$

Anomalous dimensions (~spectrum of hadrons), RG flow, OPE

Such operators also in Higgs EFT obtained by integrating heavy Top quark loop:



$m_t \to \infty$

$$\mathcal{L}_{\text{eff}} = \hat{C}_0 H\mathcal{O}_{4;0} + \sum_{k=1}^{\infty}\frac{1}{m_{\text{t}}^{2k}}\sum_i \hat{C}_i H\mathcal{O}_{4+2k;i}$$

c.f. EFT related talks by Shuang-Yong Zhou, Bo Ning ..

Basis of operators (classical)

These operators are generally not independent:

$$\mathcal{O} \sim \; c(a_1, ..., a_n)\big(D_{\mu_{11}}...D_{\mu_{1m_1}}F_{\nu_1\rho_1}\big)^{a_1} \cdots \big(D_{\mu_{n1}}...D_{\mu_{nm_n}}F_{\nu_n\rho_n}\big)^{a_n} X(\eta, \epsilon)$$

Equation of motion: $D_\mu F^{\mu\nu} = 0$

Bianchi identities: $D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$

We need to remove such relations in order to find a set of independent basis operators.

Examples:

dim-4: $\mathcal{O}_4 = \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$

dim-6: $\mathcal{O}_{6;1} = \partial^2\mathrm{Tr}(F^2), \quad \mathcal{O}_{6;2} = \mathrm{Tr}(F^3)$

dim-8: $\mathcal{O}_{8;1} = \partial^4\mathrm{Tr}(F^2), \quad \mathcal{O}_{8;2} = \partial^2\mathrm{tr}(F^3), \quad \mathcal{O}_{8;3} = \mathrm{tr}(D_1F_{23}D_4F_{23}F_{14}),$ and 8 Length-4 operators

Operator mixing (quantum)

Different operators (at same given dimension) can mixing with each other at quantum level via renormalization:

$$\mathcal{O}_{R,i} = Z_i^{\ j} \mathcal{O}_{B,j}$$

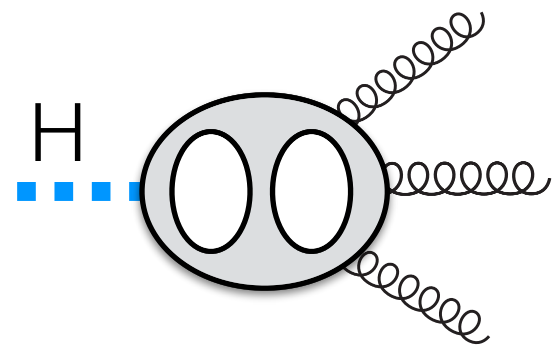
From the renormalization constant matrix, one can obtain the dilatation operator:

$$\mathcal{D} = -\frac{d \log Z}{d \log \mu}$$

The anomalous dimension is given by the eigenvalue of the dilatation operator:

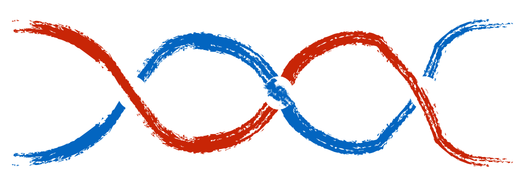
$$\mathcal{D} \cdot \mathcal{O}_{\text{eigen}} = \gamma \cdot \mathcal{O}_{\text{eigen}}$$

Loop form factor computation



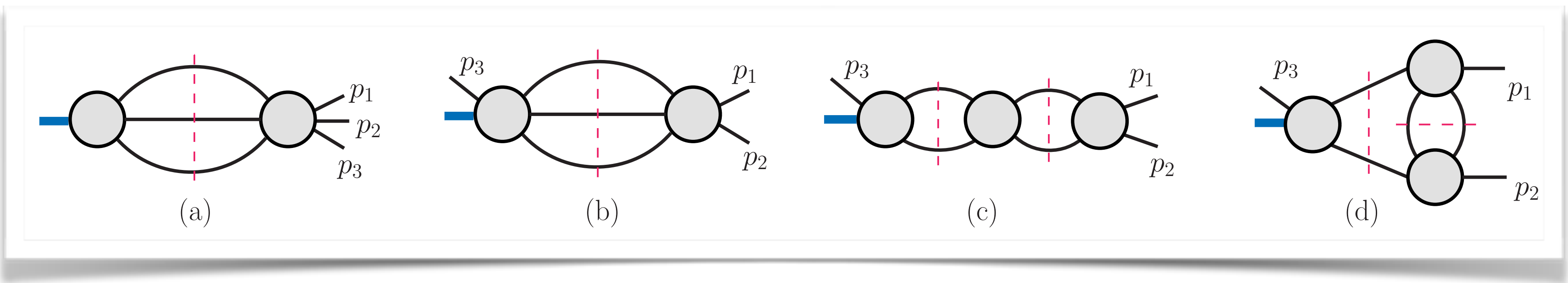
$$F^{(l)}|_{\text{cut}} = \sum_{\text{helicities}} F^{\text{tree}} \prod_j A_j^{\text{tree}} = \sum_i c_i \, M_i|_{\text{cut}}$$

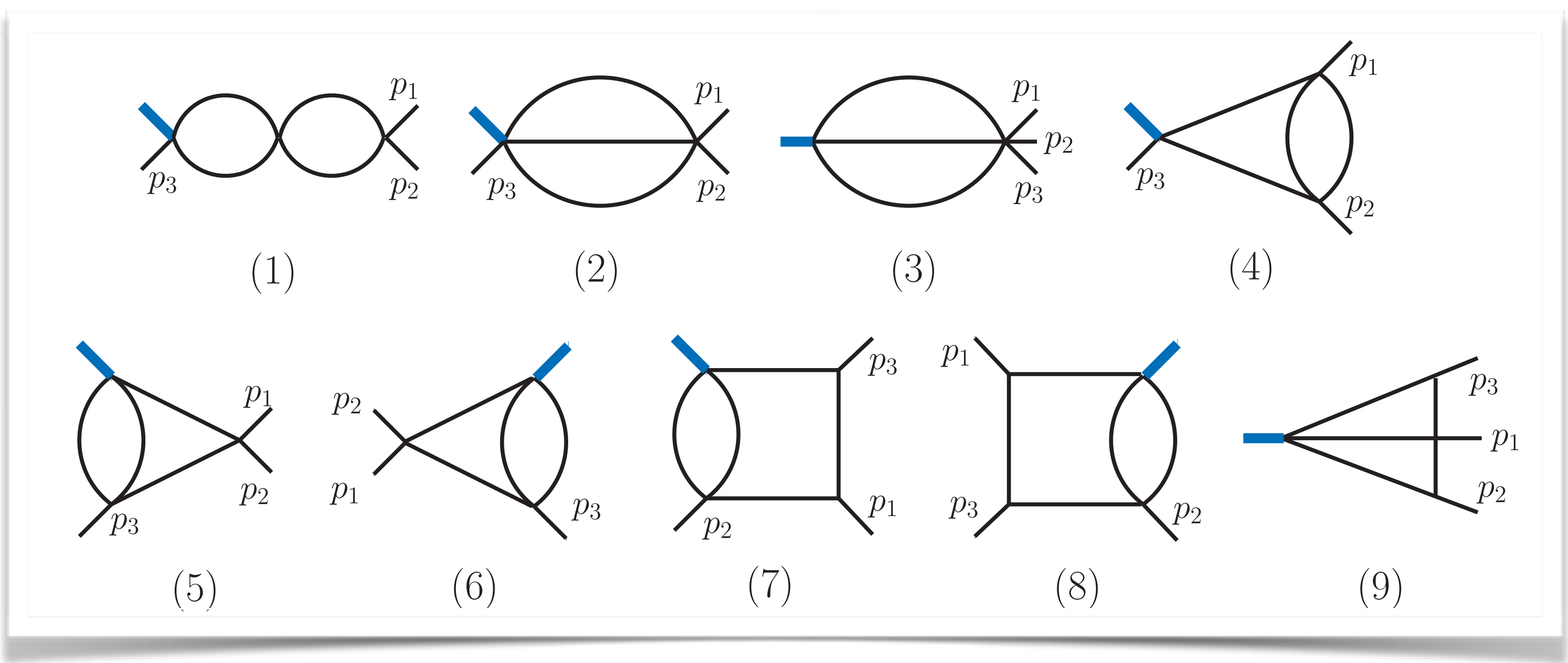
On-shell unitarity



Integration by parts

Loop form factor computation





$$F_{\mathcal{O}}^{(2)} = \Big(c_1 I_1 + c_2 I_2 + c_3 I_3 + c_4 I_4 + \left[c_5 I_5 + c_6 I_6 \right] + \left[c_7 I_7 + c_8 I_8 \right] + c_9 I_9 \Big) + \text{cyc.perm.}(1,2,3)$$

Mixing matrices and spectrum

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix.

