第一届"场论与弦论" 及相关数学物理研讨会

### Form factors of high-dimensional operators in non-planar N=4 SYM and QCD



Gang Yang



**Based on recent work:** 

- **• arXiv:2011.02494 with Qingjun Jin (靳庆军) and Ke Ren (任可)**
- **• arXiv:2011.06540 with Guanda Lin (林冠达)**

# Content

**Motivation**

**Form factors**

**QCD Non-planar N=4 SYM**

**Outlook**

# Background and Motivation

## Progress in amplitudes

Significant progress has been made in the study of amplitudes in past years.

New Structures New Methods

# Feynman diagram



Feynman diagram is a universal tool, but in practice it can be very complicated.

4-gluon tree:

s t 2 1 3 4 2  $\sim$  3  $1 \times 4$  $\overline{u}$  $1 / 4$  $2 \searrow$  3 3 4 s 2 1



#### n-gluon tree amplitudes:  $\frac{1}{2}$  $\blacksquare$  $\frac{1}{2}$ u , (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (<br>(2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1), (2.1)

s



Atree

### Parke-Taylor MHV formula ate expressions. The prime example for this is the Parke-Taylor formula [3], describing a

so that it is desirable to choose that it is desirable to choose the same reference momenta for all gluons of  $[Parke, \, \text{Taylor}, \, \text{1986}]$ colour-ordered n-gluon maximally helicity violating (MHV) scattering (MHV) scattering and trees and trees and  $\sim$  1986] scattering amplitude 1 at the scattering amplitude 1 at the scattering amplitude 1 at the scattering given helicity, and to take this momentum to be the momentum of one of the

Any n-gluon tree MHV amplitudes: Any n-giuon tree ivirty amplitudes:  $\mathbf{N}$  be-helicity gluons. This will generate the number of non-vanishing  $\mathbf{r}$ invariants. It also turns out that within the set of choices suggested by these properties, it is preferable to choose a reference momentum that is cyclicly adjacent

$$
A_n^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}
$$

results obtained by Feynman diagrams:



The reason for using the spinor helicity method is now evident; many of the dots now evident; many of the dots

### **Classical Polylogarithms for** Amplitudes sand **Wilson Leoops** Another two-loop example **Sia-gluon WHI stophyle Sight get SYM**



**A.B. Goncharov M. Spradlin C. Vergu A. Volovich [Del Duca, Duhr, Smirnov 2010]** (heroic computation)

#### Complicated results of 17 pages  $\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1$  $\overline{ }$  $\sim$   $\sqrt{ }$  $\cdot$   $\cdot$ G <sup>0</sup>, <sup>1</sup>  $\mathbf{L}$  $\mathbb{R}^3$  $\sim$  100  $\pm$  $\overline{a}$  $\mathbf{u}$  +  $\mathbf{u}$ −<br>⊢ 1"  $\mathbf{r}$  $\overline{\phantom{0}}$  $\overline{a}$  $\mathsf{I}$ u2  $\bf{l}$ ' I  $\bigcup$  $211$  $C\acute{c}$  $\mathbf l$ <u>l</u>  $\overline{\phantom{0}}$ , 1 **v**  $\lambda$ u<sup>1</sup> + u<sup>2</sup> es<br>− **J**  $\bigcup$  $\mathbf{L}$ , 1  $\Gamma$  $S_{\cdot}$  $\overline{\phantom{a}}$  + **)∣** , 1

1





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### [Del Duca, Duhr, Smirnov 2010]

u3  $, 0, 1)$  G

G

G

G

G

G

G

G

G

G

G

G

G

G

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4

4

1

4

1 − u<sup>1</sup>  $v_{123}$ , 0,  $-$ 1 − u<sup>1</sup>

− 1 1 − u<sup>1</sup>  $, v_{132}, 1, 0; 1$ 

1 − u<sup>2</sup>  $, 0, -1$ 1 − u<sup>2</sup> , v213; 1"

 $\left( v_{213};1\right)$ 

1 − u<sup>2</sup> ; 1"

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, <sup>1</sup> 1 − u<sup>2</sup> ; 1"

 $v_{213}, 0, \frac{1}{2}$ 

#### G, u3 − 1 :harov  $\mathbf{r}$  $-nc$  $\mathbf{r}$  , 1  $\mu$  in  $\mu$  , which is not the contract of  $\mu$  and  $\mu$ "multiple(Goncharov)-polylogrithm function"

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 $\div$  $0, 1; \frac{u_1 + u_3 - 1}{u_1 + u_2 - 1}$ u<sup>1</sup> − 1 <sup>+</sup> <sup>1</sup>  $\frac{1}{12}\pi^2H\left(0,1;\left(u_1+u_3\right)\right)-\frac{1}{2}$  $\frac{1}{24}\pi^2H$ 0, 1; ks + us − 1  $u_3 - 1$ 

G  $0, -1$ ; 1"  $H(1, 0; u_1) + \frac{1}{2}$ G , <sup>1</sup>

G  $\frac{1}{2}$ ; 1"  $H(1, 0; u_1) + \frac{1}{2}$ G , <sup>1</sup>

G , <sup>u</sup><sup>1</sup> <sup>−</sup> <sup>1</sup> u<sup>1</sup> + u<sup>3</sup> − 1  $H(1, 0; u_1) + \frac{1}{2}$ 

G , uzu; 1  $H(1, 0; \omega) - \frac{3}{2}$ 

H 0, 1; un + u<sub>3</sub> − 1 u<sup>1</sup> − 1  $H(1, 0; u_1) - \frac{1}{2}$ 

G  $\sim$ ; 1"  $H(1, 0; \omega) + \frac{1}{2}$ G

G  $\overline{\phantom{0}}$  $\alpha$  $H(1, 0; u_2) + \frac{1}{2}$ G , <sup>1</sup>

G  $\overline{\phantom{0}}$ ; 1"  $H(1, 0; u_2) + \frac{1}{2}$ G , <sup>1</sup>

G 1 − u<sup>1</sup> , u<sub>123</sub>; 1<sup>3</sup>  $H(1, 0; u_2) - \frac{3}{2}$ 

H <sup>0</sup>, 1; <sup>u</sup><sup>1</sup> <sup>+</sup> <sup>u</sup><sup>2</sup> <sup>−</sup> <sup>1</sup> u<sup>2</sup> − 1  $H(1, 0; u_2) - \frac{1}{2}$ 

G  $_{0}$ ,  $_{1}$ ; 1"  $H(1, 0; u_3) - \frac{1}{2}$ G  $_{0}$ ,  $^{-1}$ 

G , <sup>1</sup> ; 1"  $H(1, 0; u_3) + \frac{1}{2}$ G 1 − u<sup>2</sup>

G , <sup>1</sup> ; 1"  $H(1, 0; u_3) - \frac{1}{2}$ 

G , <sup>1</sup> ; 1"  $H(1, 0; u_3) - \frac{1}{2}$ G 1 − u<sup>2</sup>

H 0, 1; u2 + u<sub>3</sub> − 1  $H(1, 0; u_3) - \frac{1}{2}$ 

 $\frac{1}{24}\pi^2H(1,1;u_1)+\frac{1}{2}$  $\frac{1}{24}\pi^2H\left(1,1;u_2\right)+\frac{1}{2}$ 

H (0; u3) H 0, 0, 1; u<sub>1</sub> + u<sub>2</sub> − 1

 $H(0; u_3) H(0, 0, 0; u_2) + \frac{1}{2}$ 

 $H\left(0;u_{2}\right)H\left(0,0,1;\left(u_{1}+u_{2}\right)\right)-\frac{1}{2}$ 

 $H\left(0;u_{1}\right)H$ 

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3  $H\left(0;u_{1}\right)H\left(0;u_{2}\right)H\left(1,0;u_{3}\right)-\frac{3}{2}$ 

1  $H$  (0; u<sub>2</sub>) H  $0, 0, 1; \frac{u_1 + u_3 - 1}{2}$ 

 $\frac{1}{12}\pi^2H\left(0,1;\left(u_2+u_3\right)\right)-\frac{1}{2}$  $0, -1$ 



, 1; 1"  $H(0; u_2) +$ 

 $\langle \cdot \rangle$  $H(0; u3) +$ 

1 − u<sup>2</sup> , u231; 1"  $H\left(0;u_{1}\right)H\left(0;u_{2}\right)$  –

1 − u<sup>2</sup> , v231; 1"  $H$  (0; u<sub>2</sub>)  $H$  (0; u<sub>3</sub>) +

 $H$  (0; u<sub>2</sub>)  $H$  (0; u<sub>3</sub>) +



4

G 0, <u>11</u> 1 − u<sup>1</sup>  $, v_{123}; 1$  $H(0; u_2) - \frac{1}{2}$ G  $a \rightleftharpoons$ 1 − u<sup>1</sup>  $, v_{132}; 1$  $H$  (0;  $u_{2}$ ) +

u3

 $u_2 + u_3$ 

u3

u2

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 $u_2 + u_3$ 

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<sup>0</sup>, <sup>1</sup> 1 − u<sup>2</sup> , v231; 1"  $H(0; u_2) -$ 

<sup>0</sup>, <sup>1</sup> , v321; 1"  $H(0; u_2) +$ 

 $0, u_{312}, \ldots$ 1 − u<sup>3</sup>

 $0, v_{213}, \frac{1}{2}$ 

 $0, v_{312}, \frac{1}{2}$ 

! 1

1 − u<sup>1</sup>

! 1 1 − u<sup>2</sup>

! 1 1 − u<sup>2</sup>

! 1 1 − u<sup>3</sup>

! 1 1 − u<sup>3</sup>

1 − u<sup>2</sup>

1 − u<sup>1</sup>

1 − u<sup>1</sup>

1 − u<sup>2</sup> , <sup>1</sup>

1 − u3 , <sup>1</sup>

1 − u<sup>3</sup>

1 − u<sup>3</sup>

 $v_{123}$ ,  $-$  1 1 − u<sup>1</sup>

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G 1 − u<sup>3</sup>

 $v_{122}$ , 1,  $-$ 1 1 − u<sup>1</sup>  $H(0; u_2) - \frac{1}{2}$  $v_{132}$  ,  $-$  1 1 − u<sup>1</sup>

 $H(0; u2) +$ 

JHEP05(2010)084 0, u<sub>123</sub>, u<sub>2</sub> − 1  $u_1 + u_2 - 1$  $H(0; w_2) H(0; u_2) +$ 1 − u<sup>2</sup>  $H(0; u_2) +$  $H(0; u_2) +$  $, v_{132}; 1$  $H(0; u_2) +$ , <sup>1</sup> 1 − u<sup>1</sup> , v.v. 1 H (0; u2) − , u123, <sup>1</sup> 1 − u<sup>1</sup>  $H(0; u_2) +$ ! 1  $,$  vi21, 0; 1)  $H(0; u2) +$  $,$  v132, 0; 1.  $H(0; u2) (0, v_{213}; 1)$  $H(0; u2) +$ 1 − u<sup>2</sup> , v213; 1"  $H(0; u2) +$  $(v_{213}, 1; 1)$  $H(0; u_2) +$ , v231, 1; 1"  $H(0; u2) +$  $(0, v_{312}; 1)$  $H(0; u_2) +$ 1 − u<sup>3</sup>  $, v_{312}; 1$  $H(0; u_2) +$  $(u_{312}, 1; 1)$  $H$  (0;  $u_{2}$ ) +  $v_{312}$ ,  $1$ 1 − u<sup>3</sup>  $H(0; u_2) , v_{312}, \frac{1}{100}$ 1 − u<sup>3</sup>  $H(0; u_2) +$  $v_{321}$ ,  $1$ 1 − u<sup>3</sup>  $H(0; u_2) +$  $H(0; u_2) H(0; u_2) +$ 1 − u<sup>2</sup> 1 − u<sup>2</sup>  $\frac{1}{2}$ 1 − u<sup>3</sup> ; 1"  $H(0; u2) - \frac{3}{2}$  $\frac{1}{2}$ 1 − u<sup>3</sup>  $H(0; u2) +$  $v_{321}, 1, \frac{1}{2}$ 1 − u<sup>3</sup> ; 1"  $H(0; w_2) + \frac{1}{2}$ G  $v_{321}$ ,  $\frac{1}{2}$ 1 − u<sup>3</sup>  $H(0; u_2) +$ <u>, ا</u>  $H(0; u_1) H(0; u_2) + \frac{1}{2}$ G  $\frac{1}{2}$  $H(0; u_1) H(0; u_2) +$ 1 − u<sup>3</sup> , <sup>u</sup><sup>1</sup> <sup>−</sup> <sup>1</sup>  $u_1 + u_3 - 1$  $H(0; u_1) H(0; u_2) -$ 1 − u<sup>3</sup>  $, u_{312}; 1$  $H\left(0;u_{1}\right)H\left(0;u_{2}\right)-\frac{1}{2}$  $\left(1\right)$ 1 − u<sup>3</sup> , v312; 1"  $H$  (0; u<sub>1</sub>)  $H$  (0; u<sub>2</sub>) − 1 − u<sup>3</sup>  $\binom{3}{2}$  $H(0; u_1) H(0; u_2) + \frac{5}{2}$  $\frac{1}{24}\pi^{2}H$  (0; u1)  $H$  (0; u2)  $\alpha$   $^2$ , <sup>1</sup>  $H(0; u_3) - \frac{1}{2}$ <sup>0</sup>, <sup>1</sup> , <sup>1</sup>  $H(0; u_3) +$ <sup>0</sup>, <sup>u</sup><sup>2</sup> <sup>−</sup> <sup>1</sup>  $u_1 + u_2 - 1$ , <sup>1</sup> 1 − u<sup>1</sup> ; 1"  $H(0; u_3) - \frac{3}{2}$ <sup>0</sup>, <sup>1</sup> , <sup>1</sup> ; 1"  $H(0; u_3) -$ <sup>0</sup>, <sup>1</sup> u3 <u>, ا</u>  $u_2 + u_3$  $H(0; w_3) - \frac{1}{2}$ G <sup>0</sup>, <sup>u</sup><sup>3</sup> <sup>−</sup> <sup>1</sup>  $u_2 + u_3 - 1$ , <sup>1</sup> 1 − u<sup>2</sup> :1)  $H(0; u_3) -$ 1 − u<sup>1</sup>  $, 1, 1$ ; 1"  $H(0; u_3) + \frac{1}{2}$ ! 1 1 − u<sup>1</sup> , <sup>u</sup><sup>2</sup> <sup>−</sup> <sup>1</sup> u<sub>1</sub> + u<sub>2</sub> − 1  $H(0; u_3) -$ 1 − u<sup>1</sup> , <sup>u</sup><sup>2</sup> <sup>−</sup> <sup>1</sup> u<sup>1</sup> + u<sup>2</sup> − 1 , <sup>1</sup> 1 − u<sup>1</sup>  $H\left(0;u_{3}\right)+\frac{1}{2}$ G , <sup>0</sup>, <sup>1</sup>  $H$  (0;  $u_3$ ) − , <sup>0</sup>, <sup>1</sup>  $H(0; u_3) + \frac{1}{2}$ , 1 , <sup>1</sup>  $H(0; u_3) +$ 1 − u<sup>2</sup> , <sup>u</sup><sup>3</sup> <sup>−</sup> <sup>1</sup>  $u_2 + u_3 - 1$  $, 0; 1)$  $H(0; u_2) + \frac{1}{2}$ 1 − u<sup>2</sup> , <sup>u</sup><sup>3</sup> <sup>−</sup> <sup>1</sup> u<sup>2</sup> + u<sup>3</sup> − 1 , <sup>1</sup> 1 − u<sup>2</sup>  $H$  (0; u<sub>3</sub>)− 1 − u<sup>2</sup> , <sup>u</sup><sup>3</sup> <sup>−</sup> <sup>1</sup> u<sup>2</sup> + u<sup>3</sup> − 1 , <sup>u</sup><sup>3</sup> <sup>−</sup> <sup>1</sup> u<sup>2</sup> + u<sup>3</sup> − 1  $H(0; u_3) + \frac{3}{2}$ u2  $, 0, 1$ u3  $H(0; u_3) 0, -1$  $H(0; u_3) + \frac{1}{2}$ , 1 <u>, ا</u>  $H(0; u_3) -$ , <sup>0</sup>, <sup>1</sup>  $H(0; w_3) - \frac{3}{2}$ G  $, 0, 1$  $H(0; u_3) +$ , 1 , <sup>1</sup>  $H(0; u_3) + \frac{1}{2}$ G , 1 , <sup>1</sup>  $H(0; u_3) +$ , 1 , <sup>1</sup>  $H(0; u_3) + \frac{1}{2}$ G , 1 , <sup>1</sup>  $H(0; u_3) 0, -1$ 1 − u<sup>1</sup>  $, v_{123}; 1$  $H(0; u_3) + \frac{1}{2}$ 4 G <sup>0</sup>, <sup>1</sup> 1 − u<sup>1</sup>  $, v_{132}; 1$  $H(0; u_3) _{0}$   $_{1}$ 1 − u<sup>2</sup> , v<sub>213</sub>; 1<sup>"</sup>  $H(0; u_3) + \frac{1}{2}$ G <sup>0</sup>, <sup>1</sup> 1 − u<sup>2</sup> , v231; 1"  $H(0; u_3) +$  $_{0}$ ,  $_{1}$ 1 − u<sup>3</sup> , v312; 1"  $H(0; u_3) + \frac{1}{2}$ 4 <sup>0</sup>, <sup>1</sup> 1 − u<sup>3</sup>  $, v_{321}; 1$  $H(0; u_3) -$ 

