Exact solution of the supersymmetric $su(2|2)$ model

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Contents

Overview

A long history for topology in physics:

- Topological field theory, AdS/CFT, fractional charges, etc;
- Quantum Hall, topological insulator, topological order, Weyl/Majorana fermion, etc.

Two issues:

- Topological states of matter;
- Quantization in topological manifold.

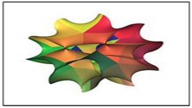
Sommerfield quantization: $\oint \vec{p} d\vec{r} = 2n\pi\hbar$.

How to quantize in topological space?

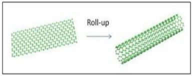
An important example of quantization in topological manifolds:

Superstring (10-d) - Calabi-Yau(6-d) $\Rightarrow$ Space-time (4d)

Calabi-Yau



Compact



The present topic: Exact quantization in topological manifold.

Our exact quantization procedure includes:

- What is the vacuum (ground state) in a topological manifold?
- What kind of particles (elementary excitations) may emerge?
- How do the particles live in a topological manifold?

Start point: topological integrable models.

The history of quantum integrable models:

- 1931, H. Bethe (coordinate Bethe Ansatz)
- 1967, C. N. Yang (Yang-Baxter equation)
- 1971, R. J. Baxter (T-Q equation)

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}.$$

- 1979, L. D. Faddeev (algebraic Bethe Ansatz)

All those methods need a vacuum state!

The first topological quantum integrable model (odd N XYZ) was discovered in 1972 but resisted solution for 40 years!

Baxter's T-Q equation encounters a big problem!

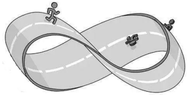
2013, off-diagonal Bethe Ansatz was proposed:

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)} + \frac{F(u)}{Q(u)}.$$

The topological quantum spin chain

$$H = -\sum_{j=1}^{N} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh\eta \sigma_j^z \sigma_{j+1}^z \right].$$

Anti-periodic boundary conditions: $\sigma_{N+1}^{\alpha} = \sigma_1^x \sigma_1^{\alpha} \sigma_1^x$ $(\alpha = x, y, z)$.



Z_2-symmetry: $\tilde{H}_j = U_j^x H U_j^x, \quad U_j^x = \prod_{l=1}^{j} \sigma_l^x, \quad [H, U^x] = 0.$

Generators and commutative relations:

$$U^{\alpha} = \sigma_1^{\alpha} \sigma_2^{\alpha} \cdots \sigma_N^{\alpha}, \quad \alpha = x, y, z,$$

$$(U^{\beta})^2 = id, \quad U^{\alpha} U^{\beta} = (-1)^N U^{\beta} U^{\alpha}, \quad \text{for} \quad \alpha \neq \beta, \alpha, \beta = x, y, z.$$

Batchelor, Frahm, Galleas, Niccoli, $\cdots$

The R-matrix

$$R_{0,j}(u) = \frac{\sinh(u+\eta)+\sinh u}{2\sinh\eta} + \frac{1}{2}(\sigma_j^x\sigma_0^x + \sigma_j^y\sigma_0^y) + \frac{\sinh(u+\eta)-\sinh u}{2\sinh\eta}\sigma_j^z\sigma_0^z.$$

Properties:

Initial condition : $R_{1,2}(0) = P_{1,2}$,

Unitarity : $R_{1,2}(u)R_{2,1}(-u) = -\frac{\sin(u+\eta)\sin(u-\eta)}{\sin^2\eta} \times \mathrm{id}$,

Crossing relation : $R_{1,2}(u) = -\sigma_1^y R_{1,2}^{t_1}(-u-\eta)\sigma_1^y$,

PT-symmetry : $R_{1,2}(u) = R_{2,1}(u) = R_{1,2}^{t_1 t_2}(u)$,

Z_2-symmetry : $\sigma_1^\alpha\sigma_2^\alpha R_{1,2}(u) = R_{1,2}(u)\sigma_1^\alpha\sigma_2^\alpha$, for $\alpha = x, y, z$,

Fusion condition : $R_{1,2}(-\eta) = -2P_{1,2}^{(-)}$,

Yang-Baxter equation :

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

The monodromy matrix

$$T_0(u) = \sigma_0^x R_{0,N}(u-\theta_N)\cdots R_{0,1}(u-\theta_1) = \begin{pmatrix} C(u) & D(u) \\ A(u) & B(u) \end{pmatrix}.$$

Inhomogeneous parameters are introduced for convenience!

The transfer matrix is

$$t(u) = tr_0 T_0(u) = B(u) + C(u).$$

From the Yang-Baxter relation, one can prove that

$$[t(u), t(v)] = 0.$$

The Hamiltonian is the first order derivative of the logarithm of the transfer matrix

$$H = -2\sinh\eta \left.\frac{\partial \ln t(u)}{\partial u}\right|_{u=0,\{\theta_j=0\}} + N\cosh\eta.$$

Conserved charges

Due to the topological boundary, the model possesses neither translational invariance nor $U(1)$ symmetry.

Nevertheless

- Basic properties of the transfer matrix:

$$\mathbf{t}(0) = \sigma_1^x P_{1,N} P_{1,N-1} \cdots P_{1,2}, \qquad \mathbf{t}^{2N}(0) = 1,$$

  is a conserved and represents the shift operator in the topological manifold.

- Topological momentum: $\mathbf{P}_q = -i \ln \mathbf{t}(0)$. The eigenvalues of $\mathbf{P}_q$ are

$$k = \frac{\pi l}{N} \ mod \, \{\pi\}, \qquad l = \{-N, -N+1, \cdots, N-1\}.$$

- String charge:

$$\mathbf{M}_q = \frac{1}{2}(\mathbf{l}_q^+ + \mathbf{l}_q^-) = \frac{1}{4}e^{-\frac{(N-1)\eta}{2}} \lim_{u\to\infty}(2\sinh\eta e^{-u})^{N-1}\mathbf{t}(u),$$

where

$$\mathbf{l}_q^{\pm} = \frac{1}{2}\sum_{j=1}^{N} e^{\mp\frac{\eta}{2}\sum_{k=j+1}^{N}\sigma_k^z}\sigma_j^{\pm}e^{\pm\frac{\eta}{2}\sum_{k=1}^{j-1}\sigma_k^z},$$

are two generators of the quantum group associated with the model.

The eigenvalues of the operator $\mathbf{M}_q$ is given by

$$M_q = \frac{1}{4}\sinh^{N-1}\eta\,\Lambda_0\,e^{-\sum_{k=1}^{N-1}z_k}.$$

When $\eta\to 0$, the model tends to an isotropic spin chain and the $U(1)$ symmetry recovers with $\mathbf{M}_q = \sum_{j=1}^{N}\sigma_j^x/2$, which is just the $U(1)$ charge.

$\mathbf{M}_q$ is not an $U(1)$ charge for generic η.