Mixing matrices and spectrum

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix.

Results were known previously only at one-loop up to dimension-8, and at two-loop up to dimension-6 operators.

Gracey 2002; Dawson, Lewis, Zeng 2014; …
Jin, GY 2019

Length-3 operators at dimension-8:

$$\mathcal{O}_{8;1} = \partial^4 \mathrm{Tr}(F^2), \quad \mathcal{O}_{8;2} = \partial^2 \mathrm{tr}(F^3), \quad \mathcal{O}_{8;3} = \mathrm{tr}(D_1 F_{23} D_4 F_{23} F_{14}),$$

$$\mathbb{D}_{\mathcal{O}_8} = \begin{pmatrix} -\frac{22}{3}\hat{\lambda} - \frac{136}{3}\hat{\lambda}^2 & 0 & 0 \\ -\frac{\hat{\lambda}^2}{\hat{g}} & \frac{7}{3}\hat{\lambda} + \frac{269}{18}\hat{\lambda}^2 & 10\hat{\lambda}^2 \\ -3\frac{\hat{\lambda}^2}{\hat{g}} & 0 & \hat{\lambda} + \frac{25}{3}\hat{\lambda}^2 \end{pmatrix}$$

$$\hat{\gamma}^{(1)}_{\mathcal{O}_8} = \left\{-\frac{22}{3}; 1; \frac{7}{3}\right\}, \qquad \hat{\gamma}^{(2)}_{\mathcal{O}_8} = \left\{-\frac{136}{3}; \frac{25}{3}; \frac{269}{18}\right\}$$

Mixing matrices and spectrum

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix.

$$\mathbb{D}_{\mathcal{O}_8} = \begin{pmatrix} -\frac{22}{3}\hat{\lambda} - \frac{136}{3}\hat{\lambda}^2 & 0 & 0 \\ -\frac{\hat{\lambda}^2}{\hat{g}} & \frac{7}{3}\hat{\lambda} + \frac{269}{18}\hat{\lambda}^2 & 10\hat{\lambda}^2 \\ -3\frac{\hat{\lambda}^2}{\hat{g}} & 0 & \hat{\lambda} + \frac{25}{3}\hat{\lambda}^2 \end{pmatrix} \qquad \hat{\gamma}^{(1)}_{\mathcal{O}_8} = \left\{-\frac{22}{3}; 1; \frac{7}{3}\right\}, \qquad \hat{\gamma}^{(2)}_{\mathcal{O}_8} = \left\{-\frac{136}{3}; \frac{25}{3}; \frac{269}{18}\right\}$$

Dim-10:

$$\mathbb{D}_{\mathcal{O}_{10,f}} = \begin{pmatrix} -\frac{22\hat{\lambda}}{3} - \frac{136}{3}\hat{\lambda}^2 & 0 & 0 & 0 & 0 \\ -\frac{\hat{\lambda}^2}{\hat{g}} & \frac{7\hat{\lambda}}{3} + \frac{269}{18}\hat{\lambda}^2 & 0 & 10\hat{\lambda}^2 & 0 \\ -\frac{209}{300}\frac{\hat{\lambda}^2}{\hat{g}} & -\frac{6\hat{\lambda}}{5} - \frac{5579\hat{\lambda}^2}{4500} & \frac{71\hat{\lambda}}{15} + \frac{2848}{125}\hat{\lambda}^2 & \frac{1493}{300}\hat{\lambda}^2 & \frac{5}{9}\hat{\lambda}^2 \\ -3\frac{\hat{\lambda}^2}{\hat{g}} & 0 & 0 & \hat{\lambda} + \frac{25}{3}\hat{\lambda}^2 & 0 \\ -\frac{19}{12}\frac{\hat{\lambda}^2}{\hat{g}} & \frac{139}{600}\hat{\lambda}^2 & \frac{499}{200}\hat{\lambda}^2 & -2\hat{\lambda} - \frac{143}{72}\hat{\lambda}^2 & \frac{17\hat{\lambda}}{3} + \frac{2195}{72}\hat{\lambda}^2 \end{pmatrix}$$

$$\hat{\gamma}^{(1)}_{\mathcal{O}_{10,f}} = \left\{-\frac{22}{3}; 1; \frac{7}{3}; \frac{71}{15}, \frac{17}{3}\right\}, \qquad \hat{\gamma}^{(2)}_{\mathcal{O}_{10,f}} = \left\{-\frac{136}{3}; \frac{25}{3}; \frac{269}{18}; \frac{2848}{125}, \frac{2195}{72}\right\}$$

Mixing matrices and spectrum

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix.

Dim-16 at 1-loop:

$$Z^{(1)}_{\mathcal{O}_{16,f}} = \frac{N_c}{\epsilon}\left(\begin{array}{c|ccccccccc|ccccc}
-\frac{11}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & \frac{7}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{5} & \frac{71}{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{5}{4} & \frac{221}{60} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & \frac{1}{10} & -\frac{19}{30} & \frac{37}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{17}{84} & -\frac{17}{28} & -\frac{47}{70} & -\frac{17}{28} & \frac{337}{84} & \frac{5}{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{20} & \frac{9}{20} & -1 & -\frac{31}{20} & -\frac{1}{4} & \frac{31}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{13}{30} & -\frac{13}{15} & \frac{13}{10} & -\frac{13}{10} & -\frac{5}{2} & \frac{13}{15} & \frac{961}{210} & \frac{8}{15} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{71}{105} & -\frac{212}{105} & \frac{141}{35} & -\frac{71}{35} & -\frac{141}{35} & \frac{79}{105} & -\frac{38}{35} & \frac{223}{35} & \frac{5}{14} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{17}{70} & \frac{19}{105} & -\frac{19}{70} & -\frac{121}{70} & -\frac{11}{42} & \frac{16}{105} & -\frac{6}{5} & \frac{127}{210} & \frac{559}{105} & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{17}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{9}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & -2 & \frac{1}{3} & \frac{43}{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{5}{2} & \frac{5}{2} & -\frac{11}{4} & \frac{67}{12}
\end{array}\right)$$

$$Z^{(1)}_{\mathcal{O}_{16,d}} = \frac{N_c}{\epsilon}\left(\begin{array}{cccccc|cc}
\frac{13}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{41}{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & -2 & \frac{301}{60} & -\frac{2}{3} & 0 & 0 & 0 & 0 \\
-1 & 1 & -\frac{3}{10} & \frac{25}{6} & 0 & 0 & 0 & 0 \\
-\frac{2}{5} & \frac{1}{5} & 0 & -\frac{1}{5} & \frac{307}{60} & \frac{7}{20} & 0 & 0 \\
\frac{1}{3} & -1 & \frac{1}{2} & -\frac{7}{3} & \frac{13}{12} & \frac{67}{12} & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{12} & \frac{67}{12}
\end{array}\right)$$

Mixing matrices and spectrum

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix.

