Area law and OPE blocks in CFT

Jiang Long

Huazhong University of Science and Technology (**HUST**)

based on arXiv: **2007.15380, 2001.05129, 1911.11487, 1907.00646**

USTC, Hefei

Nov 28, 2020

Area law in physics

- Area law
  - keypoint to understand gravitational physics & holography.
  - relates geometry to physics
- Diverse area laws in physics
  - Black hole physics: Bekenstein, Hawking, 1970'
    S.N.Solodukhin, R.Kaul, P.Majumdar, S.Carlip, A.Sen, etc

$$S_{BH}=\frac{A}{4G_N}+C\log A+\cdots . \tag{1.1}$$

  - Geometric entanglement entropy:
    Bombelli,Koul,Lee,Sorkin,Srednicki,Callan,Wilczek,etc, 1990'

$$S_{EE}=\gamma\frac{A}{\epsilon^{d-2}}+\cdots+p\log\frac{R}{\epsilon}+\cdots \tag{1.2}$$

  - Holographic description of entanglement entropy:
    Ryu & Takayanagi 2006

$$S_{RT}=\frac{A}{4G_N}+\text{qc}. \tag{1.3}$$

Q: is there any other area law in physics?

- The similarity of the area laws

$$\text{Area} \sim \text{Entropy}. \tag{1.4}$$

- New area laws

$$\text{Area} \sim \text{CCF} \tag{1.5}$$

  - Area law of entanglement entropy becomes a limit of the new area law.

Area law and modular Hamiltonian

- Entanglement entropy and modular Hamiltonian

$$S_A^{(n)} = \frac{1}{1-n}\log \mathrm{tr}_A \rho_A^n = \frac{1}{1-n}\log \mathrm{tr}_A e^{-nH_A}. \tag{1.6}$$

- Modular Hamiltonian $H_A = -\log \rho_A$.
  - Non-local operator in A.
  - Half plane J.Bisognano & E.Wichmann, 1976

$$H_A = 2\pi \int_{x>0} dx d^{d-2}\vec{y}\; x\; T_{00}. \tag{1.7}$$

  - Spherical region Σ_A (CFT) Casini, Huerta & Myers,1102.0440

$$H_A = 2\pi \int_{\Sigma_A} d^{d-1}\vec{x} \frac{R^2-(\vec{x}-\vec{x}_0)^2}{2R} T_{00}. \tag{1.8}$$

  - AdS gravity with a CFT dual Jafferis, etc, 1512.06431

$$H_A = \frac{A}{4G_N} + \mathcal{O}(G_N^0) \tag{1.9}$$

- Modular Hamiltonian is a special OPE block for spherical region.

Definition

- Primary operators $\mathcal{O}$ with quantum number Δ, J.
- Operator product expansion (OPE)
  - two scalar primary operators

$$\begin{aligned} \mathcal{O}_i(x_1)\mathcal{O}_j(x_2) &= \sum_k C_{ijk}|x_{12}|^{\Delta_k-\Delta_i-\Delta_j}(\mathcal{O}_k(x_2)+\cdots) \\ &= |x_1-x_2|^{-\Delta_i-\Delta_j}\sum_k C_{ijk}Q_k^{ij}(x_1,x_2) \end{aligned} \quad (2.1)$$

  - OPE block: $Q_k^{ij}(x_1, x_2)$. Czech, etc, 1604.03110
    - depends on the external operators
    - depends on the insertion points
    - dimension zero & non-local operator
    - special case, $i = j$, it is independent of the external operators.

$$Q_A[\mathcal{O}_k] = Q_k^{ii}(x_1, x_2). \quad (2.2)$$

    - $A \leftrightarrow (x_1, x_2)$

Timelike pair and causal diamond

- A timelike pair is in one-to-one correspondence to a causal diamond.