The off-diagonal Bethe Ansatz

From the initial condition of R-matrix, we deduce

$$\begin{aligned} t(\theta_j) &= tr_0\left\{R_{0,N}(\theta_j-\theta_N)\cdots P_{0,j}\cdots R_{0,1}(\theta_j-\theta_1)\sigma_0^x\right\} \\ &= R_{j,j-1}(\theta_j-\theta_{j-1})\cdots R_{j,1}(\theta_j-\theta_1) \\ &\quad \times\sigma_j^x R_{j,N}(\theta_j-\theta_N)\cdots R_{j,j+1}(\theta_j-\theta_{j+1}). \end{aligned}$$

From the crossing relation of R-matrix, we have

$$\begin{aligned} t(\theta_j-\eta) &= (-1)^{N-1}\, tr_0\left\{R_{0,N}^{t_0}(-\theta_j+\theta_N)\cdots R_{0,1}^{t_0}(-\theta_j+\theta_1)\sigma_0^x\right\} \\ &= (-1)^{N-1}\, tr_0\left\{\sigma_0^x R_{0,1}(-\theta_j+\theta_1)\cdots R_{0,N}(-\theta_j+\theta_N)\right\} \\ &= (-1)^{N-1} R_{j,j+1}(-\theta_j+\theta_{j+1})\cdots R_{j,N}(-\theta_j+\theta_N) \\ &\quad \times\sigma_j^x R_{j,1}(-\theta_j+\theta_1)\cdots R_{j,j-1}(-\theta_j+\theta_{j-1}). \end{aligned}$$

The product of above terms gives the important properties for operator identities

$$t(\theta_j)t(\theta_j-\eta) = -a(\theta_j)d(\theta_j-\eta)\times \mathrm{id}, \quad j=1,\cdots,N,$$

where

$$d(u) = a(u-\eta) = \prod_{j=1}^{N}\frac{\sinh(u-\theta_j)}{\sinh\eta}.$$

Functional relations

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta) = -a(\theta_j)d(\theta_j-\eta), \quad j=1,\cdots,N.$$

From the definition, we know that $\Lambda(u)$ is a trigonometrical polynomial of degree $N-1$ with the periodicity property

$$\Lambda(u+i\pi) = (-1)^{N-1}\Lambda(u).$$

The inhomogeneous $T-Q$ relation

$$\Lambda(u) = a(u)e^u\frac{Q(u-\eta)}{Q(u)} - e^{-u-\eta}d(u)\frac{Q(u+\eta)}{Q(u)} - c(u)\frac{a(u)d(u)}{Q(u)}.$$

Why?

$$d(u) = a(u-\eta) = \prod_{j=1}^{N}\frac{\sinh(u-\theta_j)}{\sinh\eta}, \qquad \Rightarrow \qquad d(\theta_j)=0, a(\theta_j-\eta)=0.$$

$$\Lambda(\theta_j) = a(\theta_j)e_j^{\theta}\frac{Q(\theta_j-\eta)}{Q(\theta_j)}, \qquad \Lambda(\theta_j-\eta) = -e^{-\theta_j}d(\theta_j-\eta)\frac{Q(\theta_j)}{Q(\theta_j-\eta)}.$$

Asymptotic behavior: $\Rightarrow$

$$c(u) = e^{u-N\eta+\sum_{l=1}^{N}(\theta_l-\lambda_l)} - e^{-u-\eta-\sum_{l=1}^{N}(\theta_l-\lambda_l)}.$$

Regularity: $\Rightarrow$ Bethe Ansatz equations

$$e^{\lambda_j} a(\lambda_j)Q(\lambda_j-\eta) - e^{-\lambda_j-\eta} d(\lambda_j)Q(\lambda_j+\eta) - c(\lambda_j)a(\lambda_j)d(\lambda_j) = 0,$$

$$j = 1, \cdots, N, \qquad Q(u) = \prod_{j=1}^{N} \sinh(u-\lambda_j)\sinh^{-1}\eta.$$

The eigenvalue of Hamiltonian is

$$E = 2\sinh\eta \sum_{j=1}^{N} [\coth(\lambda_j+\eta) - \coth(\lambda_j)] - N\cosh\eta - 2\sinh\eta.$$

Now we retrieve the eigenstates.

Separation of variables basis of the Hilbert space:

$$\begin{aligned}\langle \theta_{p_1}, \cdots, \theta_{p_n}| &= \langle 0| \prod_{j=1}^{n} C(\theta_{p_j}), \\ |\theta_{q_1}, \cdots, \theta_{q_n}\rangle &= \prod_{j=1}^{n} B(\theta_{q_j})|0\rangle,\end{aligned}$$

where $q_j, p_j \in \{1, \cdots, N\}$, $p_1 < p_2 < \cdots < p_n$ and $q_1 < q_2 < \cdots < q_n$. They are the eigenstates of $D(u)$.

The total number of linearly independent right (left) states is

$$\sum_{n=0}^{N} \frac{N!}{(N-n)!n!} = 2^N.$$

Meanwhile, the basis are orthogonal

$$\langle \theta_{p_1}, \cdots, \theta_{p_n}|\theta_{q_1}, \cdots, \theta_{q_m}\rangle = f_n(\theta_{p_1}, \cdots, \theta_{p_n})\, \delta_{m,n} \prod_{j=1}^{n} \delta_{p_j, q_j}.$$

Eigenstates

The eigenstates of the system can be decomposed as a unique linear combination of these basis.

$$\langle\Psi| = \sum_{n=0}^{N}\sum_{\{p_j\}}\chi_n(\theta_{p_1},\cdots,\theta_{p_n})\langle\theta_{p_1},\cdots,\theta_{p_n}|.$$

The expansion coefficients are

$$\langle\Psi|\theta_{q_1},\cdots,\theta_{q_n}\rangle = F_n(\theta_{q_1},\cdots,\theta_{q_n}),\quad n=0,\cdots,N.$$

We put $F_0 = 1$ by properly choosing the normalization of the eigenvector $\langle\Psi|$.

Consider the quantity $\langle\Psi|t(\theta_{q_{n+1}})|\theta_{q_1},\cdots,\theta_{q_n}\rangle$. Acting $t(\theta_{q_{n+1}})$ to the left and to the right alternately, we obtain

$$\Lambda(\theta_{q_{n+1}})F_n(\theta_{q_1},\cdots,\theta_{q_n}) = F_{n+1}(\theta_{q_1},\cdots,\theta_{q_{n+1}}),$$

and the solution

$$F_n(\theta_{q_1},\cdots,\theta_{q_n}) = \prod_{j=1}^{n}\Lambda(\theta_{q_j}).$$

The coefficients are

$$\chi_n(\theta_{p_1},\cdots,\theta_{p_n}) = \frac{\prod_{j=1}^{n}\Lambda(\theta_{q_j})}{f_n(\theta_{p_1},\cdots,\theta_{p_n})}, \quad \chi_0 = 1.$$

Thermodynamic limit

- The contribution of the third term.
- The degenerates points.

The t-W relation

For simplicity, consider the homogeneous case.