Dim-16 at 2-loop:

$$
Z^{(2)}_{\mathcal{O}_{16,f}}\Big|_{\frac{1}{\epsilon}\text{-part.}} = \frac{N_c^2}{\epsilon}\left(\begin{array}{c|ccccccccc|ccccc}
-\frac{34}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-\frac{1}{3} & \frac{269}{72} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 0 & 0 & 0 & 0 \\
-\frac{209}{900} & -\frac{5579}{18000} & \frac{712}{125} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1493}{1200} & \frac{5}{36} & 0 & 0 & 0 \\
-\frac{31}{180} & \frac{53}{3600} & -\frac{36227}{28800} & \frac{3575983}{432000} & \frac{9793}{21600} & 0 & 0 & 0 & 0 & 0 & \frac{13}{16} & \frac{16877}{14400} & -\frac{7319}{14400} & 0 & 0 \\
-\frac{181}{900} & -\frac{60979}{36000} & \frac{78487}{72000} & -\frac{2177}{2000} & \frac{704167}{72000} & 0 & 0 & 0 & 0 & 0 & \frac{1229}{1200} & \frac{115501}{43200} & -\frac{9803}{43200} & 0 & 0 \\
-\frac{523}{3920} & -\frac{2201287}{29635200} & \frac{605939}{1975680} & -\frac{64128769}{24696000} & \frac{3303367}{9878400} & \frac{332422343}{29635200} & \frac{6699071}{14817600} & 0 & 0 & 0 & \frac{37547}{78400} & \frac{75071}{39200} & -\frac{497}{576} & \frac{103}{1440} & 0 \\
-\frac{809}{5600} & -\frac{12166789}{21168000} & \frac{11202299}{7056000} & -\frac{73487}{36750} & -\frac{9182209}{7056000} & \frac{37249}{156800} & \frac{26302879}{2116800} & 0 & 0 & 0 & \frac{1613}{3360} & \frac{17401}{6720} & \frac{19}{225} & \frac{1187}{2880} & 0 \\
-\frac{269}{2520} & \frac{125599}{10584000} & \frac{50369}{1323000} & -\frac{98317}{1176000} & \frac{73489}{392000} & -\frac{8625329}{3528000} & -\frac{97913}{756000} & \frac{90760559}{7408800} & \frac{25354501}{21168000} & \frac{40519}{56448} & \frac{184259}{1058400} & \frac{65297}{23520} & -\frac{420373}{211680} & \frac{248791}{235200} & -\frac{2747}{9408} \\
-\frac{19717}{176400} & \frac{3374557}{7408800} & -\frac{102465523}{74088000} & \frac{5260289}{1764000} & -\frac{6201763}{4939200} & -\frac{115070197}{24696000} & \frac{10687837}{9261000} & \frac{6498287}{9261000} & \frac{1025255701}{74088000} & -\frac{25511}{493920} & \frac{347437}{1764000} & \frac{863371}{302400} & -\frac{230747}{105840} & \frac{938797}{705600} & -\frac{78243}{196000} \\
-\frac{19717}{176400} & -\frac{2733089}{9261000} & \frac{88146899}{74088000} & -\frac{5678651}{3528000} & -\frac{1966229}{12348000} & \frac{17842339}{18522000} & -\frac{6878309}{4630500} & -\frac{58976629}{37044000} & \frac{8569667}{9261000} & \frac{179275483}{12348000} & \frac{28489}{661500} & \frac{54403}{14700} & -\frac{228689}{88200} & \frac{687461}{264600} & -\frac{485507}{5292000} \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{12} & 0 & 0 & 0 & 0 \\
-\frac{19}{36} & \frac{139}{2400} & \frac{499}{800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{143}{288} & \frac{2195}{288} & 0 & 0 & 0 \\
-\frac{1}{3} & \frac{4}{15} & \frac{121}{400} & \frac{637}{800} & -\frac{211}{800} & 0 & 0 & 0 & 0 & 0 & \frac{119}{120} & -\frac{15643}{7200} & \frac{79313}{7200} & 0 & 0 \\
-\frac{209}{900} & \frac{6299}{21168} & \frac{6767}{35280} & \frac{71063}{88200} & -\frac{34723}{176400} & \frac{25841}{58800} & -\frac{36091}{264600} & 0 & 0 & 0 & \frac{22723}{21600} & -\frac{35}{48} & -\frac{2861}{5400} & \frac{443801}{36000} & 0 \\
-\frac{31}{180} & \frac{13843}{105840} & \frac{8317}{15120} & -\frac{797}{35280} & \frac{5477}{35280} & \frac{2417}{3528} & \frac{611}{105840} & \frac{13975}{14112} & -\frac{5377}{10584} & -\frac{3581}{10080} & \frac{114221}{151200} & \frac{6017}{15120} & \frac{121}{216} & -\frac{3661627}{1411200} & \frac{63879443}{4233600}
\end{array}\right)
$$

$$
Z^{(2)}_{\mathcal{O}_{16,d}}\Big|_{\frac{1}{\epsilon}\text{-part.}} = \frac{N_c^2}{\epsilon}\begin{pmatrix}
\frac{575}{144} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{23347}{14400} & \frac{46517}{5760} & 0 & 0 & 0 & 0 & \frac{487}{1800} & 0 \\
\frac{3883}{4032} & -\frac{171823}{37800} & \frac{36597791}{3024000} & -\frac{29581}{16800} & 0 & 0 & -\frac{1789}{4800} & 0 \\
-\frac{9271}{11200} & -\frac{35239}{50400} & \frac{74209}{168000} & \frac{188599}{18900} & 0 & 0 & \frac{2101}{4800} & 0 \\
\frac{3287}{84000} & -\frac{2048479}{1176000} & \frac{422283}{392000} & -\frac{2501309}{1764000} & \frac{49211483}{3528000} & \frac{293221}{392000} & \frac{2764807}{2116800} & -\frac{61}{20160} \\
\frac{947587}{1058400} & -\frac{1555357}{705600} & \frac{16831}{29400} & -\frac{239641}{75600} & -\frac{381527}{2116800} & \frac{5839021}{423360} & -\frac{5807}{201600} & \frac{118933}{1411200} \\
\frac{3349}{7200} & -\frac{2591}{2400} & 0 & 0 & 0 & 0 & \frac{150391}{14400} & 0 \\
-\frac{45083}{44100} & \frac{16564}{11025} & \frac{5447}{117600} & \frac{380791}{176400} & \frac{1063}{29400} & -\frac{545189}{352800} & \frac{1176541}{1058400} & \frac{174229}{12600}
\end{pmatrix}
$$

Mixing matrices and spectrum

Anomalous dimensions for length-3 operators up to dimension 16:

dim	4	6	8	10	12	14	16
$\gamma^{(1)}_{f,\alpha}$	$-\frac{22}{3}$	/	$\frac{7}{3}$	$\frac{71}{15}$	$\frac{241}{30}, \frac{101}{15}$	$\frac{61}{6}, \frac{172}{21}$	$\frac{331}{35}, \frac{1212\pm\sqrt{3865}}{105}$
$\gamma^{(2)}_{f,\alpha}$	$-\frac{136}{3}$	/	$\frac{269}{18}$	$\frac{2848}{125}$	$\frac{49901119}{1404000}, \frac{8585281}{234000}$	$\frac{4392073141}{87847200}, \frac{685262197}{15373260}$	$\frac{231568398949}{4253886000}, \frac{355106171452034\pm95588158951\sqrt{3865}}{6576507756000}$
$\gamma^{(1)}_{f,\beta}$	$-\frac{22}{3}$	1	/	$\frac{17}{3}$	9	$\frac{43}{5}$	$\frac{67}{6}$
$\gamma^{(2)}_{f,\beta}$	$-\frac{136}{3}$	$\frac{25}{3}$	/	$\frac{2195}{72}$	$\frac{79313}{1800}$	$\frac{443801}{9000}$	$\frac{63879443}{1058400}$
$\gamma^{(1)}_{d,\alpha}$	/	/	/	$\frac{13}{3}$	$\frac{41}{6}$	$\frac{551\pm3\sqrt{609}}{60}$	$\frac{321\pm\sqrt{1561}}{30}$
$\gamma^{(2)}_{d,\alpha}$	/	/	/	$\frac{575}{36}$	$\frac{46517}{1440}$	$\frac{5809305897\pm19635401\sqrt{609}}{131544000}$	$\frac{229162584707\pm225658792\sqrt{1561}}{4130406000}$
$\gamma^{(1)}_{d,\beta}$	/	/	/	/	9	/	$\frac{67}{6}$
$\gamma^{(2)}_{d,\beta}$	/	/	/	/	$\frac{150391}{3600}$	/	$\frac{174229}{3150}$