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 $v_{213}, 1, \frac{1}{2}$ 1 − u<sup>2</sup>  $\langle i \rangle$  $H(0; u_2) + \frac{1}{2}$ 4  $v_{212} =$ 1 − u<sup>2</sup>  $H(0; u_2) +$ 

 $v_{231}$ , 1,  $\frac{1}{2}$ 

; 1"  $H(0; u_2) + \frac{1}{2}$ G  $v_{231}$ ,  $\frac{1}{2}$ 

 $H(0; u_2) -$ 

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4  $0, u_{123}, \frac{1}{2}$ 1 − u<sup>1</sup>  $H(0; u_3) - \frac{1}{2}$ 4  $0, u_{223}, \frac{1}{2}$ u3  $H(0; u_3) +$ 

1  $0, v_{123}, \frac{1}{2}$ ; 1"  $H(0; u_3) + \frac{1}{2}$ 0, v<sub>231</sub>, 1

 $0, u_{231}, \frac{1}{2}$  $H(0; u_3) - \frac{1}{2}$  $0, u_{231}, \frac{ux-1}{2}$ u<sup>2</sup> + u<sup>3</sup> − 1

 $_{0,\text{cm}, \frac{1}{2}}$ : 1  $H(0; \omega_3) + \frac{1}{2}$ 0, v321, 1

1 − u<sup>1</sup>  $, v_{132}, \frac{1}{2}$ 1 − u<sup>1</sup>  $H(0; u_3) - \frac{1}{2}$ 1 − u<sup>2</sup>  $(0, v_{213}; 1)$ 

1 − u<sup>2</sup>  $(0, v_{233}; 1)$  $H(0; u_3) - \frac{1}{2}$ 1 − u<sup>2</sup> , <sup>1</sup> 1 − u<sup>2</sup>  $, v_{212}; 1$ 

1 − u<sup>2</sup> , <sup>1</sup> 1 − u<sup>2</sup> , v231; 1"  $H(0; u_3) - \frac{1}{2}$ 1 − u<sup>2</sup>  $, u_{231}, 0; 1$ 

1 − u<sup>2</sup>  $, u_2, \frac{1}{2}$ 1 − u<sup>2</sup> (r)  $H(0; u_3) + \frac{1}{2}$ 1 − u<sup>2</sup> , u<sub>231</sub>, u<sub>3</sub> − 1 u<sup>2</sup> + u<sup>3</sup> − 1

1 − u<sup>2</sup>  $, v_{213}, 0; 1$  $H\left(0;\,u_{3}\right)-\frac{1}{2}$ 1 − u<sup>2</sup>  $v_{213}$ ,  $\frac{1}{2}$ 1 − u<sup>2</sup>

1 − u<sup>2</sup>  $, v_{231}, 0; 1$  $H(0; u_3) + \frac{1}{2}$ 1 − u<sup>2</sup>  $v_{231}, \frac{1}{2}$ 1 − u<sup>2</sup>

1 − u<sup>3</sup> , <sup>0</sup>, v312; 1"  $H(0; \omega_3) + \frac{1}{2}$ 

1 − u<sup>3</sup> , <sup>1</sup> 1 − u<sup>3</sup>  $,\text{mz1}$  $H(0; u3) + \frac{1}{2}$ , <sup>1</sup>  $, \text{max1}$ 

1 − u<sup>3</sup>  $, v_{312}, 1; 1$  $H(0; u_3) + \frac{1}{2}$ 1 − u<sup>3</sup>  $v_{312}$ ,  $1$ 1 − u<sup>3</sup>

1 − u<sup>3</sup>  $, v_{321}, 1; 1)$  $H(0; u_3) + \frac{1}{2}$ 1 − u<sup>3</sup>  $v_{321}$ ,  $1$ 1 − u<sup>3</sup>

 $v_{123}$ , 1,  $\frac{1}{2}$ 1 − u<sup>1</sup> ; 1"  $H(0; u_3) - \frac{3}{2}$  $v_{22} =$  $H$  (0; u<sub>3</sub>) +

 $v_{132}$ , 1,  $\frac{1}{2}$ 1 − u<sup>1</sup> ; 1"  $H(0; u_3) + \frac{1}{2}$  $v_{13}$   $H$  (0; u<sub>3</sub>) −

 $v_{212}$ , 1,  $\frac{1}{2}$ 1 − u<sup>2</sup> ; 1"  $H(0; u_3) - \frac{1}{2}$  $v_{23}$   $\rightarrow$ 1 − u<sup>2</sup>  $H$  (0; u<sub>3</sub>) +

 $v_{23}$ , 1,  $\frac{1}{2}$ 1 − u<sup>2</sup> ; 1"  $H(0; u_3) + \frac{3}{2}$  $v_{\rm{23}}$ 1 − u<sup>2</sup>  $H$  (0; u<sub>3</sub>) +

 $(0, v_{123}; 1)$  $H(0; \omega_3) + \frac{1}{2}$ 

 $, u_{123}, 1; 1)$  $H(0; u_3) + \frac{1}{2}$ 

 $, u_{123}, \frac{1}{2}$  $H(0; u_3) - \frac{1}{2}$ 

 $v_{123}$ ,  $1$  $H(0; u_3) + \frac{1}{2}$ 

, <sup>1</sup> , v123; 1"  $H(0; u_3) + \frac{1}{2}$ , <sup>1</sup>

 $(0, v_1, u_2; 1)$ 

, u123, <sup>1</sup>

 $(0, v_{321}; 1)$  $H$  (0; u3) +

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 $, v_{123}, 0; 1$ 

 $, v_{132}, 0; 1$ 

 $H(0; u_3) H$  (0; u<sub>3</sub>) + H (0; u3) − H (0; u3) −  $v_{132}$ ; 1) H (0; u3) −  $H(0; u_2) +$  $H(0; u_3)$ v<sub>312</sub>, 1, 1 1 − u<sup>3</sup>  $H(0; u_3) + \frac{1}{2}$ G  $v_{312}$ ,  $-$ 1 − u<sup>3</sup>  $v_{321, 1, 1, 1}$ 1 − u<sup>3</sup>  $H(0; u3) + \frac{1}{2}$ G  $m<sup>2</sup>$ 1 − u<sup>3</sup> , <sup>1</sup>  $H$  (0; u1)  $H$  (0; u3) + 1 − u<sup>2</sup> , <sup>u</sup><sup>3</sup> <sup>−</sup> <sup>1</sup>  $u_2 + u_3 - 1$ ; 1"  $H(0; u_1) H(0; u_2) +$ u) , <sup>1</sup>  $u_1 + u_3$  $H(0; u_1) H(0; u_3) - \frac{1}{2}$ 4 4 1 − u<sup>2</sup> , v213; 1"  $H(0; u_1) H(0; u_2) - \frac{1}{2}$ 4 5  $\frac{5}{24}\pi^2H\left(0;u_1\right)H\left(0;u_3\right)+\frac{1}{24}$ G ! 1 , <sup>u</sup><sup>2</sup> <sup>−</sup> <sup>1</sup>  $u_1 + u_2 - 1$ , <sup>1</sup>  $H(0; u_2) H(0; u_3) + \frac{1}{2}$  $\begin{split} -\frac{1}{4} \mathcal{G}\left(r_{\text{max}}, 1, \frac{1}{1-n_0}, 1\right) H\left(0, u_0\right) \\ -\frac{1}{2} \mathcal{G}\left(r_{\text{max}}, 1, \frac{1}{1-n_0}, 1\right) H\left(0, u_0\right) \end{split}$ 

JHEP05(2010)084  $H(0; u_2) +$  $H(0; u_2) +$  $H(0; u_2) +$  $H(0;u_3)$  −  $H$  (0;  $u_3$ ) −  $H$  (0; u<sub>3</sub>) +  $H$  (0; u<sub>3</sub>) +  $H(0; u3) +$  $H(0; u_2) +$  $H$  (0; u<sub>3</sub>) – , <sup>1</sup>  $H$  (0; u2)  $H$  (0; u3) −  $, u_{123}; 1$  $H\left(0;u_{2}\right)H\left(0;u_{3}\right)-\frac{1}{2}$  $, \tau_{123}; 1$  $H$  (0; u<sub>2</sub>)  $H$  (0; u<sub>3</sub>) −  $v_{132}$ ; 1)  $H(0; u_2) H(0; u_3) + \frac{5}{2}$  $\frac{\pi}{24}\pi^2H\left(0;u_2\right)H\left(0;u_3\right)+$  $3H\left(0;u_{2}\right)H\left(0,0;u_{1}\right)H\left(0;u_{3}\right)+3H\left(0;u_{1}\right)H\left(0,0;u_{2}\right)H\left(0;u_{3}\right)+$  $H$  (0; u<sub>2</sub>)  $H$ <sup>0</sup>, 1; <sup>u</sup><sup>1</sup> <sup>+</sup> <sup>u</sup><sup>2</sup> <sup>−</sup> <sup>1</sup>  $H(0; u_3) + \frac{1}{2}$  $H(0; u_1) H(0, 1; (u_1 + u_2)) H(0; u_3) +$ H (0; u1) H 0, 1; u<sub>2</sub> + u<sub>3</sub> − 1 u<sup>3</sup> − 1  $H(0; u3) + \frac{1}{2}$  $H(0; u2) H(0, 1; (u2 + u3)) H(0; u3) +$  $H(0; u_2) H(1, 0; u_1) H(0; u_3) + \frac{3}{2}$  $H$  (0; u<sub>1</sub>)  $H$  (1, 0; u2)  $H$  (0; u3) + 1 − u<sup>2</sup>  $, v_{213}; 1$  $H(0, 0; u_1) + \frac{1}{2}$ G 1 − u<sup>2</sup> , v231; 1"  $H(0, 0; u_1) +$ 1 − u<sup>3</sup>  $v_{312}$ ; 1)  $H(0, 0; u_1) + \frac{1}{2}$ G 1 − u<sup>3</sup> , v321; 1"  $H(0, 0; u_1) - \frac{23}{24}$  $\frac{24}{24}\pi^2H(0,0;u_1)+$ 1 − u<sup>1</sup>  $v_{123}$ ; 1)  $H(0, 0; u_2) + \frac{1}{2}$ 4 G 1 − u<sup>1</sup> , v<sub>132</sub>; 1  $H(0, 0; u_2) +$ 1 − u<sup>3</sup>  $v_{312}$ ; 1  $H\left(0,0;\,u_2\right)+$ 4 G 1 − u<sup>3</sup> , v321; 1" H (0, 0; u2) − 25  $\frac{15}{4}$ H (0, 0; u<sub>1</sub>) H (0, 0; u<sub>2</sub>) −  $\frac{23}{24}$  $\frac{23}{24}\pi^2H(0,0;u_2)+\frac{1}{4}$ ! 1  $, v_{123}; 1$  $H(0, 0; u_3) +$  $v_{132}$ ; 1)  $H(0, 0; u_3) + \frac{1}{2}$  $v_{212}(1)$  $H(0, 0; u_2) +$  $, v_{231}; 1$  $H (0, 0; u_3) + 3H (0; u_1) H (0; u_2) H (0, 0; u_3) \frac{15}{4}H\left(0,0;u_{1}\right)H\left(0,0;u_{2}\right)-\frac{25}{4}$  $\frac{15}{4}H\left(0,0;u_{2}\right)H\left(0,0;u_{3}\right)-\frac{23}{24}$  $\frac{23}{24}\pi^2H(0,0;u_3)+\frac{1}{2}$  $\frac{1}{22}\pi^2H(0,1;u_1)+$  $\frac{1}{12}\pi^2H\left(0,1;u_2\right)-\frac{1}{2}$  $\frac{1}{24}r^2H$ 0, 1; u<sub>1</sub> + u<sub>2</sub> − 1  $+$   $\frac{1}{2}$  $H\left(0;u_{1}\right)H\left(0;u_{2}\right)H\left(0,1;\left(u_{1}+u_{2}\right)\right)+$ 