Using the fusion techniques, we have

$$\mathbf{t}(u)\mathbf{t}(u-\eta) = tr_{1,2}\{P^{(-)}_{1,2}\sigma^x_1\sigma^x_2\mathbf{T}_2(u)\mathbf{T}_1(u-\eta)P^{(-)}_{1,2}\}$$

$$+tr_{1,2}\{P^{(+)}_{1,2}\sigma^x_1\sigma^x_2\mathbf{T}_2(u)\mathbf{T}_1(u-\eta)P^{(+)}_{1,2}\}.$$

Thus the $t-W$ relation is

$$\mathbf{t}(u)\mathbf{t}(u-\eta) = -a(u)d(u-\eta)\times\mathbf{id} + d(u)\mathbf{W}(u),$$

where $\mathbf{W}(u)$ is an operator-valued degree N trigonometric polynomial of u.

$$[\mathbf{W}(u),\mathbf{t}(v)] = 0.$$

Functional relations

$$\Lambda(u)\Lambda(u-\eta) = -a(u)d(u-\eta) + d(u)W(u). \qquad (1)$$

Parameterization:

$$\Lambda(u) = \Lambda_0 \prod_{j=1}^{N-1} \sinh(u - z_j + \frac{\eta}{2}), \qquad \Lambda(u+i\pi) = (-1)^{N-1}\Lambda(u),$$

$$W(u) = W_0 \sinh^{-N}\eta \prod_{l=1}^{N} \sinh(u - w_l).$$

An important fact is that (1) is a degree $2N$ polynomial equation and thus gives $2N+1$ independent equations for the coefficients, which determines the $N-1$ z_j roots, N w_l roots and the two constants Λ_0 and W_0 completely.

Putting $u = z_j - \eta/2$ in (1), we obtain new Bethe ansatz equations

$$\sinh^N(z_j - \frac{3\eta}{2}) \sinh^N(z_j + \frac{\eta}{2}) = W_0 \sinh^N(z_j - \frac{\eta}{2}) \prod_{l=1}^{N} \sinh(z_j - w_l - \frac{\eta}{2}).$$

Putting $u = w_l$ in (1), we obtain

$$\Lambda_0^2 \prod_{j=1}^{N-1} \sinh(w_l - z_j + \frac{\eta}{2}) \sinh(w_l - z_j - \frac{\eta}{2}) = -\sinh^{-2N}\eta \sinh^N(w_l+\eta) \sinh^N(w_l - \eta).$$

Since $\Lambda(u)$ is a degree $N-1$ trigonometric polynomial of u, the leading terms in the right hand side of (1) must be zero. Therefore, $W_0^2 = 1$.

The coefficient Λ_0 can be determined by putting $u = 0$ in (1) as

$$\Lambda_0^2 \prod_{j=1}^{N-1} \sinh(z_j + \frac{\eta}{2}) \sinh(z_j - \frac{\eta}{2}) = (-1)^{N-1}.$$

The eigenvalue of the Hamiltonian can be expressed as

$$E = 2 \sinh \eta \sum_{j=1}^{N-1} \coth(z_j - \frac{\eta}{2}) + N \cosh \eta.$$

Ground state

From the intrinsic properties of the R-matrix, for imaginary η we have

$$\mathbf{t}^{\dagger}(u) = (-1)^{N-1}\mathbf{t}(u^* - \eta), \qquad \Lambda(u) = (-1)^{N-1}\Lambda^*(u^* - \eta).$$

The above relation implies that if z_j is a root, z_j^* must also be a root!

Therefore, z_j can be classified into 3 sets:

(1) real z_j;

(2) $\mathrm{Im} z_l = -i\pi/2$ (this is because its conjugate shifted by $i\pi$ becomes itself);

(3) complex conjugate pairs.

$W^*(u^*) = (-1)^N W(u)$ indicates if w_l is a root of $W(u)$, w_l^* must also be a root.

In the ground state, all roots take real values

$$\rho(z)+\rho^h(z)=\frac{2\cosh(\frac{\pi z}{\pi-\gamma})\sin(\frac{\pi\gamma}{2\pi-2\gamma})}{(\pi-\gamma)[\cosh(\frac{2\pi z}{\pi-\gamma})+\cos(\frac{\pi(\pi-2\gamma)}{\pi-\gamma})]}.$$

Here $\rho^h(z)$ is non-zero only in the range $|z|>D$ ($D\to\infty$ in the thermodynamic limit) with $N\int_D^\infty \rho^h(z)dz=1/2$ and $N\int_{-\infty}^{-D}\rho^h(z)dz=1/2$.

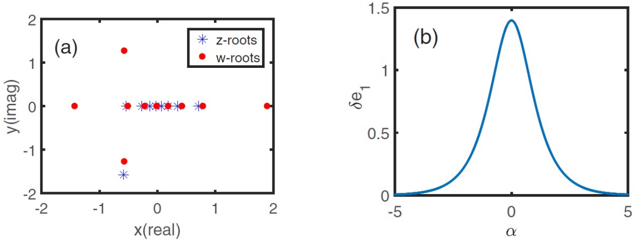
If $|\Psi\rangle$ is a common eigenstate of H and $\mathbf{P_q}$, then U_z acting on $|\Psi\rangle$ generates another degenerate eigenstate because $[H,U_z]=0$ and $[\mathbf{P_q},U_z]\neq 0$. The two half-holes contribute two half zero modes (carrying zero energy).

The ground state energy density reads

$$e_g=-\sin\gamma\int\frac{\cosh[\frac{(\pi-2\gamma)\tau}{2}]\tanh[\frac{(\pi-\gamma)\tau}{2}]}{\sinh(\frac{\pi\tau}{2})}d\tau+\cos\gamma.$$

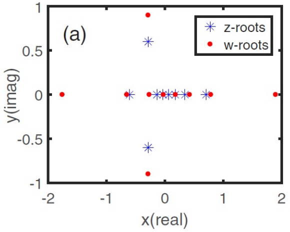
Elementary excitations I

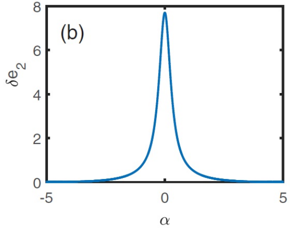
- The first kind of elementary excitations is described by a single root locating in the axis $\mathrm{Im}z = -i\pi/2$ and all the other roots remaining in the real axis.
- Accordingly, two w-roots form a conjugate pair $w_{\pm} = \beta \pm m\eta/2$ with β and m two real numbers, and all the other w-roots keep real.