Finite remainder

The finite remainders -> Higgs amplitudes with high-order top mass corrections in Higgs EFT:

$$\mathcal{L}_{\rm eff} = \hat{C}_0 H \mathcal{O}_{4;0} + \sum_{k=1}^{\infty} \frac{1}{m_{\rm t}^{2k}} \sum_i \hat{C}_i H \mathcal{O}_{4+2k;i}$$

There are “universal building blocks” that are independent of the operators:

The full transcendentality degree-4 part is universal:

$$\begin{aligned}\mathcal{R}_{\mathcal{O}}^{(2),\pm}\Big|_{\text{deg-4}} = &-\frac{3}{2}\text{Li}_4(u) + \frac{3}{4}\text{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{4}\log(w)\left[\text{Li}_3\left(-\frac{u}{v}\right) + \text{Li}_3\left(-\frac{v}{u}\right)\right] \\ &+ \frac{\log^2(u)}{32}\left[\log^2(u) + \log^2(v) + \log^2(w) - 4\log(v)\log(w)\right] \\ &+ \frac{\zeta_2}{8}\left[5\log^2(u) - 2\log(v)\log(w)\right] - \frac{1}{4}\zeta_4 + \text{perms}(u,v,w)\,,\end{aligned}$$

“maximal transcendentality principle” [Kotikov, Lipatov, Onishchenko, Velizhanin 2004]

Finite remainder

The finite remainders -> Higgs amplitudes with high-order top mass corrections in Higgs EFT:

$$\mathcal{L}_{\text{eff}} = \hat{C}_0 H \mathcal{O}_{4;0} + \sum_{k=1}^{\infty} \frac{1}{m_{\text{t}}^{2k}} \sum_i \hat{C}_i H \mathcal{O}_{4+2k;i}$$

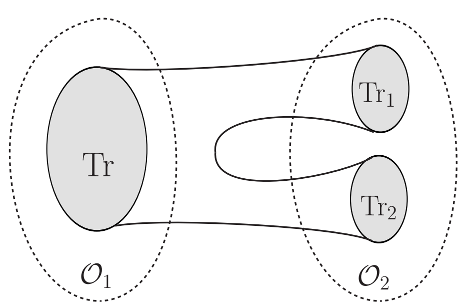
There are “universal building blocks” that are independent of the operators:

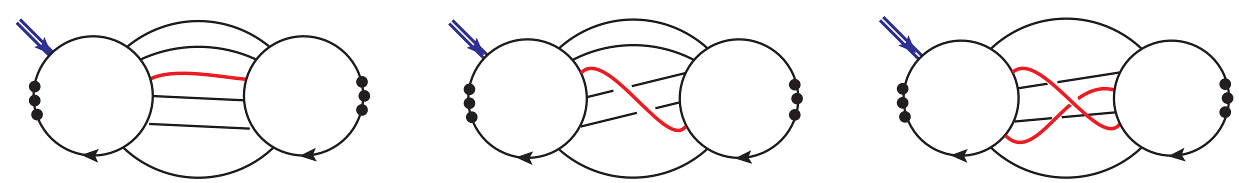
Degree-3 part and degree-2 part are consist of universal building blocks {T_3, T_2}, plus simple log functions:

$$\begin{aligned} T_3(u,v,w) :=& \Big[-\text{Li}_3\left(-\frac{u}{w}\right) + \log(u)\text{Li}_2\left(\frac{v}{1-u}\right) - \frac{1}{2}\log(u)\log(1-u)\log\left(\frac{w^2}{1-u}\right) \\ &+ \frac{1}{2}\text{Li}_3\left(-\frac{uv}{w}\right) + \frac{1}{2}\log(u)\log(v)\log(w) + \frac{1}{12}\log^3(w) + (u \leftrightarrow v)\Big] \\ &+ \text{Li}_3(1-v) - \text{Li}_3(u) + \frac{1}{2}\log^2(v)\log\left(\frac{1-v}{u}\right) - \zeta_2\log\left(\frac{uv}{w}\right) . \end{aligned}$$

$$T_2(u,v) := \text{Li}_2(1-u) + \text{Li}_2(1-v) + \log(u)\log(v) - \zeta_2 .$$

Non-planar form factors in N=4 SYM





- arXiv:2011.06540 with Guanda Lin (林冠达)

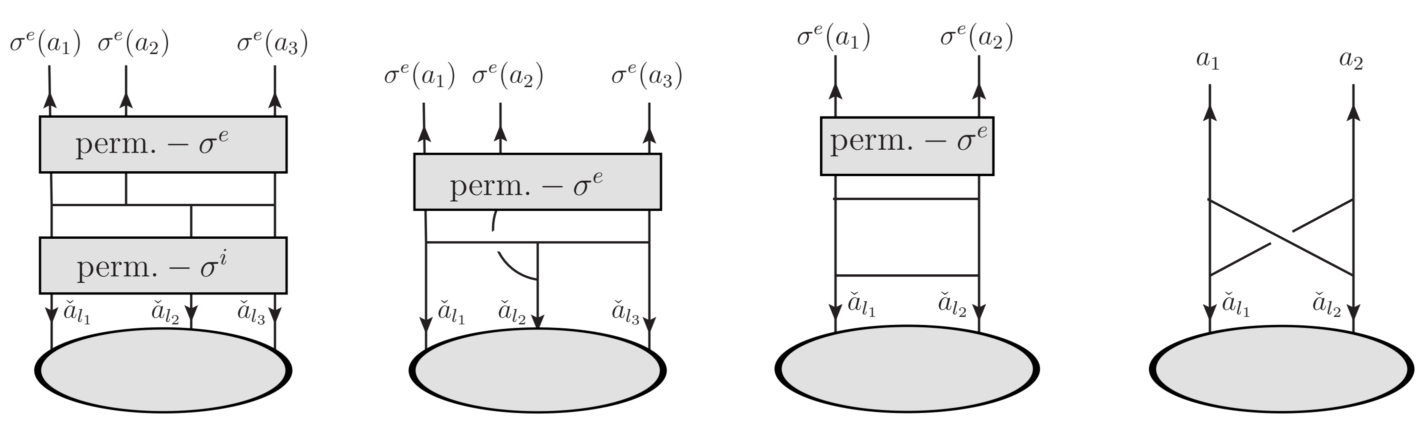
General strategy via Unitarity

In principle one may apply unitarity with full color dependence

An improved strategy is that:

Color decomposition → Unitarity with color-ordered blocks

$$\boldsymbol{F}^{(\ell)} = \sum_i \mathcal{C}_i \left[\mathcal{F}^{(\ell)}\right]_{\mathcal{C}_i}$$



(a) Cubic Graph: $\check{\mathcal{D}}_{\Delta_1}$ (b) Cubic Graph: $\check{\mathcal{D}}_{\Delta_2}$ (c) Cubic Graph: $\check{\mathcal{D}}_{\Delta_3}$ (d) Cubic Graph: $\check{\mathcal{D}}_{\Delta_4}$

$$\check{\mathcal{D}}_1 = \check{\mathcal{D}}_{\Delta_1}(\mathbf{1},\mathbf{1}), \quad \check{\mathcal{D}}_{19} = \check{\mathcal{D}}_{\Delta_2}(\mathbf{1}), \quad \check{\mathcal{D}}_{25} = \check{\mathcal{D}}_{\Delta_3}(\mathbf{1}), \quad \check{\mathcal{D}}_{27} = \check{\mathcal{D}}_{\Delta_4}(\mathbf{1})$$