> $\frac{1}{12}\pi^2H\left(0,1;u_3\right)+\frac{1}{4}$  $H(0;u_1)$   $H(0;u_2)$  H  $0, 1; \frac{u_1 + u_3 - 1}{u_1 + u_2 - 1}$ u<sup>1</sup> − 1

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 $\frac{1}{12}\pi^2H\left(0,1;\left(u_1+u_2\right)\right)+\frac{1}{12}$ 

JHEP05(2010)084  $H(1, 0; u_1) -$ ; 1"  $H(1, 0; u_1) +$ ; 1"  $H(1, 0; u_1) +$ , <sup>1</sup> H (1, 0; u1) −  $H(0, 0; u_2) H(1, 0; u_1) - \frac{3}{2}$  $H(0, 0; u3) H(1, 0; u1) +$  $\pi^2 H (1, 0; \omega) - \frac{1}{2}$  $^{\circ}$ H (1, 0; u2) − , <sup>u</sup><sup>2</sup> <sup>−</sup> <sup>1</sup> u<sup>1</sup> + u<sup>2</sup> − 1  $H(1, 0; u2) +$ ; 1"  $H(1, 0; u_2) +$ ; 1"  $H(1, 0; u_2) H\left(0,0;u_{1}\right)H\left(1,0;u_{2}\right)-\frac{3}{2}$  $H(0, 0; u_3) H(1, 0; u_2) +$  $H\left(1,0;u_{1}\right)H\left(1,0;u_{2}\right)-\frac{1}{2}$  $n^2H(1,0;u_2) -$ ; 1"  $H(1, 0; w_3) +$ , <sup>u</sup><sup>3</sup> <sup>−</sup> <sup>1</sup> u<sup>2</sup> + u<sup>3</sup> − 1  $H(1, 0; u_3) +$  $\pi^2 H(1, 0; u_3) + \frac{1}{2}$ , <sup>1</sup>  $H(1, 0; u_3) +$ , u231; 1"  $H(1, 0; u_2) +$  $H\left(0,0;u_{1}\right)H\left(1,0;u_{3}\right)-\frac{3}{2}$  $H(0, 0; u_2) H(1, 0; u_3) +$  $H(1, 0; u_1) H(1, 0; u_3) - \frac{1}{2}$  $H(1, 0; u_2) H(1, 0; u_3) +$  $\frac{1}{24}\pi^2H(1,1;u_3)+\frac{1}{2}$  $H(0; u_2) H(0, 0, 0; u_1) +$  $H(0; u1) H(0, 0, 0; u3) - \frac{1}{2}$ H (0; u2) H  $0, 0, 1; \frac{u_1 + u_2 - 1}{2}$ − H (0; u1) H (0, 0, 1; (u<sup>1</sup> + u2)) −  $0, 0, 1; \frac{u_1 + u_3 - 1}{2}$  $- H$  (0; u<sub>1</sub>)  $H$  (0, 0, 1; (u<sub>1</sub> + u<sub>3</sub>)) −  $H\left(0;u_{3}\right)H\left(0,0,1;\left(u_{1}+u_{3}\right)\right)-\frac{1}{2}$ H (0; u1) H 0, 0, 1; u2 + u3 − 1  $W(0; u_3)$  H  $0, 0, 1; \frac{u_2 + u_3 - 1}{u_3 + u_4 - 1}$  $-$  H (0; u2) H (0, 0, 1; (u2 + u3)) −  $H\left(0;u_{3}\right)H\left(0,0,1;\left(u_{2}+u_{3}\right)\right)-\frac{1}{2}$  $H(0; u_2) H(0, 1, 0; u_1) - \frac{1}{2}$  $H(0; u_1) H(0, 1, 0; u_2) + \frac{1}{2}$ H (0; u2) H  $0, 1, 1; \frac{u_1 + u_2 - 1}{u_1 + u_2 - 1}$ u<sup>2</sup> − 1 H (0; u3) H <sup>0</sup>, <sup>1</sup>, 1; <sup>u</sup><sup>1</sup> <sup>+</sup> <sup>u</sup><sup>2</sup> <sup>−</sup> <sup>1</sup> u<sup>2</sup> − 1  $+ \frac{3}{2}$ H (0; u1) H 0, 1, 1; u1 + u3 − 1 H (0; u2) H  $0, 1, 1; \frac{u_1 + u_2 - 1}{u_1 + u_2 - 1}$ − 1 H (0; u1) H  $0, 1, 1; \frac{u_2 + u_3 - 1}{u_3 + u_4 - 1}$ 1 H (0; u3) H 0, 1, 1; u<sub>2</sub> + u<sub>3</sub> − 1 + 1  $H\left(0;u_{2}\right)H\left(1,0,0;u_{1}\right)-\frac{1}{2}$  $H(0; u_1) H(1, 0, 0; u_2) + \frac{1}{2}$  $H(0; u_3) H(1, 0, 0; u_2) + \frac{1}{2}$ H (0; u1) H (1, 0, 0; u3) −  $H\left(0;u_{2}\right)H\left(1,0,0;u_{3}\right)-\frac{1}{2}$ H (0; u3) H 1, 0, 1; u<sub>1</sub> + u<sub>2</sub> - 1 u<sup>2</sup> − 1 H (0; u2) H  $1, 0, 1; \frac{u_1 + u_2 - 1}{u_1 + u_2 - 1}$ − 1 H (0; u1) H  $1, 0, 1; \frac{u_2 + u_3 - 1}{u_3 + u_4 - 1}$  $7H(0, 0, 0, 0; u_1) - 7H(0, 0, 0; u_2) - 7H(0, 0, 0; u_3) + \frac{3}{2}$  $3H(0, 0, 0, 1; (u_1 + u_2)) + \frac{3}{2}$  $0, 0, 0, 1; \frac{x_1 + x_3 - 1}{2}$ u<sup>1</sup> − 1  $+3H(0,0,0,1;(u_1+u_2))+$  $0, 0, 0, 1; \frac{n_2 + n_3 - 1}{2}$  $+ 3H (0, 0, 0, 1; (u_2 + u_3)) + \frac{9}{2}$  $W(0, 0, 1, 0; u_2) + \frac{9}{2}$  $H(0, 0, 1, 0; uv) - \frac{1}{2}$  $H(0, 1, 0, 0; u) - \frac{1}{2}$  $W(0, 1, 0, 0; u3) + \frac{1}{2}$  $0, 1, 0, 1; \frac{u_1 + u_2 - 1}{2}$ + 1  $0, 1, 0, 1; \frac{a\sqrt{2} + a\sqrt{3} - 1}{2}$ u<sup>3</sup> − 1  $+ H (0, 1, 1, 0; u<sub>1</sub>) + H (0, 1, 1, 0; u<sub>2</sub>) + H (0, 1, 1, 0; u<sub>3</sub>) 0, 1, 1, 1; \frac{n_1 + n_2 - 1}{n_1 + n_2 + n_3}$ − 1  $0, 1, 1, 1; \frac{u_1 + u_2 - 1}{2}$ u<sup>1</sup> − 1 <sup>0</sup>, <sup>1</sup>, <sup>1</sup>, 1; <sup>u</sup><sup>2</sup> <sup>+</sup> <sup>u</sup><sup>3</sup> <sup>−</sup> <sup>1</sup> u<sup>3</sup> − 1 + H 1, 0, 0, 1; u2 + u2 − 1 u<sup>2</sup> − 1 1, 0, 0, 1; u<sub>2</sub> + u<sub>3</sub> - 1  $+2H(1,0,1,0;u_1)+2H(1,0,1,0;u_2)+2H(1,0,1,0;u_2)+$ 1  $1, 1, 0, 1; \frac{u_1 + u_2 - 1}{2}$ + 1 1, 1, 0, 1; u<sub>1</sub> + u<sub>3</sub> − 1  $1, 1, 0, 1; \frac{u_2 + u_3 - 1}{2}$ + 1  $H(1, 1, 1, 0; u_1) +$  $H(1, 1, 1, 0; u_2) + \frac{1}{2}$  $\frac{1}{24}$ π<sup>2</sup>H (0; u<sub>3</sub>) N 1; <sup>1</sup>  $\frac{1}{\log 2}$ − 1  $\frac{1}{24}$ τ<sup>2</sup>Η (0; u<sub>1</sub>) Μ 1; <sup>1</sup>  $\frac{1}{2}$ <sup>π</sup>2<sup>H</sup> (0; <sup>u</sup>2) <sup>H</sup>  $\frac{1}{2}$  $\frac{1}{\sqrt{2}}$ − 1 <sup>π</sup>2<sup>H</sup> (0; <sup>u</sup>3) <sup>H</sup>  $\frac{1}{2}$  $\frac{1}{\sqrt{2}}$ <sup>+</sup> <sup>1</sup>  $\mathbf{1}$  $\frac{1}{v_{12}}$ − 1 1; <sup>1</sup> <u>21</u>)

 $\vdash$ 

 $\frac{1}{24}\pi^2H$  (0; u3)  $\mathcal{H}$ 

u<sup>3</sup> − 1

 $H(0; u_3) H(0, 1, 0; u_2) -$ 

 $0, 0, 0, 1; \frac{u_1 + u_2 - 1}{2}$ 

 $W(0, 0, 1, 0; u_1) +$ 

 $0, 1, 0, 1; \frac{u_1 + u_2 - 1}{2}$ 

+ H 1, 0, 0, 1; u1 + u3 - 1 u<sup>1</sup> − 1

− 1  $\frac{1}{24}\pi^2H\left(0;u_2\right)\mathcal{H}$ 1; <sup>1</sup> <u>2</u>)

<sup>+</sup> <sup>1</sup>  $\frac{1}{24}$ z<sup>2</sup>H (0; u3) h  $\mathbf{r}$  $\frac{1}{2}$ 

 $\frac{1}{24}\pi^2H(0;u)$  H

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 $\frac{1}{24}\pi^2H$  (0; u<sub>2</sub>)  $\mathcal H$  $_{1}$   $^{-1}$  $\frac{1}{v_{12}}$ 

H (0, 1, 0, 0; u2) −

u<sup>1</sup> − 1

 $W(0; u_3)$  H  $(1, 0, 0; u_1) H(1, 1, 1, 0; u_3)$ <sup>π</sup>2<sup>H</sup> (0; <sup>u</sup>1) <sup>H</sup> 1; <sup>1</sup> <del>2</del>)  $^{+1}$ <sup>π</sup>2<sup>H</sup> (0; <sup>u</sup>3) <sup>H</sup> 1; <sup>1</sup>  $\frac{1}{\sin \theta}$  $+$   $\overline{1}$ <sup>π</sup>2<sup>H</sup> (0; <sup>u</sup>1) <sup>H</sup> 1; <sup>1</sup>  $\frac{1}{\sin 2}$ <sup>π</sup>2<sup>H</sup> (0; <sup>u</sup>2) <sup>H</sup>  $1<sup>1</sup>$ <del>3</del>) <sup>+</sup> <sup>1</sup>  $\frac{1}{24}\sigma^2H\left(0;u_1\right)\mathcal{H}$ 1; <sup>1</sup>  $\frac{1}{\sqrt{2}}$ − 1  $\frac{1}{24}\pi^2H\left(0;u_2\right)\mathcal{H}$ 1; <sup>1</sup>  $\frac{1}{\sqrt{2}}$  $H (0; u_2) H (0; u_3)$  H  $0, 1; -1$  $\frac{1}{\pi(23)}$ − 1 H (1, 0; u2) H  $0, 1, \frac{1}{2}$  $\frac{1}{\pi(2)}$ <sup>+</sup> <sup>1</sup>  $\frac{1}{24}r^2$ N  $0, 1; -1$  $\frac{1}{\pi(23)}$ 1  $\frac{1}{24}\pi^2$ N  $0, 1;$   $1, 1$ <u>4</u>) − 1 H (0; u1) H (0; u3) H  $0, 1;$   $1$ a) − 1 H (1, 0; u3) H  $0, 1;$   $1$ <u>a</u>)  $H$   $(0; u_1)$  H  $(0; u_2)$  H  $0, 1; - \frac{1}{2}$ <u>31</u>) − 1  $H(1, 0; u_1)$  H  $0, 1; \frac{1}{2}$  $\frac{1}{2}$ <sup>+</sup> <sup>1</sup>  $\frac{1}{24}r^2$ N  $0, 1; - \frac{1}{2}$ <u>31</u>)  $H (0; u_2) H (0; u_3)$  H  $0, 1; -1$  $\frac{1}{v_{\rm{III}}}\big)$ + 1 H (0, 0; u2) H  $0, 1;$   $1$  $\frac{1}{\eta_{23}}$ H (0, 0; u3) H  $0, 1, 1$  $\frac{1}{\sin 2}$ + 1 <sup>π</sup>2<sup>H</sup>  $0.1<sup>2</sup>$  $\frac{1}{\sin 2}$ − 1 H (0; u2) H (0; u3) H  $0.1<sup>2</sup>$  $\frac{1}{\sin 2}$ H (0, 0; u2) H  $0, 1, -\frac{1}{2}$  $\frac{1}{\tau_{12}}$  $^{1}$ H (0, 0; u3) H  $0, 1, -1$  $\frac{1}{\epsilon_{122}}$  $+$   $\frac{1}{2}$ <sup>π</sup>2<sup>H</sup>  $0, 1; -\frac{1}{2}$  $\frac{1}{v_{12}}$  $H$  (0; u<sub>3</sub>)  $H$  (0; u<sub>3</sub>) H  $0, 1; -1$  $\frac{1}{v_{213}}$ + 1 H (0, 0; u1) H  $0, 1; -1$  $\frac{1}{22}$ H (0, 0; u3) H  $0, 1; -\frac{1}{2}$  $\frac{1}{\sin 2}$  $^{1}$ <sup>π</sup>2<sup>H</sup>  $0, 1; - \frac{1}{2}$  $\frac{1}{\cos \theta}$ − 1 H (0; u1) H (0; u3) H  $0, 1; - \frac{1}{2}$  $\frac{1}{\sin}$ H (0, 0; u1) H  $0, 1, -\frac{1}{2}$  $\frac{1}{\sqrt{2}}$  $^{1}$ H (0, 0; u3) H  $0, 1, -1$  $\frac{1}{\sqrt{2}}$  $+$   $\frac{1}{2}$ <sup>π</sup>2<sup>H</sup>  $0, 1; -\frac{1}{2}$ <del>2</del>)  $H$   $(0; u_1)$  H  $(0; u_2)$  H  $0, 1; -1$  $\frac{1}{\epsilon_{312}}$ + 1 H (0, 0; u1) H  $0, 1; -1$ <u>41</u>) H (0, 0; u2) H  $0, 1, 1$ <del>3</del>) + 1 <sup>π</sup>2<sup>H</sup>  $0.1<sup>2</sup>$ <u>31</u>) − 1 H (0; u1) H (0; u2) H  $0.1<sup>2</sup>$ <del>2.</del>) H (0, 0; u1) H  $0, 1, -\frac{1}{2}$  $\frac{1}{v_{22}}$  $^{1}$ H (0, 0; u2) H  $0, 1, -1$  $\frac{1}{\sqrt{2}}$  $+$   $\frac{1}{2}$ <sup>π</sup>2<sup>H</sup>  $0, 1; -\frac{1}{2}$ <del>2</del>) H (0; u2) H (0; u3) H  $1, 1; -1$  $\frac{1}{v_{\text{min}}}\big)$ + 1 H (0, 0; u2) H  $1, 1;$   $\frac{1}{2}$  $\frac{1}{\eta_B}$ H (0, 0; u3) H  $1, 1;$   $\frac{1}{\sin 2}$ <sup>+</sup> <sup>11</sup>  $\frac{11}{24}\pi^2N$  $_{1,1}$   $^{-1}$  $\frac{1}{\sin 2}$ − 1  $\frac{1}{24}$ r'n  $1, 1;$   $\frac{1}{\sin 2}$  $\frac{1}{24}r^2$ N  $_{1,1}$   $\pm$  $\frac{1}{213}$ − 1  $H\left(0;u_{1}\right)H\left(0;u_{2}\right)$  H  $1, 1; -1$  $\frac{1}{\sqrt{2}}$  $+$   $\frac{1}{2}$  $H(0, 0; u_1)$  H  $1, 1;$   $-$ <u>2</u>) H (0, 0; u3) H  $1, 1, -1$  $\stackrel{1}{\Rightarrow}$ <sup>+</sup> <sup>11</sup>  $\frac{11}{24}$ π $\frac{2}{14}$  $1, 1;$   $\frac{1}{23}$ − 1  $H\left(0;u_{1}\right)H\left(0;u_{2}\right)\mathcal{H}$  $1, 1; -1$ <u>21</u>) H (0, 0; u1) H  $1, 1, 1$ <del>3</del>1 + 1 H (0, 0; u2) H  $1, 1, 1$ <del>4</del>) <sup>+</sup> <sup>11</sup>  $\frac{21}{24}\pi^2 n$  $1, 1, 1$ <del>3</del>)  $\frac{1}{24}r^2$ N  $_{1,1}$   $\pm$  $\frac{1}{\sqrt{2}}$  $+$   $\frac{1}{2}$ H (0; u2) H  $0, 0, 1;$  1  $\frac{1}{2}$  $+$   $\frac{1}{2}$  $H$  (0; u<sub>3</sub>) H  $0, 0, 1;$  1  $\frac{1}{2}$  $H (0; u_1)$  H  $0, 0, 1;$  1  $\frac{1}{u_{23}}$ + 1  $H(0; u_3)$  H  $0, 0, 1;$  1  $\frac{1}{\pi_{23}}$  $+$   $\frac{1}{2}$  $H$  (0; u1)  $\mathcal{H}$  $0, 0, 1;$  1  $\stackrel{1}{\implies}$ H (0; u2) H  $0, 0, 1;$  1  $\frac{1}{\sin \theta}$  $^{+1}$ H (0; u3) H  $0, 1, 1; \frac{1}{1}$  $\frac{1}{\sin 2}$  $+$   $\frac{1}{2}$ H (0; u1) H  $0, 1, 1; \frac{1}{2}$  $\frac{1}{\sin}$ 