Elementary excitations II

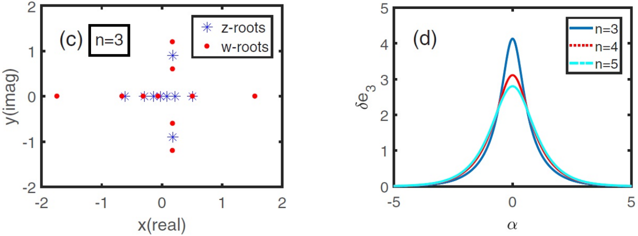
- The simplest single conjugate pair excitation is determined by $z_{\pm} \sim \alpha \pm \eta$. The corresponding w-set is formed by a w conjugate pair $w_{\pm} \sim \alpha \pm 3\eta/2$ and $N-2$ real w_l. Both the positions and the imaginary parts of the conjugate pairs are determined by convergence requirement of the density functions.





Elementary excitations III

- General complex-root excitation is described by a conjugate pair $z_\pm \sim \alpha \pm n\eta/2$ with $n \geq 3$, and all the other z-roots remaining in the real axis.
- In this case, the corresponding w-set is formed by a four-string $\sim \alpha \pm (n+1)\eta/2$, $\alpha \pm (n-1)\eta/2$ and $N-4$ real w-roots.



The supersymmetric $su(2|2)$ model

Supersymmetric fusion

- High rank quantum integrable systems
- Fusion: closed operator product identities
- Fusin: new integrable models

Main idea of fusion

R-matrix: $R_{12}(-\alpha) = P_{12}^{(d)} S_{12}$

Yang-Baxter equation:

$$R_{12}(-\alpha)R_{13}(u-\alpha)R_{23}(u) = R_{23}(u)R_{13}(u-\alpha)R_{12}(-\alpha). \tag{2}$$

$P_{12}^{(d)} \times (2)$, and using $P_{12}^{(d)} R_{12}(-\alpha) = R_{12}(-\alpha)$, we have

$$R_{12}(-\alpha)R_{13}(u-\alpha)R_{23}(u) = P_{12}^{(d)} R_{23}(u)R_{13}(u-\alpha)R_{12}(-\alpha). \tag{3}$$

Comparing the right hand sides of (2) and (3), we obtain

$$P_{12}^{(d)} R_{23}(u)R_{13}(u-\alpha)P_{12}^{(d)} = R_{23}(u)R_{13}(u-\alpha)P_{12}^{(d)} \equiv R_{\langle 12\rangle 3}(u).$$

- $V_1 \otimes V_2 \otimes V_3 \Rightarrow V_{\langle 12\rangle} \otimes V_3$,
- does not break the integrability.
- Fusion in quantum space, $R_{\langle 12\rangle\langle 34\rangle}(u)$, new integrable models.

Fusion with boundary reflection

The reflection equation at special point gives

$$R_{12}(-\alpha)K_1^-(u-\alpha)R_{21}(2u-\alpha)K_2^-(u)=K_2^-(u)R_{12}(2u-\alpha)K_1^-(u-\alpha)R_{21}(-\alpha). \tag{4}$$

$P_{12}^{(d)}\times$ (4) and using $P_{12}^{(d)}R_{12}(-\alpha)=R_{12}(-\alpha)$, we have

$$\begin{aligned} &R_{12}(-\alpha)K_1^-(u-\alpha)R_{21}(2u-\alpha)K_2^-(u)\\ &\quad=P_{12}^{(d)}K_2^-(u)R_{12}(2u-\alpha)K_1^-(u-\alpha)R_{21}(-\alpha). \end{aligned} \tag{5}$$

Comparing the right hand sides of (4) and (5), we obtain

$$\begin{aligned} P_{12}^{(d)}K_2^-(u)R_{12}(2u-\alpha)K_1^-(u-\alpha)P_{21}^{(d)} &= K_2^-(u)R_{12}(2u-\alpha)K_1^-(u-\alpha)P_{21}^{(d)}\\ &\equiv K_{\langle 12\rangle}^-(u). \end{aligned}$$

- $V_1\otimes V_2\Rightarrow V_{\langle 12\rangle}$,
- satisfied reflection equation, does not break the integrability.
- the R-matrix is inserted.

Supersymmetric $su(2|2)$ model

$$R_{12}(u) = \left(\begin{array}{cccc|cccc|cccc|cccc}
u+\eta &&&&&&&&&&&&&&& \\
& u &&& \eta &&&&&&&&&&& \\
&& u &&&&&& \eta &&&&&&& \\
&&& u &&&&&&&&& \eta &&& \\
\hline
& \eta &&& u &&&&&&&&&&& \\
&&&&& u+\eta &&&&&&&&&& \\
&&&&&& u &&& \eta &&&&&& \\
&&&&&&& u &&&&&& \eta && \\
\hline
&& \eta &&&&&& u &&&&&&& \\
&&&&&& \eta &&& u &&&&&& \\
&&&&&&&&&& u-\eta &&&&& \\
&&&&&&&&&&& u &&& -\eta & \\
\hline
&&& \eta &&&&&&&&& u &&& \\
&&&&&&& \eta &&&&&& u && \\
&&&&&&&&&&& -\eta &&& u & \\
&&&&&&&&&&&&&&& u-\eta
\end{array}\right).$$

Grassmann parities: $p(1)=0,\ p(2)=0,\ p(3)=1$ and $p(4)=1$.

Graded direct product:

$$(A\otimes B)^{\alpha\gamma}_{\beta\delta} = (-1)^{[p(\alpha)+p(\beta)]p(\gamma)} A^{\alpha}_{\beta} B^{\gamma}_{\delta}.$$

Monodromy matrix is

$$T_0(u) = R_{01}(u-\theta_1)R_{02}(u-\theta_2)\cdots R_{0N}(u-\theta_N),$$

The transfer matrix $t_p(u)$ of the system is defined as the super partial trace of the monodromy matrix in the auxiliary space

$$t_p(u) = str_0\{T_0(u)\} = \sum_{\alpha=1}^{4}(-1)^{p(\alpha)}[T_0(u)]_{\alpha}^{\alpha}.$$

$$[t_p(u), t_p(v)] = 0.$$

The model Hamiltonian is constructed by

$$H_p = \frac{\partial \ln t_p(u)}{\partial u}|_{u=0,\{\theta_j\}=0}.$$

The R-matrix has two degenerate points. The first one is $u=\eta$,

$$R_{12}(\eta)=2\eta P_{12}^{(8)},$$

where $P_{12}^{(8)}$ is a 8-dimensional supersymmetric projector

$$P_{12}^{(8)}=\sum_{i=1}^{8}|f_i\rangle\langle f_i|,$$

and the corresponding bases are

$$\begin{aligned}
&|f_1\rangle=|11\rangle,\quad |f_2\rangle=\frac{1}{\sqrt{2}}(|12\rangle+|21\rangle),\quad |f_3\rangle=|22\rangle,\\
&|f_4\rangle=\frac{1}{\sqrt{2}}(|34\rangle-|43\rangle),\quad |f_5\rangle=\frac{1}{\sqrt{2}}(|13\rangle+|31\rangle),\\
&|f_6\rangle=\frac{1}{\sqrt{2}}(|14\rangle+|41\rangle),\quad |f_7\rangle=\frac{1}{\sqrt{2}}(|23\rangle+|32\rangle),\quad |f_8\rangle=\frac{1}{\sqrt{2}}(|24\rangle+|42\rangle),\\
&p(1)=\cdots=p(4)=0,\quad p(5)=\cdots=p(8)=1.
\end{aligned}$$

The operator $P_{12}^{(8)}$ projects the original 16-dimensional tensor space $V_1\otimes V_2$ into a new 8-dimensional projected space spanned by $\{|f_i\rangle | i=1,\cdots,8\}$.