General strategy via Unitarity

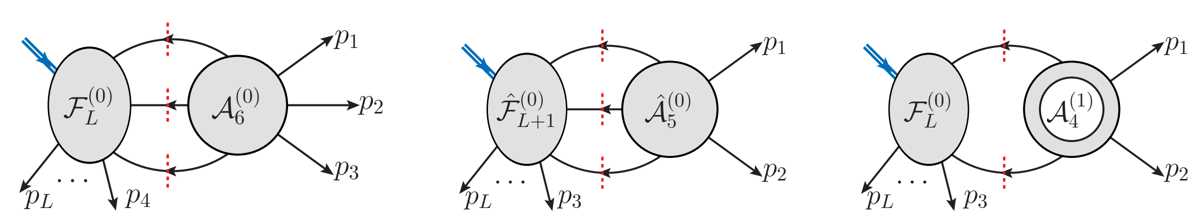
In principle one may apply unitarity with full color dependence

An improved strategy is that:

Color decomposition ⟶ Unitarity with color-ordered blocks

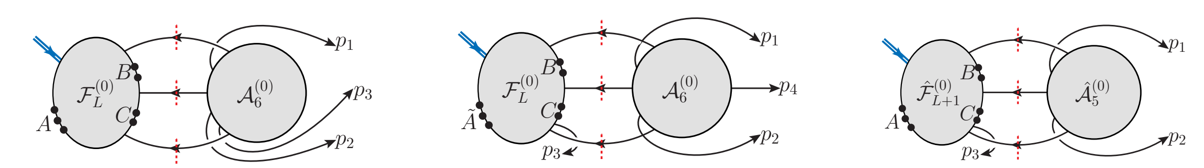
$$\boldsymbol{F}^{(\ell)} = \sum_i \mathcal{C}_i \left[\mathcal{F}^{(\ell)}\right]_{\mathcal{C}_i}$$

Planar cuts:



(a) (b) (c)

Non-planar cuts:



(a) s_{123} three-particle triple-cut. (b) s_{124} three-particle triple-cut. (c) s_{12} two-particle triple-cut.

General strategy

In principle one may apply unitarity with full color dependence

An improved strategy is that:

Color decomposition → Unitarity with color-ordered blocks

For special cases, an alternative more-power tool is:

“Color-Kinematics Duality”

Color-Kinematics duality

[Bern, Carrasco, Johansson 2008]

color factors $\longleftrightarrow$ momentum factors

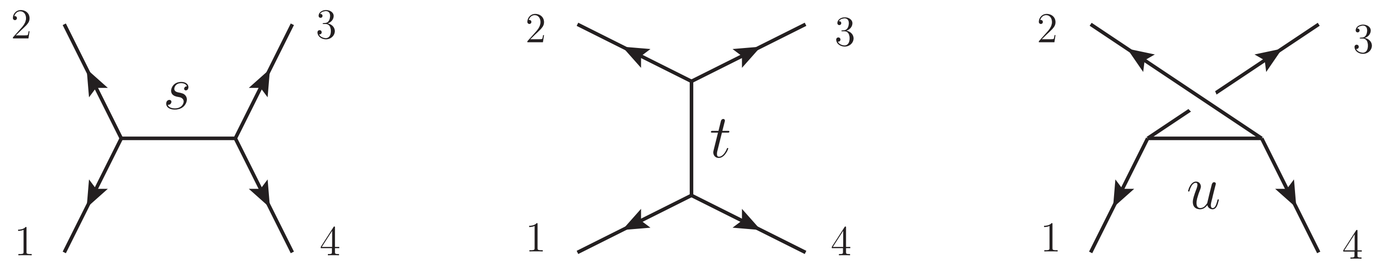
duality

$$\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b]T^c) \qquad s_{ij} = (p_i + p_j)^2$$

gauge symmetry **spacetime symmetry**

A very intriguing duality which is still not understood.

A four-point example



$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$c_s = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_2 a_3 b} \tilde{f}^{b a_4 a_1}, \quad c_u = \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_2 a_4}$$

$$c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$$

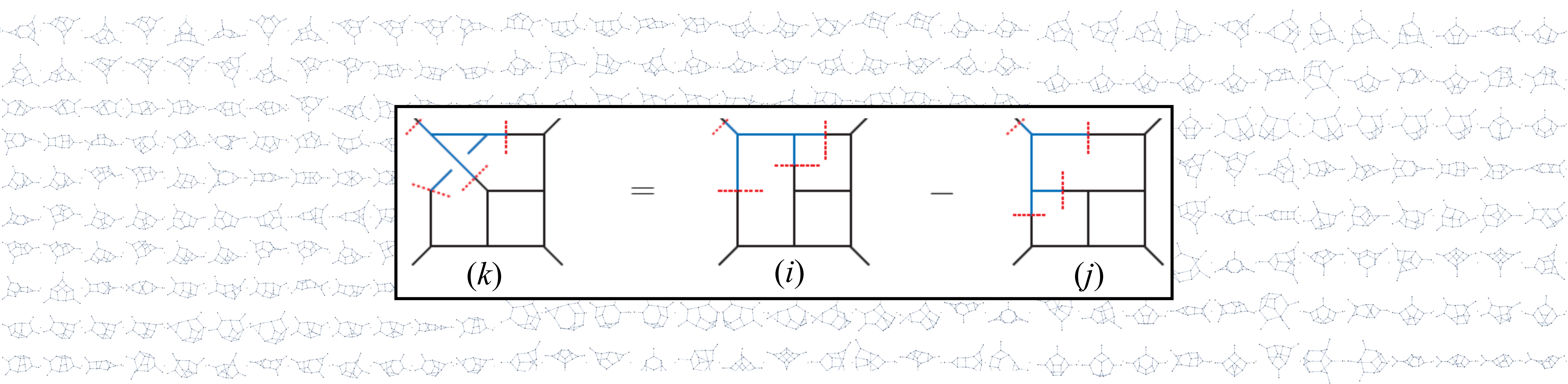
Jacobi identity

dual Jacobi relation

Form factors

For Sudakov form factor (with stress tensor operator), the construction has been obtained up to five loops.

G. Yang, PRL 117 (2016) no.27, 271602

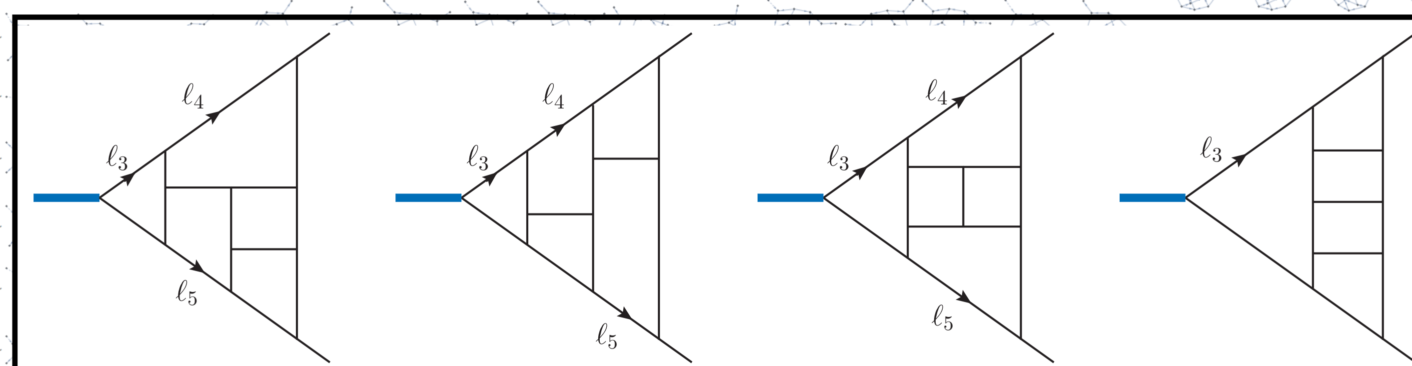


dual Jacobi relations

$$n_k = n_i - n_j$$

Form factors

For Sudakov form factor (with stress tensor operator), the construction has been obtained up to five loops.