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H (0; u2) H  $0, 1, 1; \frac{1}{2}$  $\frac{1}{w_{212}}$ + 1  $H\left(0;u_{2}\right)$  H 0, 1, 1; <sup>1</sup>  $\frac{1}{2}$ − 1  $H(0;u_3)$  H  $0, 1, 1; \frac{1}{2}$  $\frac{1}{\epsilon_{123}}$   $\frac{1}{4}H\left(0;u_2\right)$   $\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_3\right)$   $\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_1\right)$   $\mathcal{H}\left(0,1,1;\frac{1}{v_{21}}\right)$  $rac{1}{v_{213}}$ − 1 4  $H(0; u_3) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{213}}\right)$  $-\left.\frac{1}{4}H\left(0;u_1\right){\cal H}\left(0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H\left(0;u_3\right){\cal H}\left(0,1,1;\frac{1}{v_{231}}\right)\right.$  $\frac{1}{v_{231}}$ + 1 4  $H\left(0;u_1\right)$  H !  $0, 1, 1; \frac{1}{1}$  $rac{1}{v_{312}}$  $-\left.\frac{1}{4}H\left(0;u_{2}\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_{1}\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{33}}\right)\right.$  $rac{1}{v_{321}}$ + 1 4  $H(0; u_2) \mathcal{H}\left(0, 1, 1; \frac{1}{u_0} \right)$  $rac{1}{v_{321}}$  $+\left.\frac{1}{4}H\left(0;u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H\left(0;u_{1}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{23}}\right)\right.$  $rac{1}{u_{231}}$ + 1 4  $H(0; u_2) \mathcal{H}\left(1, 0, 1; \frac{1}{u_2} \right)$  $rac{1}{u_{312}}$  $+\frac{1}{4}H(0;u_2) H\left(1,0,1;\frac{1}{v_{123}}\right)$  $-\frac{1}{4}H(0;u_3)$  H !  $1, 0, 1; \frac{1}{1}$  $rac{1}{v_{123}}$ − 1 4  $H(0; u_2) \mathcal{H}\left(1, 0, 1; \frac{1}{u_1} \right)$  $rac{1}{v_{132}}$  $+\frac{1}{4}H\left(0;u_3\right)$   $\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_1\right)$   $\mathcal{H}\left(1,0,1;\frac{1}{v_{21}}\right)$  $rac{1}{v_{213}}$ − 1 4  $H(0; u_3) \mathcal{H}\left(1, 0, 1; \frac{1}{u_0}\right)$  $rac{1}{v_{213}}$  $-\left.\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(\right.\right.$  $1, 0, 1; \frac{1}{1}$  $rac{1}{v_{231}}$ + 1  $\frac{1}{4}H(0; u_2) \mathcal{H}\left(1, 0, 1; \frac{1}{v_{212}}\right) + H(0; u_2) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{123}}\right) - H(0; u_3) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{123}}\right)$  $H\left(0;u_1\right) {\cal H}\left(1,0,1;\frac{1}{v_{312}}\right) -\frac{1}{4} H\left(0;u_2\right) {\cal H}\left(1,0,1;\frac{1}{v_{312}}\right) -\frac{1}{4} H\left(0;u_1\right) {\cal H}\left(1,0,1;\frac{1}{v_{32}}\right)$  $rac{1}{v_{321}}$ + −  $H(0; u_1) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{231}}\right) + H(0; u_3) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{231}}\right) + H(0; u_1) \mathcal{H}\left(1, 1, 1; \frac{1}{v_{312}}\right)$ −  $H\left(0;u_2\right) {\cal H}\left(1,1,1;\frac{1}{v_{312}}\right)-\frac{3}{2} {\cal H}\left(0,0,0,1;\frac{1}{u_{123}}\right)-\frac{3}{2} {\cal H}\left(0,0,0,1;\frac{1}{u_{231}}\right)$ −  $\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{312}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{132}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{321}}\right)-$ 1 2  ${\cal H}\left( {0,0,1,1;\frac{1}{{u_{123}}}} \right) - \frac{1}{2}{\cal H}\left( {0,0,1,1;\frac{1}{{u_{231}}}} \right) - \frac{1}{2}{\cal H}\left( {0,0,1,1;\frac{1}{{u_{312}}}} \right)$ − 1 2 H  $\left(0,1,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right)+$ 1 4 H !  $(0, 1, 1, 1; \frac{1}{v_{123}}) + \frac{1}{4} \mathcal{H} \left(0, 1, 1, 1; \frac{1}{v_{13}}\right)$  $\frac{1}{v_{132}}$  +  $\zeta_3H(0;u_1) + \zeta_3H(0;u_2) + \zeta_3H(0;u_3) +$ 5 2  $\zeta_3H(1;u_1)+\frac{5}{2}$ 2  $\zeta_3H\left(1;u_2\right)+\frac{5}{2}\zeta_3H\left(1;u_3\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{123}}\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{23}}\right)$  $rac{1}{u_{231}}$ + 1 2 ζ3H !  $1; -1$  $rac{1}{u_{312}}$  $-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{312}}\right)$ + 1 4 ζ3H !  $_{1;}$   $\frac{1}{1}$  $rac{1}{v_{123}}$  $+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{132}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{31}}\right)$  $\frac{1}{v_{312}}$ + 1 4 ζ3H  $\left(1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{31}}\right)$  $\frac{1}{v_{312}}$ +  $\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{23}}\right)$  $rac{1}{v_{213}}$ +  $\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{312}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{123}}\right)$ +  $\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{31}}\right)$  $rac{1}{v_{312}}$ +  $\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{321}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{123}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{231}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{312}}\right)$ 

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#### Result can be remarkably simple HASI IIT CE say that recent problems at local  $\Box$ evolutionary rather than revolutionary, driven primarily by faster computers, improved algorithms (both analytic It can he rem  $\boldsymbol{\mathsf{u}}$  and  $\boldsymbol{\mathsf{v}}$  and  $\boldsymbol{\mathsf{u}}$  and  $\boldsymbol{\mathsf{v}}$

#### **17 pages =** tum twistor invariants are more natural variables. In  $\frac{1}{\sqrt{2}}$  +  $\frac{1}{\sqrt{2}}$  +  $\frac{1}{\sqrt{2}}$  +  $\frac{1}{\sqrt{2}}$  +  $\frac{1}{\sqrt{2}}$ frared and collinear behavior of general amplitudes, con- $\blacksquare$  in the ABDK and ABDK determines the n-particle method of  $\mathbf{u}$ order L ≥ 2 up to an additive finite function of kinematic function of kinematic function of kinematic function<br>Desember x<sup>±</sup>  $\mathcal{L} = \mathcal{L} \cup \mathcal{L} = \mathcal$  $\mathcal{L} = \{u_1, u_2, \ldots, u_n\}$  , we find  $u_1$

terms of

x<sup>±</sup>

### **[Goncharov, Spradlin, Vergu, Volovich 2010]**

$$
\sum_{i=1}^3 \left(L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i)\right) - \frac{1}{8} \left(\sum_{i=1}^3 \text{Li}_2(1 - 1/u_i)\right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72}
$$
  

$$
L_4(x^+, x^-) = \frac{1}{8!} \log(x^+ x^-)^4 + \sum_{m=0}^3 \frac{(-1)^m}{(2m)!!} \log(x^+ x^-)^m (\ell_{4-m}(x^+) + \ell_{4-m}(x^-)) \qquad \ell_n(x) = \frac{1}{2} (\text{Li}_n(x) - (-1)^n \text{Li}_n(1/x)) \qquad J = \sum_{i=1}^3 (\ell_1(x_i^+) - \ell_1(x_i^-)).
$$

#### a line result in terms of classical polylogarithms! 1s1 <sup>i</sup> ) − ℓ1(x<sup>−</sup>

 $\overline{\phantom{a}}$ 

i never enter the lower the lower half-plane and the lower half-plane and the lower the lower the solid and the s cal tools: **Sympol** in the Euclidean region with the understanding that the require advanced mathematical tools: "Symbol"



#### sarily been taken to ensure the proper analytic structure. For example one can easily check that J naively simplinder Goncharov ture, voids our warranty on (3). Alexander Goncharov

## Progress in amplitudes

Significant progress has been made in the study of amplitudes in past years.

New Structures New Methods

Such simplicity is unexpected and also hard to understand using traditional Feynman diagrams.

#### Lessons from modern amplitudes The computing of their anomalous dimension is one central topic in the study of integrability of *N* = 4 SYM. 4 Sudakov Form Factors

New structures and new formulations  $\frac{d}{dx}$ 

Witten's twistor theory Double-copy CHY formalism New mathematical structure

New computational methods

Spinor helicity variables

BCFW recursion relation

Unitarity cuts

New algebraic reduction and integration methods One can check the following relation using the Feynman diagram expression of *A*<sup>4</sup> and

talks hy Song He Ro Feng Junije Rao Zhihao Fu  $p_{\text{max}}$  is to put is the put is to put internal properties to be only  $p_{\text{max}}$ **c.f. talks by Song He, Bo Feng, Junjie Rao, Zhihao Fu**



is defined as a matrix element between a half-BPS operator and two on-shell particles.

Based on on-sehll methods including unitarity cuts, color-kinematics duality, we are

*<sup>s</sup>*12*A*4(1*,* <sup>2</sup>*,* <sup>3</sup>*,* 4) = <sup>X</sup> **c.f. talk by Yang Zhang**

#### Spinor helicity formalism  $\Gamma$  $\overline{\phantom{a}}$  $\alpha \sim \beta$ α-representation: G(α) = U(α) + F(α)

Massless momentum: Γi !<br>|-<br>| (1)  $\sum_{i=1}^{n}$  $\ddot{\phantom{0}}$ 

$$
p_{\mu} \rightarrow p_{\alpha\dot{\alpha}} = p_{\mu}\sigma_{\alpha\dot{\alpha}}^{\mu} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}
$$

$$
p_{\mu}p^{\mu} = 0 \quad \rightarrow \quad p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}, \qquad \alpha, \dot{\alpha} = 1, 2
$$

n ve Polarisation vector: (−) ion v λ<sup>i</sup> ˜ξ  $\bigcup$ CIC  $\mathbf{r}$ ,  $\mathbf{r}$ 

$$
\varepsilon_{i,\alpha\dot{\alpha}}^{(-)} = \frac{\lambda_i \tilde{\xi}}{\left[\tilde{\lambda}_i \tilde{\xi}\right]}, \qquad \varepsilon_{i,\alpha\dot{\alpha}}^{(+)} = \frac{\xi \tilde{\lambda}_i}{\langle \xi \lambda_i \rangle} \qquad \text{``C}
$$

**"Chinese Magic" [Xu, Zhang, Zhang, 84]**

 $\sqrt{2}r$  $\frac{1}{\sqrt{2}}$ Use good on-shell variables

# Unitarity cut method  $(2E^{\dagger +})$

"The S-matrix is an analytic function of all momentum variables with only those singularities required by unitarity." ps of all momentum .<br>itiz

"singularities": physical poles and branch cuts. s anu pra

Unitarity-cut method provides an efficient method to compute loop integrand. ℓ2(ℓ−p1−p2)<sup>2</sup> 9 <del>11</del> <del>11 01 100</del> to 00 mpato  $\mathcal{P}$  and  $\mathcal{P}$  and  $\mathcal{P}$  are  $\mathcal{P}$  and  $\mathcal{P}$  and  $\mathcal{P}$  are  $\mathcal{P}$  and  $\mathcal{P}$  are  $\mathcal{P}$ 

Cutkosky cutting rule: 
$$
\frac{1}{\ell^2} \to 2\pi\delta^{(+)}(\ell^2)
$$
  
\n $\ell - p_1 - p_2$   
\n $\ell$   
\nX  
\n $\ell$ 

# Progress in planar N=4 SYM

In the planar limit, N=4 SYM is believed to be exactly solvable.