Taking the fusion by the operator $P_{12}^{(8)}$, we construct a fused R-matrices

$$R_{\langle 12\rangle 3}(u) = (u+\frac{1}{2}\eta)^{-1}P_{12}^{(8)}R_{23}(u-\frac{1}{2}\eta)R_{13}(u+\frac{1}{2}\eta)P_{12}^{(8)} \equiv R_{\bar{1}3}(u),$$

$$R_{3\langle 21\rangle}(u) = (u+\frac{1}{2}\eta)^{-1}P_{21}^{(8)}R_{32}(u-\frac{1}{2}\eta)R_{31}(u+\frac{1}{2}\eta)P_{21}^{(8)} \equiv R_{3\bar{1}}(u).$$

The fused R-matrix $R_{\bar{1}2}(u)$ is a 32×32 matrix defined in the product spaces $V_{\bar{1}}\otimes V_2$.

At the point of $u=-\frac{3}{2}\eta$, the fused R-matrix $R_{\bar{1}2}(u)$ can also be written as a 20-dimensional supersymmetric projector

$$R_{\bar{1}2}(-\frac{3}{2}\eta) = -3\eta P_{\bar{1}2}^{(20)}.$$

The operator $P_{\bar{1}2}^{(20)}$ projects the 32-dimensional product space $V_{\bar{1}}\otimes V_2$ into a 20-dimensional subspace spanned by $\{|\phi_i\rangle, i=1,\cdots,20\}$.

Taking the fusion by the projector $P_{\bar{1}2}^{(20)}$, we obtain the new fused R-matrices

$$R_{\langle \bar{1}2\rangle 3}(u) = (u-\eta)^{-1}P_{2\bar{1}}^{(20)}R_{\bar{1}3}(u+\frac{1}{2}\eta)R_{23}(u-\eta)P_{2\bar{1}}^{(20)} \equiv R_{\tilde{1}3}(u),$$

$$R_{3\langle 2\bar{1}\rangle}(u) = (u-\eta)^{-1}P_{\bar{1}2}^{(20)}R_{3\bar{1}}(u+\frac{1}{2}\eta)R_{32}(u-\eta)P_{\bar{1}2}^{(20)} \equiv R_{3\tilde{1}}(u).$$

The second degenerate point of R-matrix is $u = -\eta$,

$$R_{12}(-\eta) = -2\eta \bar{P}^{(8)}_{12} = -2\eta(1 - P^{(8)}_{12}).$$

Taking the fusion by the projector $\bar{P}^{(8)}_{12}$, we obtain the fused R-matrices

$$R_{\langle 12\rangle' 3}(u) = (u - \frac{1}{2}\eta)^{-1} \bar{P}^{(8)}_{12} R_{23}(u + \frac{1}{2}\eta) R_{13}(u - \frac{1}{2}\eta) \bar{P}^{(8)}_{12} \equiv R_{\bar{1}'3}(u),$$

$$R_{3\langle 21\rangle'}(u) = (u - \frac{1}{2}\eta)^{-1} \bar{P}^{(8)}_{21} R_{32}(u + \frac{1}{2}\eta) R_{31}(u - \frac{1}{2}\eta) \bar{P}^{(8)}_{21} \equiv R_{3\bar{1}'}(u).$$

At the point $u = \frac{3}{2}\eta$, we have

$$R_{\bar{1}'2}(\frac{3}{2}\eta) = 3\eta P^{(20)}_{\bar{1}'2}.$$

Taking the fusion by the projector $P^{(20)}_{\bar{1}'2}$, we obtain

$$R_{\langle \bar{1}'2\rangle 3}(u) = (u + \eta)^{-1} P^{(20)}_{2\bar{1}'} R_{\bar{1}'3}(u - \frac{1}{2}\eta) R_{23}(u + \eta) P^{(20)}_{2\bar{1}'} \equiv R_{\tilde{1}'3}(u),$$

$$R_{3\langle 2\bar{1}'\rangle}(u) = (u + \eta)^{-1} P^{(20)}_{\bar{1}'2} R_{3\bar{1}'}(u - \frac{1}{2}\eta) R_{32}(u + \eta) P^{(20)}_{\bar{1}'2} \equiv R_{3\tilde{1}'}(u).$$

Then we find

$$R_{\tilde{1}2}(u) = R_{\tilde{1}'2}(u),$$

which will help us to close the recursive fusion processes.

Now, we are ready to extend the fusion from one site to the whole system.

From above fused R-matrices, we construct the fused monodromy matrices as

$$T_{\bar{0}}(u) = R_{\bar{0}1}(u-\theta_1)R_{\bar{0}2}(u-\theta_2)\cdots R_{\bar{0}N}(u-\theta_N),$$

$$T_{\bar{0}'}(u) = R_{\bar{0}'1}(u-\theta_1)R_{\bar{0}'2}(u-\theta_2)\cdots R_{\bar{0}'N}(u-\theta_N),$$

$$T_{\tilde{0}}(u) = R_{\tilde{0}1}(u-\theta_1)R_{\tilde{0}2}(u-\theta_2)\cdots R_{\tilde{0}N}(u-\theta_N),$$

$$T_{\tilde{0}'}(u) = R_{\tilde{0}'1}(u-\theta_1)R_{\tilde{0}'2}(u-\theta_2)\cdots R_{\tilde{0}'N}(u-\theta_N),$$

where the subscripts $\bar{0}$, $\bar{0}'$, $\tilde{0}$ and $\tilde{0}'$ mean the auxiliary spaces, and the quantum spaces in all the monodromy matrices are the same.