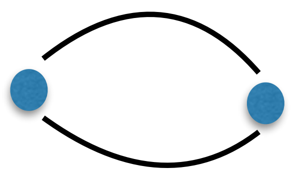


Master graphs

Form factors

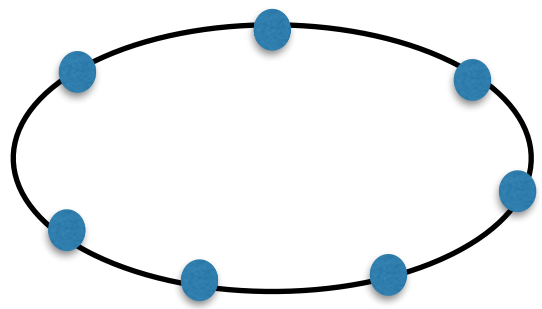
For Sudakov form factor (with stress tensor operator), the construction has been obtained up to five loops.

$$\mathcal{O}_{\mathrm{L=2}} = \mathrm{tr}(\phi^2)$$

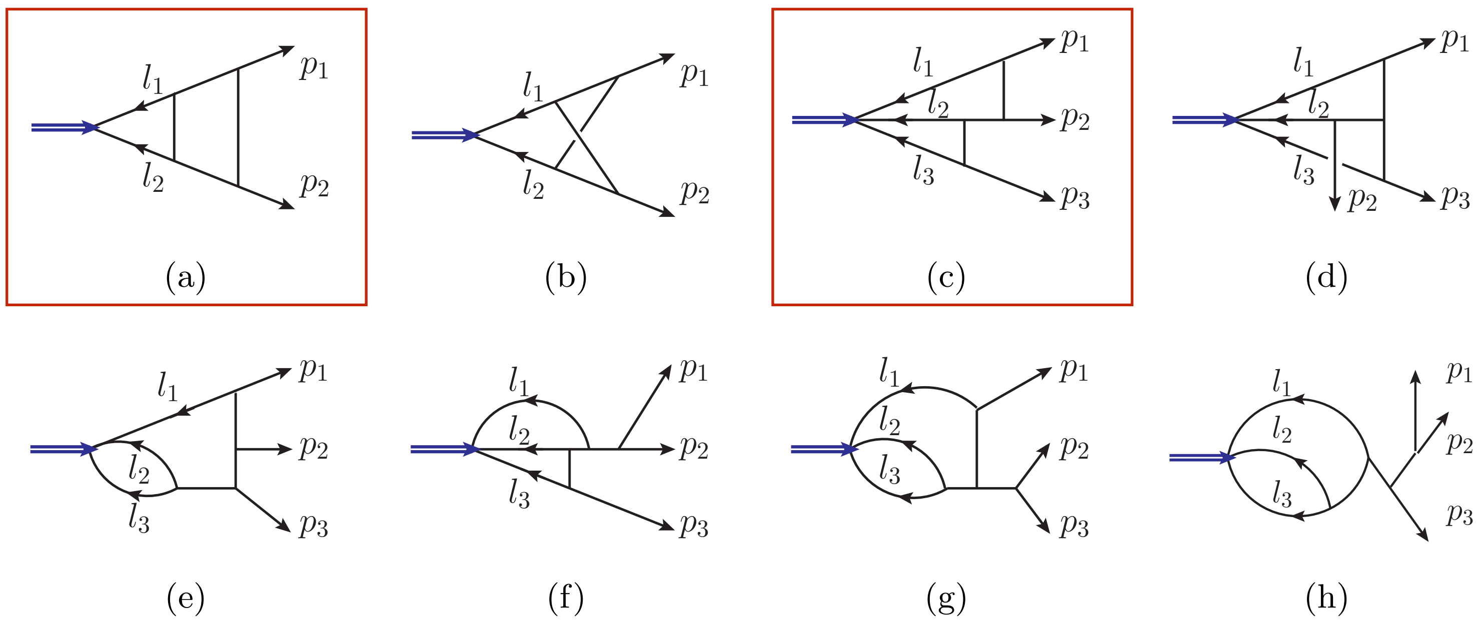


How about more general high dimensional operators?

$$\mathcal{O}_{\mathrm{L}} = \mathrm{tr}(\phi\phi\ldots\phi)$$

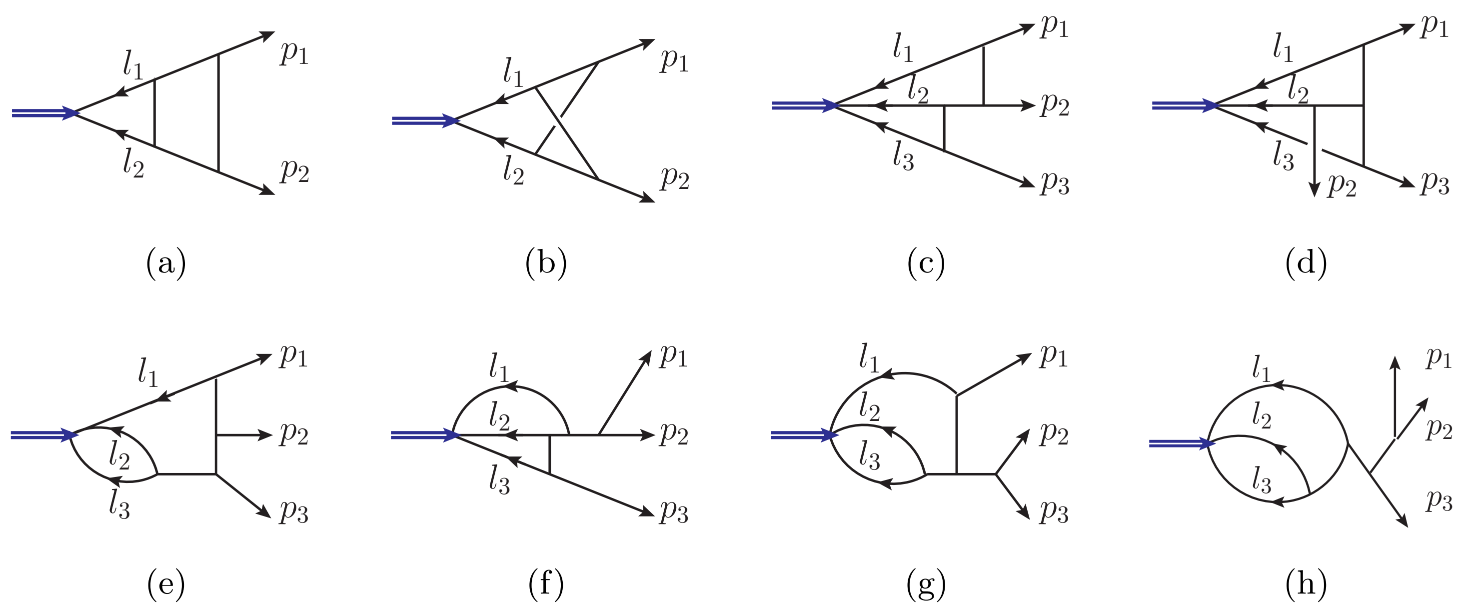


Two-loop solution of BPS form factors



Two-planar master topologies

Two-loop solution of BPS form factors



Γ_i	N_i	S_i
(a)	s_{12}^2	2
(b)	s_{12}^2	4
(c)	$s_{12}s_{l_2l_3} - \frac{1}{2}s_{123}s_{l_112} - \frac{1}{2}l_2^2 s_{12} + \frac{1}{2}l_1^2 s_{23}$	1
(d)	$2(l_1\cdot p_2)s_{13} - s_{l_11}s_{12} + \frac{1}{2}s_{l_11}s_{123} + s_{13}s_{12} - \frac{1}{2}l_1^2 s_{23}$+cyc.	6
(e)	$\frac{1}{2}s_{12}(l_2^2 - l_3^2)$	2
(f)	$\frac{1}{2}(s_{13}-s_{12})l_1^2$	2
(g)	$\frac{1}{2}(s_{12}-s_{13})(l_2^2-l_3^2)$	2
(h)	$\frac{1}{2}(s_{12}-s_{13})(l_2^2-l_3^2)$	4

$$\boldsymbol{I}_{12}^{(2)} = \sum_{\sigma\in S_2\times S_2}\sum_{i=\mathrm{a}}^{\mathrm{b}}\int\prod_{j=1}^{2}\frac{\mathrm{d}^D l_i}{i(\pi)^{\frac{D}{2}}}\frac{1}{S_i^{12}}\sigma\cdot\frac{\check{\mathcal{C}}_i^{12}N_i^{12}}{\prod_a d_{i,a}}$$

$$\boldsymbol{I}_{123}^{(2)} = \sum_{\sigma\in S_3\times S_3}\sum_{i=\mathrm{c}}^{\mathrm{h}}\int\prod_{j=1}^{2}\frac{\mathrm{d}^D l_i}{i(\pi)^{\frac{D}{2}}}\frac{1}{S_i^{123}}\sigma\cdot\frac{\check{\mathcal{C}}_i^{123}N_i^{123}}{\prod_a d_{i,a}}$$