### Thanks to: **Integrability and AdS/CFT correspondence**

(Dual conformal and Yangian symmetries)

**See also other theories, Fishnet theory, 3D ABJM, c.f. talk by Junbao Wu**

Many exact solutions were found:

- anomalous dimensions
- scattering amplitudes/Wilson loops **Pentagon OPE program, [Basso, Pedro, Sever; …]**
- correlation functions... **Hexagon form factor program, [Basso, Komatsu, Pedro; …]**

### Operator mixing and spectrum

Different operators can mixing with each other at quantum level via renormalization:

$$
\mathcal{O}_{R,i} = Z_i^j \mathcal{O}_{B,j}
$$

From the renormalization constant matrix, one can obtain the dilatation operator:

$$
\mathcal{D} = -\frac{d \log Z}{d \log \mu}
$$

The anomalous dimension is given by the eigenvalue of the dilatation operator.

$$
\mathcal{D} \cdot \mathcal{O}_{\text{eigen}} = \gamma \cdot \mathcal{O}_{\text{eigen}}
$$

### Integrability en die operator integra



 $\tan \theta$  site there is an  $\theta$ property of the trace, we should include the further restriction that the Hilbert space be invariant under the shift Example: the scalar operators at 1-loop:

$$
\mathfrak{D}^{(1)}=\mathbb{H}_{\mathrm{SO}(6)}=\sum_{i}2(1-\mathbb{P})_{i\,i+1}+\mathbb{T}_{i\,i+1}\qquad\text{[Minahan, Zarembo 02]}
$$

the shift. Because *P*`*,*`+1 and *K*`*,*`+1 act on neighboring fields, the spin-chain Hamiltonian only has nearest neighbor interactions between the spins. Direct evidence that N=4 SYM is integrable.

#### Cusp anomalous dimension nai<br>T  $\sim$ ,usp anonialous uinension 中还有关于经典弦的可积性,但没有时间介绍了。

Non-perturbative result via integrability method:

**[Beisert, Eden, Staudacher '06]** 然后

native result via method:	\n $K_{ij} = j(-1)^{i(j+1)} \int_0^\infty \frac{dt}{t} \frac{J_i(2gt)J_j(2gt)}{e^t - 1}$ \n
\n $\Gamma_{\text{cusp}} = 4g^2 \left( \frac{1}{1+K} \right)_{11}$ \n	

Weak coupling expansion: <sup>K</sup>ij <sup>=</sup> <sup>j</sup>(−1)<sup>i</sup>(j+1) ! <sup>∞</sup> de de la posta de la posta<br>Jegovina de la posta de la

$$
\Gamma_{\rm cusp} = 4g^2 - \frac{4\pi^2}{3}g^4 + \frac{44\pi^4}{45}g^6 - 8\left(4\zeta_3^2 + \frac{73}{630}\pi^6\right) + \mathcal{O}(g^{10})
$$

**[Belitsky, Gorsky, Korchemsky'03], [Kotikov,Lipatov,Onishchenko,Velizhanin'04] [Bern,Czakon,Dixon,Kosower,Smirnov'06] [Cachazo,Spradlin,Volovich'06]……**

Strong coupling expansion (AdS/CFT):	[Ben,Czakon, Dixon, Kosower, Smirnov'06]
$\Gamma_{\text{cusp}} = 2g - \frac{3 \log 2}{2\pi} + \mathcal{O}(1/g)$	[Gubser, Klebanov, Polyakov'02], [Kruczenski'02], [Kruczenski'02].

**[Makeenko'02] ……** 解析结果时,发现结果和数值的不一致才发现这一错误的,不然隐藏得可够深的呢!

# Underlying picture

 $\mathcal{L}$  $Cine$ gιa τις Single trace operators can be viewed as states of a dynamic, cyclic, pure = pure = pure = p quantum spin chain. The latter can be related to a string picture.



 $\bullet$  *See the review by Beisert et.al, 2010* 

### Motivation





Form factor: A probe to the off-shell World

### Form factors  $\Box$ 3.5 Structure of  $S$





#### Gauge invariant operators Gauge filval and Gaune inveriant operat  $T_{\rm eff}$  single-trace local operators:  $T_{\rm eff}$  $\boldsymbol{\mathcal{A}}$  $\gamma$ re  $\overline{\phantom{a}}$

Local gauge invariant operators are constructed as traces of covariant fields. Local gauge invariant operators are constructed as traces of  $\mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{M})$ where the gauge in family specifies on a

 $\mathcal{O}(x) = \text{Tr}(\mathcal{W}^{(m_1)}_1 \mathcal{W}^{(m_2)}_2 \dots \mathcal{W}^{(m_n)}_n)(x)$   $\mathcal{W} \rightarrow \mathcal{UV}$  $\frac{1}{2}$ and a set of the set o Furthermore we can dress each letter with covariant derivative  $\mathcal{W} \rightarrow U \mathcal{W} U^\dagger$  $\frac{1}{2}$  $\frac{1}{2}$  $\psi \nu \rightarrow U \nu \nu U$ 

$$
\mathcal{W}^{(m)} := D^m \mathcal{W}, \qquad D_{\alpha \dot{\alpha}} \mathcal{W} = \partial_{\alpha \dot{\alpha}} \mathcal{W} - ig_{\text{YM}} [A_{\alpha \dot{\alpha}}, \mathcal{W}]
$$

 $F(1111 - 4 \cup 1111, 111 \cup 111 \$  $W_i \subseteq \{ \varphi_{AB}, \; L_{\alpha\beta}, \; L'_{\dot{\alpha}\dot{\beta}}, \; \varphi_{\dot{\alpha}A}, \; \varphi_{\alpha ABCJ} \qquad A \equiv 1,$  $\overline{\phantom{a}}$ In N=4 SYM, there are following 'letters':  $\mathcal{W}_i \in \{\phi_{AB}, F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}, \bar{\psi}_{\dot{\alpha}A}, \psi_{\alpha ABC}\}\qquad A=1,$  $A = 1, 2, 3, 4$ g 'letters':  $\alpha, \dot{\alpha} = 1, 2$  $\overline{AB}$ 

 $\frac{1}{2}$ 

 $\sum_{i=1}^{n}$ 

### Operators and on-shell kinematics are and an ahall king THE MILL OF STREET SHIP OF STREET

In terms of spinor helicity variables: Dαα˙ λ<sup>α</sup>λ˜<sup>α</sup>˙ (3.13)

**[Beisert 10] [Zwiebel 11] [Wilhelm 14]**



#### Operators and form factors GIOIS 20 A Dilatation operator and primary 6  $\sum_{n=1}^{\infty}$ Derators and t  $\sum_{n=1}^{\infty}$ Jperators and to

Applying the rules:

$$
\text{tr}(\bar{F}_{\alpha\beta}F^{\alpha\beta}) \to \lambda_1^{\alpha}\lambda_1^{\beta}\lambda_{2\alpha}\lambda_{2\beta}(\eta_1)^4(\eta_2)^4 = \langle 1 \, 2 \rangle^2(\eta_1)^4(\eta_2)^4 \qquad \xrightarrow{\psi_{\alpha ABC}} \xrightarrow{\psi_{\alpha ABC}} \xrightarrow{\psi_{\alpha ABC}} \xrightarrow{\psi_{\alpha ABC}} \lambda_1^{\alpha}\lambda_1^{\alpha}\lambda_1^{\beta}\lambda_1^{\beta}\lambda_1^{\beta}\lambda_2^{\beta}\lambda_2^{\gamma}\lambda_1^{\gamma}\lambda_3^{\gamma}\lambda_4^{\gamma} = [1 \, 2][2 \, 3][3 \, 1] \qquad \xrightarrow{\psi_{\alpha ABC}} \xrightarrow{\psi_{\alpha ABC}} \lambda_1^{\alpha}\lambda_1^{\beta}\lambda_1^{\beta}\lambda_1^{\gamma}\lambda_1^{\gamma}\lambda_2^{\gamma}\lambda_3^{\gamma}\lambda_4^{\gamma}\lambda_5^{\gamma}\lambda_6^{\gamma} = [1 \, 2][2 \, 3][3 \, 1]
$$



The RHS exactly reproduce the minimal form factor results: The RHS exactly reproduce the minimal form factor rea The RHS exactly reproduce the minimal form factor rest

$$
F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x \, e^{-iq\cdot x} \, \langle p_1 \ldots p_n | \mathcal{O}(x) | 0 \rangle = \delta^{(4)} \left( \sum_{i=1}^n p_i - q \right) \langle p_1 \ldots p_n | \mathcal{O}(0) | 0 \rangle
$$



### Operators and form factors orm tactors  $\bigcup_{i=1}^n$  duality between  $\bigcup_{i=1}^n$  duality between  $\bigcup_{i=1}^n$  and  $\bigcup_{i=1}^n$  duality between  $\bigcup_{i=1}^n$



 $F_{n,\mathcal{O}}(1,\ldots,n)$  $\mathcal{O}(x) = \text{Tr}(\mathcal{W}^{(m_1)}_1 \mathcal{W}^{(m_2)}_2 \dots \mathcal{W}^{(m_n)}_n)$ 

One can translate any local operator into the "on-shell" language!

2.1 Starting from the minimal form factors, one can consider the form factors. Starting from tree minimal form factors, one can construct non-minimal form factors and loop form factors.

## Simplicity of MHV Form factors

Parke-Taylor structure of form factors: [Brandhuber, Spence, Travaglini, GY 2011]

$$
F_n^{\text{MHV}}(1^+,...,i_\phi,...,j_\phi,...,n^+; \text{tr}(\phi^2)) = \delta^4(\sum_{i=1}^n p_i - q) \frac{\langle ij \rangle^2}{\langle 12 \rangle \cdots \langle n1 \rangle}
$$

Recall the Parke-Taylor formula for amplitudes:

$$
A_n^{\text{MHV}}(1^+,...,i^-,..,j^-,..,n^+) = \delta^4(\sum_{i=1}^n p_i) \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}
$$



#### Form factor via Unitarity  $\sim$  2π −→  $\overline{a}$ ℓ2(ℓ−p1−p2)<sup>2</sup> 12 UNITAR  $\frac{1}{2}$ 1 ℓ2(ℓ−p1−p2)<sup>2</sup>  $\mu$ s Hoitarity ϵ → 0 PSU(2, 2|4) α, α˙ |A A = 1, 2, 3, 4  $\mathcal{P}$  $P\cap r$ m ta $\cap$ t $\cap$ ϵ → 0 PSU(2, 2|4) α, α˙ |A A = 1, 2, 3, 4  $\lambda$  dightatically divergences can continuously. Given the renormalization constants *Z*, the anomalous dimension matrix , also called the different operator D, can be obtained as

At one-loop, there are only 'range-2' interactions: ←−−−−−−−− blahblahblah g− ←−−−−−−−− blahblahblah g− −−−−→  $\cos \alpha \cdot \Omega^2$  int −→ x שפ−ב <sub>וו</sub> anan 2' interactions:  $\mathsf{p}_1, \mathsf{p}_2$  $\overline{\phantom{a}}$ any<del>c</del>-2 interactions.  $\blacksquare$ 

$$
F_n^{(1)} = \sum_{i=1}^n \bigoplus_{\nu=1}^n \bigotimes_{p_{i+1}}^{p_i}
$$

i=1 The basis is very simple: ←−−−−−−−− blahblahblah JI∖ <sup>ℓ</sup><sup>2</sup> → 2πδ(+)(ℓ<sup>2</sup>)  $\mathsf{h}$  $\cup$ 



One-loop anomalous dimension is given by the bubble coefficients: the given by the babble coemercine.

$$
Z^{(1)} = -\frac{C_{\text{bub}}}{\epsilon}, \quad \mathcal{D}^{(1)} = 2\epsilon Z^{(1)} = -2C_{\text{bub}} \qquad {}^{(\mathbb{D}^{(1)})_{\text{SO}(6)} = \sum_{i} 2(1 - \mathbb{P})_{i} + 1 + \mathbb{T}_{i} + 1}
$$

## Loop structure of form factors

 $U_{\rm{H}}$  is infrared divergences. The infrared dividending and renormalization and renormalization. At higher loops, the IR and UV are mixed:

General structure of (bare) form factors:  $\mathbb{R}$  $\cot$  (cture of (bare) for form fac  $\mathcal{L}$  $\overline{\phantom{a}}$ 



# QCD Spectrum and Higgs Amplitudes

**• arXiv:2011.02494 with Qingjun Jin (靳庆军) and Ke Ren (任可)** 

#### QCD operators  $\bigcap_{n=1}^{\infty}$  computations. up to dimension 16, and in later sections we will compute their anomalous dimension and  $\blacksquare$ theory method and then apply the on-shell form factor method. Besides counting the number 2 Constructing operator basis of basis, a central goal is to explain how to construct a convenient set of basis operators that  $f(\mathbf{A} \cap \mathbf{A})$

A gauge invariant scalar operator takes the following form We consider scalar gauge invariant local opers field strength *Fµ*⌫ and covariant derivatives *Dµ*. The field strength carries an adjoint color We consider scalar gauge invariant local operators: of basis, a central goal is to explain how to explain how to construct a construction of basis operators that up to dimension 16, and in later sections we will compute the indicated will compute the indicated will compute plies. A good knowledge of such perturbative information is helpful to understand the RG iar  $\ddot{\phantom{0}}$ 2*p*<sup>2</sup> *T ,*  $\frac{1}{2}$ 

$$
\mathcal{O} \sim c(a_1,...,a_n) (D_{\mu_{11}}...D_{\mu_{1m_1}} F_{\nu_1 \rho_1})^{a_1} \cdots (D_{\mu_{n1}}...D_{\mu_{nm_n}} F_{\nu_n \rho_n})^{a_n} X(\eta, \epsilon)
$$