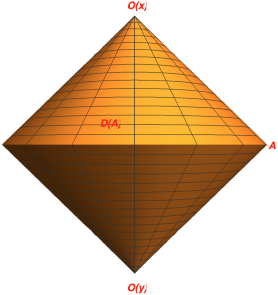


Figure: A timelike pair and causal diamond

- Diamond $\mathcal{O}(A)$ is invariant under the action of conformal Killing vector

$$K^{\mu} = \frac{1}{2R}(R^2 - (\vec{x} - \vec{x}_0)^2, -2t(\vec{x} - \vec{x}_0)) \tag{2.3}$$

OPE block

- OPE block with equal external primary operator
  $\Delta_1 = \Delta_2,\ J_1 = J_2 = 0$

de Boer, etc, 1606.03307

- In general, $\partial \cdot \mathcal{O} \neq 0 \to$ Type-O OPE block

$$Q_A[\mathcal{O}_{\mu_1 \cdots \mu_J}] = \int_A d^d x K^{\mu_1} \cdots K^{\mu_J} |K|^{\Delta - d - J} \mathcal{O}_{\mu_1 \cdots \mu_J}, \tag{2.4}$$

- conserved current $\partial \cdot \mathcal{J} = 0 \to$ Type-J OPE block

$$Q_A[\mathcal{J}_{\mu_1 \cdots \mu_J}] = \int_{\Sigma_A} d^{d-1}\vec{x} (K^0)^{J-1} \mathcal{J}_{0 \cdots 0}. \tag{2.5}$$

  - Modular Hamiltonian is a special Type-J OPE block for spherical region
  - generated by stress tensor, $\partial_\mu T^{\mu\nu} = 0$.

Deformed reduced density matrix

- Reduced density matrix is the exponential operator of modular Hamiltonian $\rho_A = e^{-H_A}$.
- Replace modular Hamiltonian by a general OPE block $\to$ deformed reduced density matrix

$$\rho_A = e^{-\mu Q_A}. \tag{3.1}$$

  - Q_A could be a linear superposition of OPE blocks.
  - when $[H_A, Q_A] = 0$, μ could be interpreted as the chemical potential which is dual to Q_A.
  - not always well defined if Q_A has no lower bound
  - a formal generator of connected correlation function (CCF)

$$T_A(\mu) = \log\langle e^{-\mu Q_A}\rangle \tag{3.2}$$

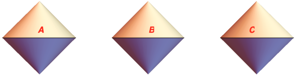
  - (m)-type CCF

$$\langle Q_A^m\rangle_c = (-1)^m \partial_\mu^m T_A(\mu)|_{\mu=0}. \tag{3.3}$$

Connected correlation function (CCF)

- several spacelike separated regions $A, B, C, \cdots$
- m_1 OPE blocks in A, m_2 OPE blocks in B, etc.
- Y-type CCF ,$Y = (m_1, m_2, \cdots, m_n)$, $m_1 \geq m_2 \geq \cdots \geq m_n \geq 1$.

$$\langle Q_A^{m_1} Q_B^{m_2} \cdots \rangle_c \tag{3.4}$$



- conformal symmetry constrains $(m, 1)$-type CCF

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_m] Q_B[\mathcal{O}] \rangle_c = D[\mathcal{O}_1, \cdots, \mathcal{O}_m, \mathcal{O}] G_{\Delta,J}(z). \tag{3.5}$$

  - $G_{\Delta,J}$ is the conformal block associated with primary operator $\mathcal{O}$.
  - z denotes the cross ratio related to two diamonds A and B.