The fused transfer matrices are

$$t_p^{(1)}(u) = str_{\bar{0}} T_{\bar{0}}(u), \quad t_p^{(2)}(u) = str_{\bar{0}'} T_{\bar{0}'}(u),$$

$$\tilde{t}_p^{(1)}(u) = str_{\tilde{0}} T_{\tilde{0}}(u), \quad \tilde{t}_p^{(2)}(u) = str_{\tilde{0}'} T_{\tilde{0}'}(u).$$

These fused transfer matrices with certain spectral difference must satisfy some intrinsic relations. We first consider the quantity

$$
\begin{aligned}
t_p(u)t_p(u+\eta) &= str_{12}\{T_1(u)T_2(u+\eta)\} \\
&= str_{12}\{(P_{12}^{(8)}+\bar{P}_{12}^{(8)})T_1(u)T_2(u+\eta)(P_{12}^{(8)}+\bar{P}_{12}^{(8)})\} \\
&= str_{12}\{P_{12}^{(8)}T_1(u)T_2(u+\eta)P_{12}^{(8)}\}+str_{12}\{\bar{P}_{12}^{(8)}\bar{P}_{12}^{(8)}T_1(u)T_2(u+\eta)\bar{P}_{12}^{(8)}\} \\
&= str_{12}\{P_{12}^{(8)}T_1(u)T_2(u+\eta)P_{12}^{(8)}\}+str_{12}\{\bar{P}_{12}^{(8)}T_2(u+\eta)T_1(u)\bar{P}_{12}^{(8)}\bar{P}_{12}^{(8)}\} \\
&= \prod_{j=1}^{N}(u-\theta_j+\eta)t_p^{(1)}(u+\frac{1}{2}\eta)+\prod_{j=1}^{N}(u-\theta_j)t_p^{(2)}(u+\frac{1}{2}\eta).
\end{aligned}
$$

$$t_p^{(1)}(u+\frac{1}{2}\eta)t_p(u-\eta) = str_{\bar{1}2}\{(P_{2\bar{1}}^{(20)}+\tilde{P}_{2\bar{1}}^{(12)})T_{\bar{1}}(u+\frac{1}{2}\eta)T_2(u-\eta)(P_{2\bar{1}}^{(20)}+\tilde{P}_{2\bar{1}}^{(12)})\}$$

$$= str_{\bar{1}2}\{P_{2\bar{1}}^{(20)}T_{\bar{1}}(u+\frac{1}{2}\eta)T_2(u-\eta)P_{2\bar{1}}^{(20)}\} + str_{\bar{1}2}\{\tilde{P}_{2\bar{1}}^{(12)}T_{\bar{1}}(u+\frac{1}{2}\eta)T_2(u-\eta)\tilde{P}_{2\bar{1}}^{(12)}\}$$

$$= \prod_{j=1}^{N}(u-\theta_j-\eta)\tilde{t}_p^{(1)}(u) + \prod_{j=1}^{N}(u-\theta_j)\bar{t}_p^{(1)}(u),$$

$$t_p^{(2)}(u-\frac{1}{2}\eta)t_p(u+\eta) = str_{\bar{1}'2}\{(P_{2\bar{1}'}^{(20)}+\tilde{P}_{2\bar{1}'}^{(12)})T_{\bar{1}'}(u-\frac{1}{2}\eta)T_2(u+\eta)(P_{2\bar{1}'}^{(20)}+\tilde{P}_{2\bar{1}'}^{(12)})\}$$

$$= str_{\bar{1}'2}\{P_{2\bar{1}'}^{(20)}T_{\bar{1}'}(u-\frac{1}{2}\eta)T_2(u+\eta)P_{2\bar{1}'}^{(20)}\} + str_{\bar{1}'2}\{\tilde{P}_{2\bar{1}'}^{(12)}T_{\bar{1}'}(u-\frac{1}{2}\eta)T_2(u+\eta)\tilde{P}_{2\bar{1}'}^{(12)}\}$$

$$= \prod_{j=1}^{N}(u-\theta_j+\eta)\tilde{t}_p^{(2)}(u) + \prod_{j=1}^{N}(u-\theta_j)\bar{t}_p^{(2)}(u).$$

During the derivation, we have used the relations

$$P_{2\bar{1}}^{(20)}+\tilde{P}_{2\bar{1}}^{(12)}=1,\;\; P_{2\bar{1}}^{(20)}\tilde{P}_{2\bar{1}}^{(12)}=0,\;\; P_{2\bar{1}'}^{(20)}+\tilde{P}_{2\bar{1}'}^{(12)}=1,\;\; P_{2\bar{1}'}^{(20)}\tilde{P}_{2\bar{1}'}^{(12)}=0.$$

Then we arrive at the closed operator product identities among the transfer matrices

$$t_p(\theta_j)t_p(\theta_j+\eta)=\prod_{l=1}^{N}(\theta_j-\theta_l+\eta)t_p^{(1)}(\theta_j+\frac{1}{2}\eta),$$
$$t_p(\theta_j)t_p(\theta_j-\eta)=\prod_{l=1}^{N}(\theta_j-\theta_l-\eta)t_p^{(2)}(\theta_j-\frac{1}{2}\eta),$$
$$t_p^{(1)}(\theta_j+\frac{1}{2}\eta)t_p(\theta_j-\eta)=\prod_{l=1}^{N}\frac{\theta_j-\theta_l-\eta}{\theta_j-\theta_l+\eta}t_p^{(2)}(\theta_j-\frac{1}{2}\eta)t_p(\theta_j+\eta).$$

Above equations constitute the closed recursive fusion relations. From the definitions, we know that the transfer matrices $t_p(u)$, $t_p^{(1)}(u)$ and $t_p^{(2)}(u)$ are the operator polynomials of u with degree $N-1$. Then, above $3N$ conditions are sufficient to solve them.

From graded Yang-Baxter equations, the transfer matrices $t_p(u)$, $t_p^{(1)}(u)$ and $t_p^{(2)}(u)$ commutate with each other, namely,

$$[t_p(u), t_p^{(1)}(u)] = [t_p(u), t_p^{(2)}(u)] = [t_p^{(1)}(u), t_p^{(2)}(u)] = 0.$$

Therefore, they have common eigenstates and can be diagonalized simultaneously. Functional relations among these eigenvalues

$$\Lambda_p(\theta_j)\Lambda_p(\theta_j+\eta) = \prod_{l=1}^{N}(\theta_j-\theta_l+\eta)\Lambda_p^{(1)}(\theta_j+\frac{1}{2}\eta),$$

$$\Lambda_p(\theta_j)\Lambda_p(\theta_j-\eta) = \prod_{l=1}^{N}(\theta_j-\theta_l-\eta)\Lambda_p^{(2)}(\theta_j-\frac{1}{2}\eta),$$

$$\Lambda_p^{(1)}(\theta_j+\frac{1}{2}\eta)\Lambda_p(\theta_j-\eta) = \prod_{l=1}^{N}\frac{\theta_j-\theta_l-\eta}{\theta_j-\theta_l+\eta}\Lambda_p^{(2)}(\theta_j-\frac{1}{2}\eta)\Lambda_p(\theta_j+\eta),$$

where $j = 1, 2, \cdots N$. Because the eigenvalues $\Lambda_p(u)$, $\Lambda_p^{(1)}(u)$ and $\Lambda_p^{(2)}(u)$ are the polynomials of u with degree $N-1$, the above $3N$ conditions can determine these eigenvalues completely.