 $P_1: \mathbb{R}^2 \times \mathbb{R}$  $D_{\mu} \star = \partial_{\mu} + ig[A_{\mu}, \star]$ ,  $[D_{\mu}, D_{\nu}] \star = ig[F_{\mu\nu}, \star]$   $F_{\mu\nu} = F_{\mu\nu}^a T^a$ ,  $[T^a, T^b] = i f^{abc} T^c$ 

 $A$ pomalous dimensions  $\ell$ , speatrum of badrons). PC flow, ODE for simplicity we will consider the parity even operators where  $\mathcal{X}$  contains only  $\mathcal{X}$ .  $\mu$ irialous differisions (~spectrum or naurons), nu now, OFE *n*<sub>*i*</sub></sup>, *ni*, *b*<sub>*i*</sub>, *b*<sub>*i*</sub>, *b*<sub>*i*</sub>, *b*<sub>*i*</sub>, *b*<sub>*i*</sub>, *b*<sub>*i*</sub>, *d*<sub>*i*</sub>, Anomalous dimensions (~spectrum of hadrons), RG flow, OPE e $\mathcal{L}_{\text{eff}}$  action, which describes the Higgs production via gluon fusion production  $\mathcal{L}_{\text{eff}}$ 

Such operators also in Higgs FFT obtained by heavy Top quark loop:<br> for simplicity we will consider the parity even operators where *X* contains only ⌘'s. *c*(*a*1*, ..., an*) *Dµ*<sup>11</sup> *...Dµ*1*m*<sup>1</sup> *F*⌫1⇢<sup>1</sup> *a*<sup>1</sup> *··· Dµn*<sup>1</sup> *...Dµnmn F*⌫*n*⇢*<sup>n</sup>* Such operators also in Higgs EFT obtained by integrating The coupling is proportional to the mass of quarks, which is dominated by the heaviest top the heaviest top th addii operators aiso in mggs Li Tobialited by integrating for *gg* ! *gh* receives n operators als so in Filggs EFT optained by integrating vy Top quark roop.



$$
\mathcal{L}_{\text{eff}} = \hat{C}_0 H \hat{O}_{4;0} + \sum_{k=1}^{\infty} \frac{1}{m_t^2 k} \sum_i \hat{C}_i H \hat{O}_{4+2k;i}
$$

of canonical dimension 0. For the Higgs plus one jet production, the contribution of higher **c.f. EFT related talks by Shuang-Yong Zhou, Bo Ning ..** 

### Basis of operators (classical) Dasis UI UPETAIUI

A gauge invariant scalar operator the following form the following form that is the following form of the following formulation of the following formula in the following formula in the following formula in the following fo These operators are generally not independent:

$$
\mathcal{O} \sim c(a_1, ..., a_n) (D_{\mu_{11}}...D_{\mu_{1m_1}} F_{\nu_1 \rho_1})^{a_1} \cdots (D_{\mu_{n1}}...D_{\mu_{nm_n}} F_{\nu_n \rho_n})^{a_n} X(\eta, \epsilon)
$$

Equation of motion:

Bianchi identities:

 $D_{\mu}F^{\mu\nu}=0$ 

 $D_{\mu}F_{\nu\rho} + D_{\nu}F_{\rho\mu} + D_{\rho}F_{\mu\nu} = 0$ 

 $\boxed{\text{Equation of motion:}}$   $D_{\mu}F^{\mu\nu}=0$  We need to remove such  $\overline{u}$   $\overline{u}$   $\overline{u}$   $\overline{u}$   $\overline{u}$   $\overline{u}$  relations in order to find a For sign childentities  $D.F_{xx} + D.F_{yy} + D.F_{yy} = 0$  set of independent basis  $F_{\mu}$   $F_{\mu}$   $F_{\mu}$   $\mu$   $\mu$   $\mu$   $\mu$   $\mu$   $\mu$   $\mu$   $\sigma$   $\sigma$   $\sigma$   $\sigma$   $\sigma$  $\sigma$  $\sigma$  $\sigma$  $\sigma$  $\sigma$  $\sigma$  $\sigma$ operators.

### Examples:

 $d\mathcal{L}(F^{\mu\nu})$ dim-4:  $\mathcal{O}_4 = \text{Tr}(F_{\mu\nu}F^{\mu\nu})$ 

dim-6:  $\mathcal{O}_{6;1} = \partial^2 \text{Tr}(F^2), \quad \mathcal{O}_{6;2} = \text{Tr}(F^3)$  $0,1$  and  $0,2$  dimension typical dimension typical dimension typical dimension typically receives  $0,2$ 

dim-8:  $\mathcal{O}_{8,1} = \partial^4 \text{Tr}(F^2)$ ,  $\mathcal{O}_{8,2} = \partial^2 \text{tr}(F^3)$ ,  $\mathcal{O}_{8,3} = \text{tr}(D_1 F_{23} D_4 F_{23} F_{14})$ , and 8 Length-4 operators

### Operator mixing (quantum)

Different operators (at same given dimension) can mixing with each other at quantum level via renormalization:

$$
\mathcal{O}_{R,i} = Z_i^j \mathcal{O}_{B,j}
$$

From the renormalization constant matrix, one can obtain the dilatation operator:

$$
\mathcal{D} = -\frac{d \log Z}{d \log \mu}
$$

The anomalous dimension is given by the eigenvalue of the dilatation operator:

$$
\mathcal{D} \cdot \mathcal{O}_{\text{eigen}} = \gamma \cdot \mathcal{O}_{\text{eigen}}
$$

### Loop form factor computation







 $+$  $c_3I_3 +$  $2414$  |  $2515$  |  $6$  $\frac{1}{\sqrt{2}}$  $-c_8I_8$  $(949)$   $(99)$ <sup>1</sup>\$3*, d*-sector *.* (3.2)  $F_{\mathcal{O}}^{(2)} = (c_1I_1 + c_2I_2 + c_3I_3 + c_4)$  $(c_1I_1 + c_2I_2 + c_3I_3 + c_4I_4 + [c_5I_5 + c_6I_6] + [c_7I_7 + c_8I_8] + c_9I_9$  $\setminus$ + cyc*.*perm*.*(1*,* 2*,* 3)*,*

## Mixing matrices and spectrum

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix.

### Mixing matrices and spectrum

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix. the announcethe one-loop renormalization matrix is: *ization*  $\frac{1}{2}$ matr riali **A** . *.* **(4.40)** 

> *Z*(1) = **D** MONT PION *N<sup>c</sup>* n previ n<br>Pud ) dim *caory* ormy at one loop ap to Results were known previously only at one-loop up to dimension-8, <u>.</u><br>2002: Dawson, Lewis, Zeng and at two-loop up to dimension-6 operators.

is, Zen<mark>g</mark> 2 **n**  $\lambda$  . **Gracey 2002; Dawson, Lewis, Zeng 2014; … Jin, GY 2019**

th-: *O*8  $\overline{\phantom{0}}$ Ę o opei  $\overline{a}$  $\overline{a}$ ors at d  $\overline{2}$  $\prod$  $\overline{v}$ Length-3 operators at dimension-8:

 $U_{\mathbf{y}}$ , the distribution operator is given as  $U_{\mathbf{y}}$  $\mathcal{O}_{8;1} = \partial^4 \text{Tr}(F^2), \quad \mathcal{O}_{8;2} = \partial^2 \text{tr}(F^3), \quad \mathcal{O}_{8;3} = \text{tr}(D_1 F_{23} D_4 F_{23} F_{14}),$ 

$$
\mathbb{D}_{\mathcal{O}_{8}} = \begin{pmatrix} -\frac{22}{3}\hat{\lambda} - \frac{136}{3}\hat{\lambda}^2 & 0 & 0 \\ -\frac{\hat{\lambda}^2}{\hat{g}} & \frac{7}{3}\hat{\lambda} + \frac{269}{18}\hat{\lambda}^2 & 10\hat{\lambda}^2 \\ -3\frac{\hat{\lambda}^2}{\hat{g}} & 0 & \hat{\lambda} + \frac{25}{3}\hat{\lambda}^2 \end{pmatrix} \qquad \hat{\gamma}_{\mathcal{O}_{8}}^{(1)} = \left\{ -\frac{22}{3}; 1; \frac{7}{3} \right\}, \qquad \hat{\gamma}_{\mathcal{O}_{8}}^{(2)} = \left\{ -\frac{136}{3}; \frac{25}{3}; \frac{269}{18} \right\}
$$

#### Mixing matrices and spectrum At two-loop level, the *Z*(2) matrix is:  $\overline{1}$ Using (4.25), the dilation operator is given as

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix. *r*<br>*L O*8  $\ddot{\mathbf{r}}$ tors c = *c* ain bc 269<br>269 - 269  $\mathbf{a}$  $\mathbf{r}$ and UV divergences, b pour in 1<br>220 obt , subtractio ,<del>c</del>ɔ, vy ɔuvuɑcun CA *.* (4.42)

$$
\mathbb{D}_{\mathcal{O}_{8}} = \begin{pmatrix} -\frac{22}{3}\hat{\lambda} - \frac{136}{3}\hat{\lambda}^2 & 0 & 0 \\ -\frac{\hat{\lambda}^2}{\hat{g}} & \frac{7}{3}\hat{\lambda} + \frac{269}{18}\hat{\lambda}^2 & 10\hat{\lambda}^2 \\ -3\frac{\hat{\lambda}^2}{\hat{g}} & 0 & \hat{\lambda} + \frac{25}{3}\hat{\lambda}^2 \end{pmatrix} \qquad \hat{\gamma}_{\mathcal{O}_{8}}^{(1)} = \left\{ -\frac{22}{3}; 1; \frac{7}{3} \right\} , \qquad \hat{\gamma}_{\mathcal{O}_{8}}^{(2)} = \left\{ -\frac{136}{3}; \frac{25}{3}; \frac{269}{18} \right\}
$$

 $\overset{(2)}{\mathcal{O}_{10}}$ 

 $\frac{1}{3}$  10, j

 $=$  {

 $\frac{1}{\sqrt{2}}$ 

 $\frac{1}{3}$ 

<sup>125</sup> *,*

The computation of renormalization constant is the same as explained in the dimension-8

 $\frac{1}{2}$  (4.48)

Note that the o↵-diagonal elements of the first column belong to *Z*(2)  $\Box$ iiii-i $\Box$ . The dilation operator matrix can be obtained from (4.25), and for *f abc* sector it is Dim-10:

$$
\mathbb{D}_{\mathcal{O}_{10,f}} = \begin{pmatrix}\n-\frac{22\hat{\lambda}}{3} - \frac{136}{3}\hat{\lambda}^2 & 0 & 0 & 0 & 0 \\
-\frac{\hat{\lambda}^2}{9} & \frac{7\hat{\lambda}}{3} + \frac{269}{18}\hat{\lambda}^2 & 0 & 10\hat{\lambda}^2 & 0 \\
-\frac{230}{300}\frac{\hat{\lambda}^2}{\hat{g}} & -\frac{6\hat{\lambda}}{5} - \frac{5579\hat{\lambda}^2}{4500} & \frac{71\hat{\lambda}}{15} + \frac{2848}{125}\hat{\lambda}^2 & \frac{1493}{300}\hat{\lambda}^2 & \frac{5}{9}\hat{\lambda}^2 \\
-3\frac{\hat{\lambda}^2}{2} & 0 & 0 & \hat{\lambda} + \frac{25}{3}\hat{\lambda}^2 & 0 \\
-\frac{19}{12}\frac{\hat{\lambda}^2}{\hat{g}} & \frac{139}{600}\hat{\lambda}^2 & \frac{499}{200}\hat{\lambda}^2 & -2\hat{\lambda} - \frac{143}{72}\hat{\lambda}^2 & \frac{17\hat{\lambda}}{3} + \frac{2195}{72}\hat{\lambda}^2 \\
\hat{\gamma}^{(1)}_{\mathcal{O}_{10,f}} = \left\{-\frac{22}{3}; 1; \frac{7}{3}; \frac{71}{15}, \frac{17}{3}\right\}, \qquad \hat{\gamma}^{(2)}_{\mathcal{O}_{10,f}} = \left\{-\frac{136}{3}; \frac{25}{3}; \frac{269}{18}; \frac{2848}{125}, \frac{2195}{72}\right\}
$$

3

 $\int$   $\frac{1}{3}$ ,  $\frac{1}{3}$ ,  $\frac{1}{2}$ 

#### Mixing matrices and spectrum and specuant <sup>3</sup> 71 <sup>30</sup> 0 0 0 0 0 00 0 00 0 0

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix. )<br>B<br>B divergence iv subtracting  $\frac{8110000}{100000}$ 20 April 2010 - Andrew March 2010 JV ren<sub>'</sub> rmaliza  $\mathbf{i}$   $\overline{D}$ natrix. <sup>71</sup> <sup>105</sup> <sup>212</sup> 141 <sup>35</sup> <sup>71</sup> <sup>35</sup> <sup>141</sup> 79 <sup>105</sup> <sup>38</sup> 223 5

Dim-16 at 1-loop:

*Z*(1) *<sup>O</sup>*16*,f* <sup>=</sup> *<sup>N</sup><sup>c</sup>* ✏ BBBBBBBBBBBBBBBBBBBBBBBBBBBB@ 0 0 0 0 0 0 0 00 0 00 0 0 <sup>7</sup> 0 0 0 0 0 0 00 0 00 0 0 <sup>3</sup> 0 0 0 0 0 00 0 00 0 0 <sup>0</sup> <sup>5</sup> <sup>1</sup> 0 0 0 00 0 00 0 0 <sup>1</sup> <sup>1</sup> <sup>19</sup> 0 0 0 00 0 00 0 0 <sup>17</sup> <sup>17</sup> <sup>47</sup> <sup>17</sup> 0 00 0 00 0 0 <sup>3</sup> <sup>1</sup> <sup>31</sup> <sup>1</sup> 0 00 0 00 0 0 <sup>13</sup> <sup>13</sup> <sup>13</sup> <sup>5</sup> 0 0 00 0 0 <sup>71</sup> <sup>212</sup> <sup>71</sup> <sup>141</sup> <sup>38</sup> 0 00 0 0 <sup>17</sup> <sup>19</sup> <sup>121</sup> <sup>11</sup> <sup>6</sup> 0 00 0 0 0 0 0 0 0 0 0 00 <sup>1</sup> 00 0 0 0 0 0 0 0 0 0 00 <sup>1</sup> <sup>17</sup> 00 0 0 0 0 0 0 0 0 00 <sup>0</sup> <sup>2</sup> <sup>9</sup> 0 0 0 0 0 0 0 0 0 00 <sup>1</sup> <sup>2</sup> <sup>1</sup> 0 0 0 0 0 0 0 0 00 <sup>1</sup> <sup>2</sup> <sup>5</sup> 2 <sup>2</sup> <sup>11</sup> 4 CCCCCCCCCCCCCCCCCCCCCCCCCCCCA *Z*(1) *<sup>O</sup>*16*,d* <sup>=</sup> *<sup>N</sup><sup>c</sup>* ✏ BBBBBBBBBBBB@ 0 0 0 00 0 0 0 0 00 0 0 <sup>2</sup> <sup>301</sup> <sup>2</sup> 0 0 0 0 1 1 <sup>3</sup> 0 0 0 0 <sup>0</sup> <sup>1</sup> 0 0 <sup>1</sup> <sup>1</sup> <sup>7</sup> 0 0 0 0 0 0 00 <sup>9</sup> 0 0 0 0 0 00 <sup>7</sup> The intrinsic two-loop renormalization matrices of *f abc* and *dabc* sector are