(m)-type CCF of modular Hamiltonian (I)

- Rényi entanglement entropy is the generator of (m)-type CCF for modular Hamiltonian.

$$\langle H_A^m \rangle_c = (-1)^m \partial_n^m (1-n) S_A^{(n)}|_{n\to 1} \tag{3.6}$$

- Area law of Rényi EE

$$S_A^{(n)} = \gamma(n) \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + p(n) \log \frac{R}{\epsilon} + \cdots . \tag{3.7}$$

- (m)-type CCF of modular Hamiltonian should also obey area law

$$\langle H_A^m \rangle_c = \tilde{\gamma} \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + \tilde{p} \log \frac{R}{\epsilon} + \cdots . \tag{3.8}$$

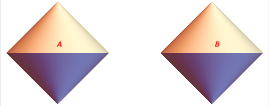
(m)-type CCF of modular Hamiltonian (II)

- An argument without reference to Rényi EE
- $(m-1,1)$-type CCF is always conformal block, especially for modular Hamiltonian

$$\langle H_A^{m-1} H_B \rangle_c = D[T_{\mu_1\nu_1}, \cdots, T_{\mu_m\nu_m}] G_{d,2}(z). \tag{3.9}$$

- Choose A and B as follows

$$A = \{(0,\vec{x}) | (\vec{x} - \vec{x}_A)^2 \leq R^2\}, \quad B = \{(0,\vec{x}) | \vec{x}^2 \leq R'^2\} \tag{3.10}$$



  - A and B are spacelike separated, the cross ratio $0 < z < 1$.

$$z = \frac{4RR'}{x_A^2 - (R - R')^2} \tag{3.11}$$

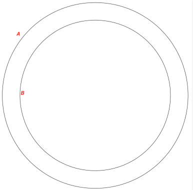
(m)-type CCF of modular Hamiltonian (III)

- consider the limit $B \to A$,

$$\langle H_A^{m-1} H_B \rangle_c \to \langle H_A^m \rangle_c, \quad B \to A \tag{3.12}$$

- need a way to continue conformal block
- $x_A = 0$, then $R' \to R$ through

$$R' = R - \epsilon, \quad z = \frac{4R(R-\epsilon)}{-\epsilon^2} \sim -\frac{R^2}{\epsilon^2} \to -\infty \tag{3.13}$$



- continue conformal block $G_{d,2}(z)$ to the region $z \to -\infty$

(m)-type CCF of modular Hamiltonian (IV)

- area law from the continuation of conformal block

$$\langle H_A^m \rangle_c = \lim_{B\to A} \langle H_A^{m-1} H_B \rangle_c = \lim_{z\to -\frac{R^2}{\epsilon^2}} D[T_{\mu_1\nu_1}, \cdots, T_{\mu_m\nu_m}] G_{d,2}(z)$$
$$= \gamma \frac{R^2}{\epsilon^2} + \cdots + p_1^e \log \frac{R}{\epsilon} + \cdots, \quad d = 4. \tag{3.14}$$

with

$$p_1^e = -120 D. \tag{3.15}$$

- we obtain area law from continuation of $(m-1, 1)$-type CCF.
- D: leading behaviour when A and B are far away (IR)
- p_1^e: cutoff independent coefficient when A and B are the same (UV).
- -120 is from the continuation of conformal block which is fixed by conformal symmetry.
- $p \sim E \times D$, a typical UV/IR relation

(m)-type CCF of OPE block (I)

- Modular Hamiltonian H_A is a type-J OPE block for spherical region
- (m)-type CCF of modular Hamiltonian obeys area law
- More general (m)-type CCF

$$\langle Q_A[\mathcal{O}_1]\cdots Q_A[\mathcal{O}_m]\rangle_c \tag{4.1}$$

where $Q_A[\mathcal{O}_i]$ belong to the same type of OPE block.