$T-Q$ relations

$$\Lambda_p(u)=\sum_{l=1}^{4} z_p^{(l)}(u),$$

$$\Lambda_p^{(1)}(u)=\left[Q_p^{(0)}(u+\frac{1}{2}\eta)\right]^{-1}\Big\{\sum_{l=1}^{2} z_p^{(l)}(u+\frac{1}{2}\eta)z_p^{(l)}(u-\frac{1}{2}\eta)+\sum_{l=2}^{4}\sum_{m=1}^{l-1} z_p^{(l)}(u+\frac{1}{2}\eta)z_p^{(m)}(u-\frac{1}{2}\eta)\Big\},$$

$$\Lambda_p^{(2)}(u)=\left[Q_p^{(0)}(u-\frac{1}{2}\eta)\right]^{-1}\Big\{\sum_{l=3}^{4} z_p^{(l)}(u+\frac{1}{2}\eta)z_p^{(l)}(u-\frac{1}{2}\eta)+\sum_{l=2}^{4}\sum_{m=1}^{l-1} z_p^{(l)}(u-\frac{1}{2}\eta)z_p^{(m)}(u+\frac{1}{2}\eta)\Big\}.$$

Here

$$z_p^{(l)}(u)=\begin{cases}(-1)^{p(l)}Q_p^{(0)}(u)\dfrac{Q_p^{(l-1)}(u+\eta)Q_p^{(l)}(u-\eta)}{Q_p^{(l)}(u)Q_p^{(l-1)}(u)}, & l=1,2,\\ (-1)^{p(l)}Q_p^{(0)}(u)\dfrac{Q_p^{(l-1)}(u-\eta)Q_p^{(l)}(u+\eta)}{Q_p^{(l)}(u)Q_p^{(l-1)}(u)}, & l=3,4,\end{cases}$$

$$Q_p^{(0)}(u)=\prod_{l=1}^{N}(u-\theta_j),\qquad Q_p^{(m)}(u)=\prod_{j=1}^{L_m}(u-\lambda_j^{(m)}),\qquad m=1,2,3,\qquad Q_p^{(4)}(u)=1,$$

Bethe ansatz equations.

Open boundary condition

The general solution of reflection matrix $K_0^-(u)$ is

$$K_0^-(u) = \xi + uM, \quad M = \begin{pmatrix} 1 & c_1 & 0 & 0 \\ c_2 & -1 & 0 & 0 \\ 0 & 0 & -1 & c_3 \\ 0 & 0 & c_4 & 1 \end{pmatrix},$$

and the dual reflection matrix $K^+(u)$ can be obtained by the mapping

$$K_0^+(u) = K_0^-(-u)|_{\xi, c_i \to \tilde{\xi}, \tilde{c}_i},$$

where the ξ, $\tilde{\xi}$ and $\{c_i, \tilde{c}_i | i = 1, \cdots, 4\}$ are the boundary parameters which describe the boundary interactions, and the integrability requires

$$c_1 c_2 = c_3 c_4, \quad \tilde{c}_1 \tilde{c}_2 = \tilde{c}_3 \tilde{c}_4.$$

$[K^-(u), K^+(v)] \neq 0$, which means that they cannot be diagonalized simultaneously.

The two 8-dimensional fusion of the super projectors gives

$$
\begin{aligned}
K^-_{\bar{1}}(u) &= (u + \frac{1}{2}\eta)^{-1} P^{(8)}_{21} K^-_1(u - \frac{1}{2}\eta) R_{21}(2u) K^-_2(u + \frac{1}{2}\eta) P^{(8)}_{12}, \\
K^+_{\bar{1}}(u) &= (u - \frac{1}{2}\eta)^{-1} P^{(8)}_{12} K^+_2(u + \frac{1}{2}\eta) R_{12}(-2u) K^+_1(u - \frac{1}{2}\eta) P^{(8)}_{21}, \\
K^-_{\bar{1}'}(u) &= (u - \frac{1}{2}\eta)^{-1} \bar{P}^{(8)}_{21} K^-_1(u + \frac{1}{2}\eta) R_{21}(2u) K^-_2(u - \frac{1}{2}\eta) \bar{P}^{(8)}_{12}, \\
K^+_{\bar{1}'}(u) &= (u + \frac{1}{2}\eta)^{-1} \bar{P}^{(8)}_{12} K^+_2(u - \frac{1}{2}\eta) R_{12}(-2u) K^+_1(u + \frac{1}{2}\eta) \bar{P}^{(8)}_{21}.
\end{aligned}
$$

- All the fused K-matrices are the 8×8 ones.
- Their matric elements are the polynomials of u with maximum degree two.

The reflection matrices $K_{\bar{1}}^{\pm}(u)$ [or $K_{\bar{1}'}^{\pm}(u)$] and $K_{2}^{\pm}(u)$ can be fused by the the 20-dimensional projector $P_{\bar{1}2}^{(20)}$ or $P_{\bar{1}'2}^{(20)}$, and we have

$$K_{\tilde{1}}^{-}(u)=(u-\eta)^{-1}P_{2\bar{1}}^{(20)}K_{\bar{1}}^{-}(u+\frac{1}{2}\eta)R_{2\bar{1}}(2u-\frac{1}{2}\eta)K_{2}^{-}(u-\eta)P_{\bar{1}2}^{(20)},$$

$$K_{\tilde{1}}^{+}(u)=(2u+\eta)^{-1}P_{\bar{1}2}^{(20)}K_{2}^{+}(u-\eta)R_{\bar{1}2}(-2u+\frac{1}{2}\eta)K_{\bar{1}}^{+}(u+\frac{1}{2}\eta)P_{2\bar{1}}^{(20)},$$

$$K_{\tilde{1}'}^{-}(u)=(u+\eta)^{-1}P_{2\bar{1}'}^{(20)}K_{\bar{1}'}^{-}(u-\frac{1}{2}\eta)R_{2\bar{1}'}(2u+\frac{1}{2}\eta)K_{2}^{-}(u+\eta)P_{\bar{1}'2}^{(20)},$$

$$K_{\tilde{1}'}^{+}(u)=(2u-\eta)^{-1}P_{\bar{1}'2}^{(20)}K_{2}^{+}(u+\eta)R_{\bar{1}'2}(-2u-\frac{1}{2}\eta)K_{\bar{1}'}^{+}(u-\frac{1}{2}\eta)P_{2\bar{1}'}^{(20)}.$$

- All the fused K-matrices are the 20×20 ones.
- Their matric elements are the polynomials of u with maximum degree three.

The transfer matrices are defined as

$$t(u) = str_0\{K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\},$$

$$t^{(1)}(u) = str_{\bar{0}}\{K_{\bar{0}}^+(u)T_{\bar{0}}(u)K_{\bar{0}}^-(u)\hat{T}_{\bar{0}}(u)\},$$

$$t^{(2)}(u) = str_{\bar{0}'}\{K_{\bar{0}'}^+(u)T_{\bar{0}'}(u)K_{\bar{0}'}^-(u)\hat{T}_{\bar{0}'}(u)\}.$$

where

$$\hat{T}_0(u) = R_{N0}(u+\theta_N)\cdots R_{20}(u+\theta_2)R_{10}(u+\theta_1),$$

$$\hat{T}_{\bar{0}}(u) = R_{N\bar{0}}(u+\theta_N)\cdots R_{2\bar{0}}(u+\theta_2)R_{1\bar{0}}(u+\theta_1),$$

$$\hat{T}_{\bar{0}'}(u) = R_{N\bar{0}'}(u+\theta_N)\cdots R_{2\bar{0}'}(u+\theta_2)R_{1\bar{0}'}(u+\theta_1).$$

The transfer matrices $t(u)$, $t^{(1)}(u)$ and $t^{(2)}(u)$ commutate with each other

$$[t(u), t^{(1)}(u)] = [t(u), t^{(2)}(u)] = [t^{(1)}(u), t^{(2)}(u)] = 0.$$

Therefore, they have common eigenstates and can be diagonalized simultaneously.