 $\sum_{i=1}^{n}$ 

CCCCCCCCCCCCA

#### Mixing matrices and spectrum 3 0 0 0 0 00 00  $\overline{a}$ Mixing matrices and <sup>5579</sup> 18000  $1100$   $1000$   $10110$  $M_{11}$ <u>Mixing</u> ma 0 0 0 0 0 0 0 00 <sup>1</sup> <sup>3</sup> <sup>2</sup> <sup>1</sup> 3  $\overline{a}$  0 0 0 0 0 0 0 00 <sup>1</sup> <sup>5</sup> CCCCCCCCCCCCCCCCCCCCCCCCCCCCA

Form factors contain both IR and UV divergences, by subtracting the universal IR, one can obtain the UV renormalization matrix.  $\overline{b}$  $\overline{u}$  cubtro  $\gamma$ tin $\alpha$  <sup>269</sup> 2520 12 J I I T ONIT Ractors contain both in and OV divergences, by subtracting ion matrix.  $\times$  . C<br>il<br>il rm fact rs contain h 'n and IIV diver <sup>269</sup> 2520 7 U  $\overline{3}$  corrected  $\overline{3}$  $\overline{1}$ 9 ).  $\cdot$ l $\,$  $\sim$  544  $\sim$  $\sim$  111  $200,000$   $\overline{\phantom{a}}$  16<br>10 e universal IR, one can obtain the UV rend al In, one ca 1 ODIAIN LITE  $\overline{\mathbf{3}}$  $\overline{\phantom{a}}$ 17 17 711 L C<br>C<br>C<br>C tho universal ID  $\prod$ universal in

Dim-16 at 2-loop:  $\epsilon$  ا - $\mathbf{S}$  $\overline{1}$  $\epsilon$ 7. September 2005. September 2006. September 2006. September 2006. September 2006. September 2006. September 2<br>September 2006. September 2006. September 2006. September 2006. September 2006. September 2006. September 2006  $\lim_{\rightarrow} 16$ at  $2$ -lool 7111-TO di 2100



### Mixing matrices and spectrum

Anomalous dimensions for length-3 operators up to dimension 16:



#### **Finite remainder**  $\overline{a}$ nite remainder Higgs particle has no direct interaction with gluons but through Yukawa coupling with quarks.

The finite remainders -> Higgs amplitudes with high-order top mass corrections in Higgs EFT: 5.1 Transcendent structure of the structur<br>Transcendent of the structure of the struc  $\mathcal{L}_{\text{eff}} = \hat{C}_0 H \mathcal{O}_{\text{A-0}} + \sum_{i=1}^{N} \sum_{i=1}^{N} \hat{C}_i H \mathcal{O}_{\text{A-0}}$  $\mathsf{r}\mathsf{s}\to\mathsf{H}$ iggs amplitudes with high-order top  $\blacksquare$  $\mathcal{L}_{\text{eff}} = \hat{C}_0 H \mathcal{O}_{4;0} + \sum$  $\frac{\infty}{\sqrt{2}}$ 1  $m_{\rm t}^{2k}$  $\sum$  $\hat{C}_i H \mathcal{O}_{4+2k;i}$ 

*k*=1

*i*

There are "universal building blocks" that are independent of the operators: together with this paper. As an example, the result of *O*8;↵;*f*;1 is explicitly given in Appendix une operatore. I building blocks" that are independent of of canonical dimension of canonical dimension one jet production, the contribution, the contribution, the contribution of higher production, the contribution, the contribution, the contribution, the contribution of higher

 $\sum_{k=1}^{\infty} m_t^{2n}$ 

The full transcendentality degree-4 part is universal: Tho full transcondentality degree 4 part is universal of two-loop models (under many degree of partie annoted). ntality degree-4 part is universal: which dimension-

$$
\mathcal{R}_{\mathcal{O}}^{(2),\pm}\Big|_{\text{deg-4}} = -\frac{3}{2} \text{Li}_4(u) + \frac{3}{4} \text{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{4} \log(w) \left[\text{Li}_3\left(-\frac{u}{v}\right) + \text{Li}_3\left(-\frac{v}{u}\right)\right] \n+ \frac{\log^2(u)}{32} \left[\log^2(u) + \log^2(v) + \log^2(w) - 4\log(v)\log(w)\right] \n+ \frac{\zeta_2}{8} \left[5\log^2(u) - 2\log(v)\log(w)\right] - \frac{1}{4}\zeta_4 + \text{perms}(u, v, w),
$$

"maximal transcendentality principle" **[Kotikov, Lipatov, Onishchenko, Velizhanin 2004**]

#### Finite remainder  $\blacksquare$  -inite remainder poles of the same degree originate from the same term in the master integral coecients, so can be traced back to the absence of one-loop divergence. As mentioned in section 4.2, under The coupling is proportional to the mass of  $\mathsf{F}\mathsf{I}\mathsf{M}$ rational term, so divergence subtraction formula from (4.8) and (4.10) becomes Higgs particle has no direct interaction with gluons but through Yukawa coupling with quarks.

can be tracted back to the absence of the absence of the section of one- $\sigma$  mnte refriam form  $\sim$  riggs amplitudes with ingri-onemass corrections in Higgs EFT: *inite remainc*  $\overline{d}$  $\overline{\phantom{a}}$  $>$  Higgs amplitudes with *<sup>B</sup>* , and no term can The finite remainders -> Higgs amplitudes with high-order top  $\infty$ 1

$$
\mathcal{L}_{\text{eff}} = \hat{C}_0 H \mathcal{O}_{4;0} + \sum_{k=1} \frac{1}{m_\text{t}^{2k}} \sum_i \hat{C}_i H \mathcal{O}_{4+2k;i}
$$

which explicitly shows the leading singularity is of *<sup>O</sup>*(✏2) from *<sup>I</sup>*(1)(✏)*F*(1) *<sup>B</sup>* , and no term can icic dic <u>Ulliv</u> the operators:<br>
<u>
</u> There are "universal building blocks" that are in Libro dio <u>Cinivorodi</u><br>... There are "universal building blocks" that are independent of of canonical dimension of canonical dimension one jet production, the contribution, the contribution, the contribution of higher production, the contribution, the contribution, the contribution, the contribution of higher

Degree-3 part and degree-2 part are consist of universal building blocks {T<sub>3</sub>, T<sub>2</sub>}, plus simple log functions: be absorbed into a set of universal building blocks, and no other polylogarithm functions like with and with  $\frac{1}{3}$  is, is j, plus  $\frac{1}{3}$ ⇣ *u* ⌘  $\alpha$ cke (T<sub>2</sub> T<sub>2</sub>) plue eimple log fung  $\lceil$  can  $\lceil$ degree-2 part are consist of universal<br>← Tal in lug aimple leg functione: 1 ⇣ *uv* ⌘ 1 <sup>2</sup> log(*u*) log(*v*) log(*w*) + <sup>1</sup> i  $\overline{\phantom{a}}$ <sup>12</sup> log3(*w*)+(*<sup>u</sup>* \$ *<sup>v</sup>*)  $\lceil 2 \rceil$  olus simple loo functions. operators may be used to improve the precision for the precision for the cross section  $\mathcal{L}_\mathcal{S}$ 

$$
T_3(u, v, w) := \Big[ -\text{Li}_3\left(-\frac{u}{w}\right) + \log(u)\text{Li}_2\left(\frac{v}{1-u}\right) - \frac{1}{2}\log(u)\log(1-u)\log\left(\frac{w^2}{1-u}\right) + \frac{1}{2}\text{Li}_3\left(-\frac{uv}{w}\right) + \frac{1}{2}\log(u)\log(v)\log(w) + \frac{1}{12}\log^3(w) + (u \leftrightarrow v)\Big] + \text{Li}_3(1-v) - \text{Li}_3(u) + \frac{1}{2}\log^2(v)\log\left(\frac{1-v}{u}\right) - \zeta_2\log\left(\frac{uv}{w}\right).
$$

 $T_2(u, v) :=$ Li<sub>2</sub>(1 *- u*) + Li<sub>2</sub>(1 *- v*) + log(*u*) log(*v*) -  $\zeta_2$ . degree-2 part are three functions *{T*2[(*x*)*,* (*y*)]*|* <sup>2</sup> *<sup>Z</sup>*3*}* together with log<sup>2</sup> and ⇡2, where  $\det(1-v) + \det(u) \log(v) - \zeta_2$ .  $\frac{1}{25}$  for an interval inter





where x*i,* y*<sup>i</sup>* " 1*,...N<sup>c</sup>* are (anti)fundamental indices. We choose normalization: trp*TaTb*q " **• arXiv:2011.06540 with Guanda Lin (林冠达)**

# General strategy via Unitarity

In principle one may apply unitarity with full color dependence

An improved strategy is that: results of examples will also be used in later calculation in Section 5. Some further details

Color decomposition  $\longrightarrow$  Unitarity with color-ordered blocks  $\boldsymbol{F}^{(\ell)} = \sum \limits$ *i Ci*  $\left[ \mathcal{F}^{(\ell)} \right]$ *Ci* soure. amplitude *A*p0<sup>q</sup>



 $\check{D}$   $\check{D}$   $(1,1)$   $\check{D}$   $\check{D}$   $(1)$  $\check{\mathcal{D}}_1 = \check{\mathcal{D}}_{\Delta_1}(1,1)\,,\quad \check{\mathcal{D}}_{19} = \check{\mathcal{D}}_{\Delta_2}(1)\,,\quad \check{\mathcal{D}}_{25} = \check{\mathcal{D}}_{\Delta_3}(1)\,,\quad \check{\mathcal{D}}_{27} = \check{\mathcal{D}}_{\Delta_4}(1)$ Below we will first consider the color decomposition for *I* <sup>p</sup>2<sup>q</sup> which will help to identify

# General strategy via Unitarity

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# General strategy

In principle one may apply unitarity with full color dependence

An improved strategy is that:

Color decomposition  $\longrightarrow$  Unitarity with color-ordered blocks

For special cases, an alternative more-power tool is:

"Color-Kinematics Duality"

### Color-Kinematics duality  $\mathfrak{\text{1}}$ Tics duality

√ **[Bern, Carrasco, Johansson 2008]** 



A very intriguing duality which is still not understood.

#### A TOUI-DOINT EXAMPLE t <u>2</u> 3 3 4 4 5  $\pm$  3 4 5  $\pm$  3  $\pm$ A four noint avampl  $\sf t$ example A four-point example



$$
A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}
$$

$$
c_s = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_2 a_3 b} \tilde{f}^{b a_4 a_1}, \quad c_u = \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_2 a_4}
$$

$$
c_s = c_t + c_u \Rightarrow n_s = n_t + n_u
$$
  
Jacobi identity  
dual Jacobi relation

## Form factors

For Sudakov form factor (with stress tensor operator), the construction has been obtained up to five loops.

**G. Yang, PRL 117 (2016) no.27, 271602**



 **To get:** 

### **gauge theory gravity theory**  dual Jacobi relations

$$
n_k=n_i-n_j
$$

## Form factors

For Sudakov form factor (with stress tensor operator), the construction has been obtained up to five loops.



(a) (b) (c) (d) End the bound of the Master graphs  $\{\hat{\phi}$  in the blue line of

## Form factors

For Sudakov form factor (with stress tensor operator), the construction has been obtained up to five loops.

$$
\mathcal{O}_{L=2} = \text{tr}(\phi^2)
$$

How about more general high dimensional operators?

$$
\mathcal{O}_{\mathrm{L}} = \mathrm{tr}(\phi\phi...\phi)
$$



### Two-loop solution of BPS form factors



tions are defined according the labeling of momenta as functions *Ni*p*p*1*, p*2*, l*1*, l*2q for *i* P*{*a,b*}*, and *N*<sub>1</sub>, *N*<sub>2</sub>, para*, master* topologies are defined with the rule to reduce the rule to Two-planar master topologies in the *Two-planar* master topologies

#### Two-loop solution of BPS form factors *l* 2 <sup>2</sup>*s*<sup>12</sup> `  $\mathbf{I}$ *l* 2 <sup>1</sup>*s*<sup>23</sup> ` *l*  $\frac{1}{2}$ After using (6.20) and (6.22), there are still two parameters, chosen as *c*4*, c*6. It turns out





*l*1  $l_2$ 

 $l_3$ 

(f)





(h)

*p*1

 $p_{2}$ 

*p*3





(e)

(g)  $(e)$  (t)  $(g)$  (g) (1)



$$
\mathbf{I}_{12}^{(2)} = \sum_{\sigma \in S_2 \times S_2} \sum_{i=1}^{b} \int \prod_{j=1}^{2} \frac{d^D l_i}{i(\pi)^{\frac{D}{2}}} \frac{1}{S_i^{12}} \sigma \cdot \frac{\tilde{C}_i^{12} N_i^{12}}{\prod_a d_{i,a}}
$$

 $\mathcal{A}$  is that numerators  $\mathcal{A}$  with  $\mathcal{A}$  are not equal to  $\mathcal{A}$  are not equal

$$
\boldsymbol{I}_{123}^{(2)} = \sum_{\sigma \in S_3 \times S_3} \sum_{i=c}^{h} \int \prod_{j=1}^{2} \frac{d^D l_i}{i(\pi)^{\frac{D}{2}}} \frac{1}{S_i^{123}} \sigma \cdot \frac{\check{C}_i^{123} N_i^{123}}{\prod_a d_{i,a}}
$$