New features of non-planar results

BDS ansatz is no-longer enough:

Planar

$$\underline{\mathcal{I}}^{(2)} = \frac{1}{2}\left(\underline{\mathcal{I}}^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\underline{\mathcal{I}}^{(1)}(2\epsilon) + \mathcal{R}^{(2)} + \mathcal{O}(\epsilon)$$

↓

non-planar

$$\underline{\boldsymbol{I}}^{(2)} = \frac{1}{2}\left(\underline{\boldsymbol{I}}^{(1)}(\epsilon)\right)^2 + \tilde{f}^{(2)}(\epsilon)\underline{\boldsymbol{I}}^{(1)}(2\epsilon) + \boldsymbol{H}^{(2)}(\epsilon) + \boldsymbol{R}^{(2)} + \mathcal{O}(\epsilon)$$

IR:

$$\boldsymbol{H}^{(2)} = \sum_{i<j<k} \boldsymbol{H}^{(2)}_{ijk}\,,$$

$$\boldsymbol{H}^{(2)}_{123} = \left(\sum_{\sigma\in S_3}(-1)^{\sigma}\check{\mathcal{D}}_{\Delta_2}(\sigma)\right)\frac{1}{4\epsilon}\log\left(\frac{-s_{12}}{-s_{23}}\right)\log\left(\frac{-s_{23}}{-s_{13}}\right)\log\left(\frac{-s_{13}}{-s_{12}}\right)$$

Bern, De Freitas and Dixon, 2002, ….
Explained by dipole formula: Becher and M. Neubert; Gardi and L. Magnea 2009

Form factor computation provides an independent check for non-planar IR structure two-loop amplitudes with general n-point.

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Finite remainder:

$$\boldsymbol{R}^{\rm NP} = \sum_{i<j<k} \boldsymbol{R}^{\rm NP}_{ijk}\,,$$

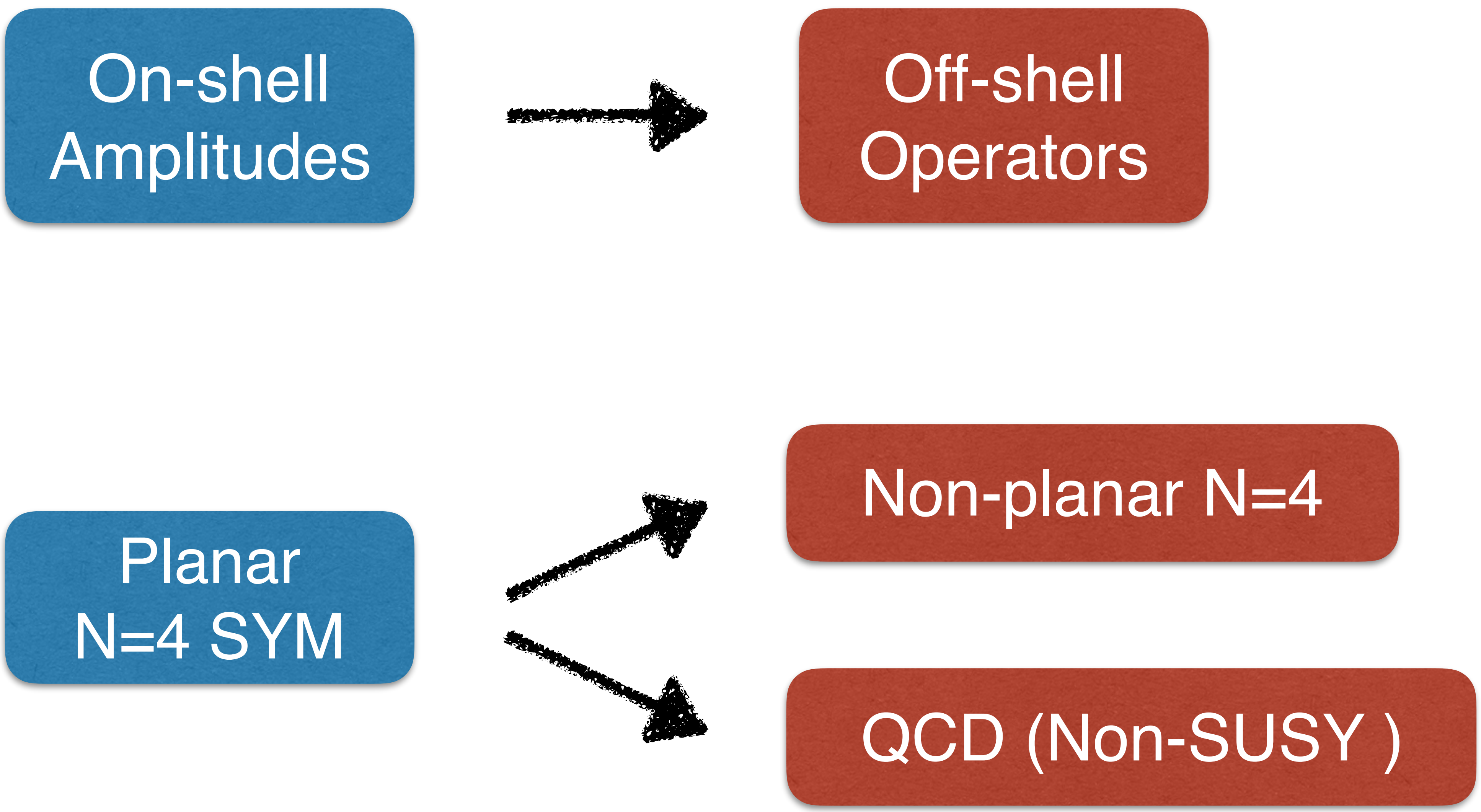
$$\boldsymbol{R}^{\rm NP}_{123} = \sum_{\sigma\in S_3}\check{\mathcal{D}}_{\Delta_2}(\sigma)\mathcal{R}^{\rm NP}_{\rm basis}(\sigma(u,v,w)) = \left(\sum_{\sigma\in S_3}(-1)^{\sigma}\check{\mathcal{D}}_{\Delta_2}(\sigma)\right)\mathcal{R}^{\rm NP}_{\rm basis}(u,v,w)$$

New non-planar maximally transcendental part

$$\mathcal{R}^{\rm NP}_{\rm basis}(u,v,w) = {\rm Li}_3\left(1-\frac{1}{u}\right)\log\left(\frac{v}{w}\right) + \frac{1}{12}\log(u)^3\log\left(\frac{v}{w}\right) + \zeta_2\log(1-u)\log\left(\frac{v}{w}\right)$$

Summary and Outlook

Summary


On-shell Amplitudes
Off-shell Operators
Planar N=4 SYM
Non-planar N=4
QCD (Non-SUSY)

Outlook

- Classification and two-loop renormalization for more generic operators (fermions, CP-odd, and finally in generic EFT)

- Integrability for QCD high-twist operators (planar 2-loop)

$\mathrm{tr}(D_+^{n_1} F D_+^{n_2} F \ldots D_+^{n_L} F)$ In progress with Qingjun Jin, Ke Ren and Rui Yu