Using the method we have used in the periodic case, we can obtain the operator product identities among the fused transfer matrices as

$$
\begin{aligned}
t(\pm\theta_j)t(\pm\theta_j+\eta) &= -\frac{1}{4}\frac{(\pm\theta_j)(\pm\theta_j+\eta)}{(\pm\theta_j+\frac{1}{2}\eta)^2} \\
&\quad\times\prod_{l=1}^{N}(\pm\theta_j-\theta_l+\eta)(\pm\theta_j+\theta_l+\eta)t^{(1)}(\pm\theta_j+\frac{1}{2}\eta), \\
t(\pm\theta_j)t(\pm\theta_j-\eta) &= -\frac{1}{4}\frac{(\pm\theta_j)(\pm\theta_j-\eta)}{(\pm\theta_j-\frac{1}{2}\eta)^2} \\
&\quad\times\prod_{l=1}^{N}(\pm\theta_j-\theta_l-\eta)(\pm\theta_j+\theta_l-\eta)t^{(2)}(\pm\theta_j-\frac{1}{2}\eta), \\
t(\pm\theta_j-\eta)t^{(1)}(\pm\theta_j+\frac{1}{2}\eta) &= \frac{(\pm\theta_j+\frac{1}{2}\eta)^2(\pm\theta_j-\eta)}{(\pm\theta_j+\eta)(\pm\theta_j-\frac{1}{2}\eta)^2} \\
&\quad\times\prod_{l=1}^{N}\frac{(\pm\theta_j-\theta_l-\eta)(\pm\theta_j+\theta_l-\eta)}{(\pm\theta_j-\theta_l+\eta)(\pm\theta_j+\theta_l+\eta)}t(\pm\theta_j+\eta)t^{(2)}(\pm\theta_j-\frac{1}{2}\eta).
\end{aligned}
$$

From the definitions, we know that the transfer matrix $t(u)$ is a operator polynomial of u with degree $2N+2$ while the fused ones $t^{(1)}(u)$ and $t^{(2)}(u)$ are the operator polynomials of u both with degree $2N+4$. Thus they can be completely determined by $6N+13$ independent conditions. The above recursive fusion relations gives $6N$ constraints and we still need 13 ones, which can be achieved by analyzing the values of transfer matrices at some special points.

Let $|\Phi\rangle$ be a common eigenstate. Acting the transfer matrices on this eigenstate, we have

$$t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle,$$

$$t^{(1)}(u)|\Psi\rangle = \Lambda^{(1)}(u)|\Psi\rangle,$$

$$t^{(2)}(u)|\Psi\rangle = \Lambda^{(2)}(u)|\Psi\rangle.$$

The inhomogeneous T-Q relations

$$\Lambda(u)=\sum_{l=1}^{4}z^{(l)}(u)+x_1(u)+x_2(u),$$

$$\Lambda^{(1)}(u)=-4u^2[Q^{(0)}(u+\frac{1}{2}\eta)(u+\frac{1}{2}\eta)(u-\frac{1}{2}\eta)]^{-1}\Big\{\sum_{l=1}^{4}\sum_{m=1}^{2}\tilde{z}^{(l)}(u+\frac{1}{2}\eta)\tilde{z}^{(m)}(u-\frac{1}{2}\eta)$$

$$-z^{(1)}(u+\frac{1}{2}\eta)z^{(2)}(u-\frac{1}{2}\eta)+z^{(4)}(u+\frac{1}{2}\eta)z^{(3)}(u-\frac{1}{2}\eta)\Big\},$$

$$\Lambda^{(2)}(u)=-4u^2[Q^{(0)}(u-\frac{1}{2}\eta)(u+\frac{1}{2}\eta)(u-\frac{1}{2}\eta)]^{-1}\Big\{\sum_{l=1}^{4}\sum_{m=3}^{4}\tilde{z}^{(l)}(u+\frac{1}{2}\eta)\tilde{z}^{(m)}(u-\frac{1}{2}\eta)$$

$$+z^{(1)}(u+\frac{1}{2}\eta)z^{(2)}(u-\frac{1}{2}\eta)-z^{(4)}(u+\frac{1}{2}\eta)z^{(2)}(u-\frac{1}{2}\eta)\Big\},$$

where

$$\tilde{z}^{(1)}(u)=z^{(1)}(u)+x_1(u),\ \tilde{z}^{(2)}(u)=z^{(2)}(u),\ \tilde{z}^{(3)}(u)=z^{(3)}(u),\ \tilde{z}^{(4)}(u)=z^{(4)}(u)+x_2(u).$$

$$z^{(l)}(u) = \begin{cases} (-1)^{p(l)}\alpha_l(u)Q^{(0)}(u)K^{(l)}(u)\dfrac{Q^{(l-1)}(u+\eta)Q^{(l)}(u-\eta)}{Q^{(l)}(u)Q^{(l-1)}(u)}, & l = 1, 2, \\ \\ (-1)^{p(l)}\alpha_l(u)Q^{(0)}(u)K^{(l)}(u)\dfrac{Q^{(l-1)}(u-\eta)Q^{(l)}(u+\eta)}{Q^{(l)}(u)Q^{(l-1)}(u)}, & l = 3, 4, \end{cases}$$

$$x_1(u) = u^2 Q^{(0)}(u+\eta)Q^{(0)}(u)\frac{f^{(1)}(u)Q^{(2)}(-u-\eta)}{Q^{(1)}(u)},$$

$$x_2(u) = u^2 Q^{(0)}(u+\eta)Q^{(0)}(u)Q^{(0)}(-u)\frac{f^{(2)}(u)Q^{(2)}(-u-\eta)}{Q^{(3)}(u)},$$

$$Q^{(0)}(u) = \prod_{l=1}^{N}(u-\theta_l)(u+\theta_l), \quad Q^{(m)}(u) = \prod_{j=1}^{L_m}(u-\lambda_j^{(m)})(u+\lambda_j^{(m)}+m\eta), \quad m = 1, 2,$$

$$Q^{(3)}(u) = \prod_{j=1}^{L_3}(u-\lambda_j^{(m)})(u+\lambda_j^{(m)}+\eta), \quad Q^{(4)}(u) = 1,$$

Bethe ansatz equations.

Concluding Remarks and Perspective

- Open boundary problems
- Quantum Toda
- Chiral Potts
- High rank systems
- Thermodynamics

To be continued!