#### New features of non-planar results ature: 2 s of non-pl p1 ´ ✏q  $\overline{2}$  $\overline{\phantom{0}}$ *I*nar i *eE*✏  $\cup$ ا ا 115 **.**  $\overline{a}$ *<sup>I</sup>*p1<sup>q</sup> *<sup>I</sup>*p1<sup>q</sup> p*p*1*, p*2*, p*3q *<sup>D</sup>*ˇ<sup>19</sup>  $\mathbf{r}$ Note that the range-4 part is straightforward to write down and we will not give them here. **I**PP IS 1991 IS 2012 IS 20 ISBN 1991 IS 2014 IS 20 ISBN 6-100 ISBN 6-2015 ISBN 0-2015 ISBN 0-2016 ISB is cusp anomalous dimension [92, 93] and *<sup>G</sup>*coll is collinear anomalous dimension [94, 95].<sup>17</sup>

 $S_{\rm{max}}$  theory by simply dropping lower transcendental pieces (including the beta function beta function beta function  $\sigma$  ansatz is no-ionger enough: and  $\sigma$  $\sum$  operators of Irportune  $U$ sing two-loop form factor  $\mathcal{O}$ *g*YM rather than *g* and explicitly write down the *N*<sup>c</sup> dependence. BDS ansatz is no-longer enough:

Planar

\n
$$
\underline{\mathcal{I}}^{(2)} = \frac{1}{2} \left( \underline{\mathcal{I}}^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) \underline{\mathcal{I}}^{(1)}(2\epsilon) + \mathcal{R}^{(2)} + \mathcal{O}(\epsilon)
$$
\nnon-planar

\n
$$
\underline{\mathbf{I}}^{(2)} = \frac{1}{2} \left( \underline{\mathbf{I}}^{(1)}(\epsilon) \right)^2 + \tilde{f}^{(2)}(\epsilon) \underline{\mathbf{I}}^{(1)}(2\epsilon) + \mathbf{H}^{(2)}(\epsilon) + \mathbf{R}^{(2)} + \mathcal{O}(\epsilon)
$$

$$
\begin{aligned}\n\boxed{\mathbf{R}}: \quad \mathbf{H}^{(2)} &= \sum_{i < j < k} \mathbf{H}_{ijk}^{(2)}, \\
\mathbf{H}_{123}^{(2)} &= \left(\sum_{\sigma \in S_3} (-1)^{\sigma} \check{\mathcal{D}}_{\Delta_2}(\sigma)\right) \frac{1}{4\epsilon} \log\left(\frac{-s_{12}}{-s_{23}}\right) \log\left(\frac{-s_{23}}{-s_{13}}\right) \log\left(\frac{-s_{13}}{-s_{12}}\right)\n\end{aligned}
$$

.<br>21 **Bern, De Freitas and Dixon, 2002, .…**<br> **Bern, De Freitas and Dixon, 2002, .…** Bern, De Freitas and Dixon, 2002, ....<br>Explained by dipole formula: Becher and M. Neubert; Gardi and L. Magnea 2009 2d harmonic polylogarithms [105, 106]. <sup>p</sup>✏q*F*p0<sup>q</sup> ` *<sup>F</sup>*p1q*,*fin ` *<sup>O</sup>*p✏q*,* (7.4)

Form factor computation provides an independent check for non-planar<br>IB structure two loop amplitudes with general p point IR structure two-loop amplitudes with general n-point. *x* design independent check for non-planaries an independent check for non-planaries – 52 – 52 –<br>2 <sup>17</sup>The non-planar corrections of cusp and collinear anomalous dimensions only start at fourth loop order. Tonn ractor computation provides an independent check for non-planar<br>*N* IR structure two-loop amplitudes with general n-point.

#### New features of non-planar results ature: 2 s of non-pl p1 ´ ✏q  $\overline{2}$  $\overline{\phantom{0}}$ *I*nar i *eE*✏  $\cup$ ا ا 115 **.**  $\overline{a}$ *<sup>I</sup>*p1<sup>q</sup> *<sup>I</sup>*p1<sup>q</sup> p*p*1*, p*2*, p*3q *<sup>D</sup>*ˇ<sup>19</sup>  $\mathbf{r}$ Note that the range-4 part is straightforward to write down and we will not give them here. we know that non-classical part of " *<sup>R</sup>*p2<sup>q</sup>  $\bigcap$ can be chosen as the same *G* functions as in  $\operatorname{\mathsf{dll}}$ **I**PP IS 1991 IS 2012 IS 20 ISBN 1991 IS 2014 IS 20 ISBN 6-100 ISBN 6-2015 ISBN 0-2015 ISBN 0-2016 ISB is cusp anomalous dimension [92, 93] and *<sup>G</sup>*coll is collinear anomalous dimension [94, 95].<sup>17</sup>

BDS ansatz is no-longer enough: contribution of the BDS and Catalogue versatz is no-longe *R*NP " *<sup>R</sup>*p2<sup>q</sup>  $\overline{a}$ ` *R*basisp*u, v, w*q ` *R*basisp*v, w, u*q ` *R*basisp*w, u, v*q ´ 3 ˆ as no nongononough. *g*YM rather than *g* and explicitly write down the *N*<sup>c</sup> dependence. andale to no forgot onought. divergence we also add auxiliary function *U*p*ij*q. The range-2 remainder is defined as n Unbuyn

Planar

\n
$$
\underline{\mathcal{I}}^{(2)} = \frac{1}{2} \left( \underline{\mathcal{I}}^{(1)}(\epsilon) \right)^2 + f^{(2)}(\epsilon) \underline{\mathcal{I}}^{(1)}(2\epsilon) + \mathcal{R}^{(2)} + \mathcal{O}(\epsilon)
$$
\nnon-planar

\n
$$
\underline{\mathbf{I}}^{(2)} = \frac{1}{2} \left( \underline{\mathbf{I}}^{(1)}(\epsilon) \right)^2 + \tilde{f}^{(2)}(\epsilon) \underline{\mathbf{I}}^{(1)}(2\epsilon) + \mathbf{H}^{(2)}(\epsilon) + \mathbf{R}^{(2)} + \mathcal{O}(\epsilon)
$$

$$
\begin{array}{ll}\n\mathbf{H}^{(2)} = \sum_{i < j < k \\
\mathbf{H}_{123}^{(2)} = \left( \sum_{\sigma \in S_3} (-1)^{\sigma} \check{\mathcal{D}}_{\Delta_2}(\sigma) \right) \frac{1}{4\epsilon} \log \left( \frac{-s_{12}}{-s_{23}} \right) \log \left( \frac{-s_{23}}{-s_{13}} \right) \log \left( \frac{-s_{13}}{-s_{12}} \right)\n\end{array}
$$

Finite remainder:  $\begin{bmatrix} R^{\text{NP}} = \sum_{i \leq i \leq k} I_i \end{bmatrix}$ 

**maximally** 

Finite remainder:	\n $\mathbf{R}^{\text{NP}} = \sum_{i < j < k} \mathbf{R}^{\text{NP}}_{ijk},$ \n
New non-planar	\n $\mathbf{R}^{\text{NP}}_{123} = \sum_{\sigma \in S_3} \check{\mathcal{D}}_{\Delta_2}(\sigma) \mathcal{R}^{\text{NP}}_{\text{basis}}(\sigma(u, v, w)) = \left( \sum_{\sigma \in S_3} (-1)^{\sigma} \check{\mathcal{D}}_{\Delta_2}(\sigma) \right) \mathcal{R}^{\text{NP}}_{\text{basis}}(u, v, w)$ \n
transcendental part	\n $\mathcal{R}^{\text{NP}}_{\text{basis}}(u, v, w) = \text{Li}_3 \left( 1 - \frac{1}{u} \right) \log \left( \frac{v}{w} \right) + \frac{1}{12} \log(u)^3 \log \left( \frac{v}{w} \right) + \zeta_2 \log(1 - u) \log \left( \frac{v}{w} \right)$ \n

# Summary and Outlook

## Summary





# Outlook

- Classification and two-loop renormalization for more generic operators (fermions, CP-odd, and finally in generic EFT)
- Integrability for QCD high-twist operators (planar 2-loop)  ${\rm tr}(D_+^{n_1}FD_+^{n_2}F...D_+^{n_L}F)$  . In progress with Qingjun Jin, Ke Ren and Rui Yu
- CK duality for higher loop cases and for non-BPS operators 3-loop solution also found **To appear with Guanda Lin and Siyuan Zhang**
- Explore hidden structures from the non-planar data

# Outlook

- Classification and two-loop renormalization for more generic operators (fermions, CP-odd, and finally in generic EFT)
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# Thank you!

# Backup

## Hierarchy of simplicity





### **Non-supersymmetric QCD**

# Loop structure of form factors

Form factors have divergences:

IR divergences

soft and collinear divergences

UV divergences

renormalization of coupling g and operators O

The IR and UV are mixed in general in a non-trivial way.

General structure of (bare) amplitudes/form factors:



#### IR structure in QCD  $\sim$  $|H \rangle$  $\overline{\phantom{a}}$ ture ll  $\overline{a}$ b . (2.20) . (2.20) . (2.20) . (2.20) . (2.20) . (2.20) . (2.20) . (2.20) . (2.20) . (2.20) . (2.20) . (2.20)  $\blacksquare$  $\overline{\phantom{a}}$  $\circ$   $\circ$   $\circ$ F(2) = S−<sup>2</sup> <sup>ϵ</sup> <sup>F</sup>(2) <sup>b</sup> <sup>+</sup> <sup>S</sup>−<sup>1</sup> \$ Z(1) − " 1 +  $\bigcup \bigcup \bigcup$

Universal IR structure: The renormalized form factor contains IR divergences, which take a universal structure  $\mathcal{L}$  divergences, which take a universal structure  $\mathcal{L}$  $\overline{U}$  (see also  $\overline{U}$ ) in  $\overline{V}$ **[Catani 1998]**  $\overline{\phantom{a}}$  $\overline{a}$ 

$$
F^{(1)} = I^{(1)}(\epsilon)F^{(0)} + F^{(1),\text{fin}} + \mathcal{O}(\epsilon),
$$
  

$$
F^{(2)} = I^{(2)}(\epsilon)F^{(0)} + I^{(1)}(\epsilon)F^{(1)} + F^{(2),\text{fin}} + \mathcal{O}(\epsilon)
$$

where for the form factor with n external gluons, we have  $\mathcal{L}_{\text{max}}$ e.g. for pure external gluons:

$$
I^{(1)}(\epsilon) = -\frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left( \frac{C_A}{\epsilon^2} + \frac{\beta_0}{2\epsilon} \right) \sum_{i=1}^n (-s_{i,i+1})^{-\epsilon},
$$
  
\n
$$
I^{(2)}(\epsilon) = -\frac{1}{2} \left[ I^{(1)}(\epsilon) \right]^2 - \frac{\beta_0}{\epsilon} I^{(1)}(\epsilon)
$$
  
\n
$$
+ \frac{e^{-\gamma_E \epsilon} \Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left[ \frac{\beta_0}{\epsilon} + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) C_A - \frac{10}{9} n_f \right] I^{(1)}(2\epsilon)
$$
  
\n
$$
+ n \frac{e^{\gamma_E \epsilon}}{\epsilon \Gamma(1-\epsilon)} \left[ \left( \frac{\zeta_3}{2} + \frac{5}{12} + \frac{11\pi^2}{144} \right) C_A^2 + \frac{5n_f^2}{27} - \left( \frac{\pi^2}{72} + \frac{89}{108} \right) C_A n_f - \frac{n_f}{4C_A} \right]
$$

#### Finite remainder  $\sim$  $\frac{1}{2}$  $\overline{1}$   $\overline{1}$   $\overline{1}$   $\overline{1}$  $\overline{\phantom{a}}$  $\frac{1}{2}$   $\frac{1}{2}$  $\overline{\phantom{a}}$

The intrinsic information is contained in the finite part:

$$
F^{(1)} = I^{(1)}(\epsilon)F^{(0)} + F^{(1),\text{fin}} + \mathcal{O}(\epsilon)
$$
  

$$
F^{(2)} = I^{(2)}(\epsilon)F^{(0)} + I^{(1)}(\epsilon)F^{(1)} + F^{(2),\text{fin}} + \mathcal{O}(\epsilon)
$$

There are six different color factors: are six different color facts **n**  $\epsilon$ ,  $\alpha$  $\mathscr{R}^{(l)}_{\scriptscriptstyle\cap}$  $\mathcal O$  $=\mathscr{F}_{\mathscr{O}}^{(l),\mathrm{fin}}/\mathscr{F}_{\mathscr{O}}^{(0)}$ 

$$
\mathcal{R}_{\odot}^{(2)} = N_c^2 \mathcal{R}_{\odot}^{(2),N_c^2} + N_c^0 \mathcal{R}_{\odot}^{(2),N_c^0} + \frac{1}{N_c^2} \mathcal{R}_{\odot}^{(2),N_c^{-2}} + n_f N_c \mathcal{R}_{\odot}^{(2),n_fN_c} + \frac{n_f}{N_c} \mathcal{R}_{\odot}^{(2),n_f/N_c} + n_f^2 \mathcal{R}_{\odot}^{(2),n_f^2}
$$

 $\overline{C}$ ransion<sup>.</sup> ansion: A different expansion:

 $E_{F}^{\mathscr{R}_{\widehat{C}}^{\Sigma,\Sigma_{A}C_{F}}+C_{F}^{\Sigma}\mathscr{R}_{\widehat{C}}^{\Sigma,\Sigma_{F}}+n_{f}C_{A}\mathscr{R}_{\widehat{C}}^{\Sigma}}$ <sup>12</sup> <sup>+</sup>  $\frac{1}{2}$  $\sim$   $\sim$   $\sim$   $\sim$   $\sim$   $\sim$  ${\mathscr R}^{(2)}_\varcap$  $\mathcal O$  $=C_A^2\mathscr{R}_{\mathscr{O}}^{(2),\,mathcal{C}_A^2}$  $\alpha_{\beta}^{(2),C_A^2}$  +  $C_A C_F \mathcal{R}_{\Theta}^{(2),C_A C_F}$  +  $C_F^2 \mathcal{R}_{\Theta}^{(2),C_F^2}$ *F*  $\frac{f(2), C_F^2}{f_0} + n_f C_A \mathcal{R}_{\odot}^{(2), n_f C_A} + n_f C_F \mathcal{R}_{\odot}^{(2), n_f C_F} + n_f^2 \mathcal{R}_{\odot}^{(2), n_f^2}$ *f*  $\mathcal O$  $C_A = N_c$ ,  $C_F =$  $N_c^2 - 1$ 2*Nc*