- Consider $(m-1,1)$-type CCF

$$\langle Q_A[\mathcal{O}_1]\cdots Q_A[\mathcal{O}_{m-1}]Q_B[\mathcal{O}_m]\rangle_c = D[\mathcal{O}_1,\cdots,\mathcal{O}_m]G_{\Delta_m,J_m}(z). \tag{4.2}$$

- Continuation for general conformal block

(m)-type CCF of OPE block (II)

- We obtain the following behaviour

$$\langle Q_A[\mathcal{O}_1]\cdots Q_A[\mathcal{O}_m]\rangle_c = \gamma\frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + p_q[\mathcal{O}_1,\cdots,\mathcal{O}_m]\log^q\frac{R}{\epsilon} + \cdots \quad (4.3)$$

- If the coefficient D is finite,
  - Leading term obeys area law
  - q:maximal power of the logarithmic term, degree of the (m)-type CCF.

$$q = \begin{cases} 1, & \text{type-J \& d even.} \\ 2, & \text{type-O \& d even.} \\ 0, & \text{type-J \& d odd.} \\ 1, & \text{type-O \& d odd.} \end{cases} \quad (4.4)$$

  - q is fixed by the conformal block associated with $\mathcal{O}_m$.
  - coefficient p_q is cutoff independent.
  - UV/IR relation

$$p_q[\mathcal{O}_1,\cdots,\mathcal{O}_m] = E[\mathcal{O}_m]D[\mathcal{O}_1,\cdots,\mathcal{O}_m] \quad (4.5)$$

UV/IR relation

- $p = E \times D$.
- D: leading behaviour of $(m-1,1)$-type CCF when A and B are far apart (IR aspect)
- p: cutoff independent coefficient in the subleading term of (m)-type CCF, A and B should be the same (UV aspect)
- E: encodes kinematic information
  - E can be obtained from analytic continuation of conformal block. For example, for type-J OPE block in four dimensions

$$E[\mathcal{O}] = \begin{cases} 12, & \Delta = 3, J = 1. \\ -120, & \Delta = 4, J = 2. \\ \dots \end{cases} \tag{4.6}$$

UV/IR relation: $p = E \times D$

- For type-O OPE block in four dimensions

$$E[\mathcal{O}] = \begin{cases} -\frac{2^{2\Delta-1}\Gamma(\frac{\Delta-1}{2})\Gamma(\frac{\Delta+1}{2})}{\pi\Gamma(\frac{\Delta-2}{2})^2}, & \Delta > 1, \quad J = 0. \\ \frac{2^{2\Delta-1}\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta+2}{2})}{\pi\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta+1}{2})}, & \Delta > 3, \quad J = 1. \\ -\frac{4^{\Delta-1}(\Delta-2)\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta+3}{2})}{\pi\Gamma(\frac{\Delta-4}{2})\Gamma(\frac{\Delta+2}{2})}, & \Delta > 4, \quad J = 2. \\ \dots \end{cases} \tag{4.7}$$

  - Unitary bound of scalar operator in four dimensions $\Delta \geq 1$
    - Note: $\Delta \to 1$, E is divergent.
  - Unitary bound of a non-conserved primary current in four dimensions $\Delta > J + 2$. S.Minwalla, 9712074
    - limiting behaviour: $\Delta \to J + 2$, $E[\mathcal{O}] \to 0$, $p_2 \to 0$ for finite D. The degree $q = 2$ becomes $q = 1$.

UV/IR relation: $p = E \times D$

- The relation is 'asymmetric' since E just depends on operator $\mathcal{O}_m$.

$$p_q[\mathcal{O}_1, \cdots, \mathcal{O}_m] = E[\mathcal{O}_m] D[\mathcal{O}_1, \cdots, \mathcal{O}_m] \tag{4.8}$$

- There should be m different ways to move one OPE block to region B, for example

$$\langle Q_A[\mathcal{O}_2] \cdots Q_A[\mathcal{O}_m] Q_B[\mathcal{O}_1] \rangle_c \tag{4.9}$$

leads to another UV/IR relation

$$p_q[\mathcal{O}_2, \cdots, \mathcal{O}_m, \mathcal{O}_1] = E[\mathcal{O}_1] D[\mathcal{O}_2, \cdots, \mathcal{O}_m, \mathcal{O}_1]. \tag{4.10}$$