- CK duality for higher loop cases and for non-BPS operators

3-loop solution also found To appear with Guanda Lin and Siyuan Zhang

- Explore hidden structures from the non-planar data

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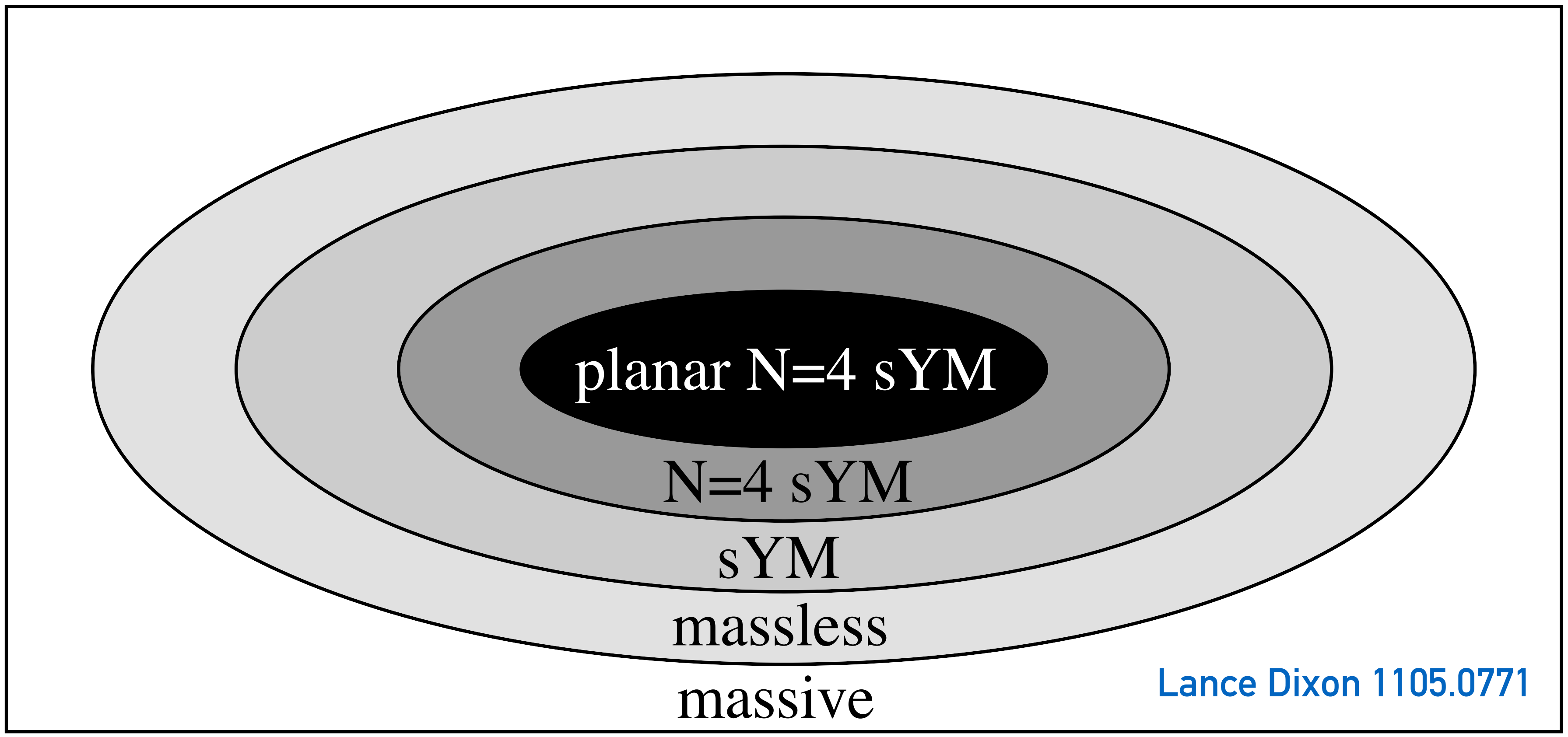
Thank you!

Backup

Hierarchy of simplicity



Non-planar N=4 SYM

Non-supersymmetric QCD

Loop structure of form factors

Form factors have divergences:

IR divergences	soft and collinear divergences
UV divergences	renormalization of coupling g and operators O

The IR and UV are mixed in general in a non-trivial way.

General structure of (bare) amplitudes/form factors:

full result = IR + UV + finite remainder

Universal infrared divergences

wanted UV divergences and finite parts

IR structure in QCD

Universal IR structure: [Catani 1998]

$$F^{(1)} = I^{(1)}(\epsilon)F^{(0)} + F^{(1),\text{fin}} + \mathcal{O}(\epsilon)\,,$$
$$F^{(2)} = I^{(2)}(\epsilon)F^{(0)} + I^{(1)}(\epsilon)F^{(1)} + F^{(2),\text{fin}} + \mathcal{O}(\epsilon)$$

e.g. for pure external gluons:

$$I^{(1)}(\epsilon) = -\frac{e^{\gamma_E\epsilon}}{\Gamma(1-\epsilon)}\left(\frac{C_A}{\epsilon^2} + \frac{\beta_0}{2\epsilon}\right)\sum_{i=1}^{n}(-s_{i,i+1})^{-\epsilon}\,,$$
$$I^{(2)}(\epsilon) = -\frac{1}{2}\left[I^{(1)}(\epsilon)\right]^2 - \frac{\beta_0}{\epsilon}I^{(1)}(\epsilon)$$
$$+\frac{e^{-\gamma_E\epsilon}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left[\frac{\beta_0}{\epsilon} + \left(\frac{67}{9} - \frac{\pi^2}{3}\right)C_A - \frac{10}{9}n_f\right]I^{(1)}(2\epsilon)$$
$$+\,n\frac{e^{\gamma_E\epsilon}}{\epsilon\Gamma(1-\epsilon)}\left[\left(\frac{\zeta_3}{2} + \frac{5}{12} + \frac{11\pi^2}{144}\right)C_A^2 + \frac{5n_f^2}{27} - \left(\frac{\pi^2}{72} + \frac{89}{108}\right)C_A n_f - \frac{n_f}{4C_A}\right]$$

Finite remainder

The intrinsic information is contained in the finite part:

$$F^{(1)} = I^{(1)}(\epsilon)F^{(0)} + F^{(1),\text{fin}} + \mathcal{O}(\epsilon),$$
$$F^{(2)} = I^{(2)}(\epsilon)F^{(0)} + I^{(1)}(\epsilon)F^{(1)} + F^{(2),\text{fin}} + \mathcal{O}(\epsilon)$$

There are six different color factors: $\mathscr{R}^{(l)}_{\mathcal{O}} = \mathscr{F}^{(l),\text{fin}}_{\mathcal{O}} / \mathscr{F}^{(0)}_{\mathcal{O}}$

$$\mathscr{R}^{(2)}_{\mathcal{O}} = N_c^2 \mathscr{R}^{(2),N_c^2}_{\mathcal{O}} + N_c^0 \mathscr{R}^{(2),N_c^0}_{\mathcal{O}} + \frac{1}{N_c^2}\mathscr{R}^{(2),N_c^{-2}}_{\mathcal{O}} + n_f N_c \mathscr{R}^{(2),n_f N_c}_{\mathcal{O}} + \frac{n_f}{N_c}\mathscr{R}^{(2),n_f/N_c}_{\mathcal{O}} + n_f^2 \mathscr{R}^{(2),n_f^2}_{\mathcal{O}}$$

A different expansion:

$$\mathscr{R}^{(2)}_{\mathcal{O}} = C_A^2 \mathscr{R}^{(2),C_A^2}_{\mathcal{O}} + C_A C_F \mathscr{R}^{(2),C_A C_F}_{\mathcal{O}} + C_F^2 \mathscr{R}^{(2),C_F^2}_{\mathcal{O}} + n_f C_A \mathscr{R}^{(2),n_f C_A}_{\mathcal{O}} + n_f C_F \mathscr{R}^{(2),n_f C_F}_{\mathcal{O}} + n_f^2 \mathscr{R}^{(2),n_f^2}_{\mathcal{O}}$$

$$C_A = N_c, \qquad C_F = \frac{N_c^2 - 1}{2N_c}$$