- p_q is cutoff independent
- Cyclic identity ($m = 3$ as an exmaple)

$$p_q[\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_1] = p_q[\mathcal{O}_3, \mathcal{O}_1, \mathcal{O}_2] = p_q[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3]. \tag{4.11}$$

- The leading term coefficient γ is cutoff dependent, it doesn't satisfy cyclic identity

$$\gamma[\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_1] \neq \gamma[\mathcal{O}_3, \mathcal{O}_1, \mathcal{O}_2] \neq \gamma[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3]. \tag{4.12}$$

Example 1: Chiral operators in CFT_2 (I)

- In two dimensions, the leading term of (m)-type CCF is already the logarithmic term.
- A chiral operator $\mathcal{O}$ in CFT_2 only depends on (anti-)holomorphic coordinate

$$\bar{\partial}\mathcal{O}(z) = 0 \tag{5.1}$$

- The ball in one spatial dimension is an interval, we assume the length is 2 and the center is 0.
- The corresponding OPE block is type-J.

$$Q_A[\mathcal{O}] = \int_{-1}^{1} dz(\frac{1-z^2}{2})^{h-1}\mathcal{O}(z). \tag{5.2}$$

- degree $q = 1$.

Chiral operators in CFT_2 (II)

- (2)-type, $\surd$; UV/IR relation $\surd$

$$p_1[\mathcal{O},\mathcal{O}]=\frac{(-1)^{-h}\sqrt{\pi}\Gamma(h)}{\Gamma(h+\frac{1}{2})}N_{\mathcal{O}}.. \tag{5.3}$$

- (3)-type, using UV/IR relation

$$p_1[\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3]=\frac{\pi^{3/2}2^{3-h_1-h_2-h_3}(-1)^{\frac{h_1+h_2+h_3}{2}}\Gamma(h_1)\Gamma(h_2)\Gamma(h_3)\kappa C_{123}}{\Gamma(\frac{1+h_1+h_2-h_3}{2})\Gamma(\frac{1+h_1+h_3-h_2}{2})\Gamma(\frac{1+h_2+h_3-h_1}{2})\Gamma(\frac{h_1+h_2+h_3}{2})} \tag{5.4}$$

with $\kappa=\frac{1+(-1)^{h_1+h_2+h_3}}{2}$.

  - $\surd$ for h_i are integers and no larger than 6.
  - cyclic identity $\surd$

$$p_1[\mathcal{O}_2,\mathcal{O}_3,\mathcal{O}_1]=p_1[\mathcal{O}_3,\mathcal{O}_1,\mathcal{O}_2]=p_1[\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3] \tag{5.5}$$

- (4)-type, free scalar theory and theory with $\mathcal{W}$ symmetry $\surd$

Example 2: Non-conserved primary operator in CFT_4 (I)

- For a primary operator which is non-conserved, $Q_A[\mathcal{O}]$ is type-O.
- Degree $q = 2$.
- (2)-type

$$p_2[\mathcal{O},\mathcal{O}] = \begin{cases} -\frac{4\pi^2(\Delta-1)\Gamma(\Delta-2)^2\Gamma(\frac{\Delta}{2})^4}{\Gamma(\Delta)^2\Gamma(\Delta-1)^2} N_{\mathcal{O}}, & J=0, \Delta \geq 1. \\ -\frac{4^{1-\Delta}\pi^3\Delta\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta+1}{2})}{\Gamma(\frac{\Delta}{2}+1)^2} N_{\mathcal{O}}, & J=1, \Delta > 3. \\ -\frac{3\pi^4(\Delta-2)\Delta^2\Gamma(\frac{\Delta}{2}-2)^2\Gamma(\frac{\Delta}{2}-1)^2}{64\Gamma(\Delta-4)\Gamma(\Delta+2)} N_{\mathcal{O}}, & J=2, \Delta > 4. \\ \dots \end{cases} \tag{5.6}$$

- They are obtained from UV/IR relation
- They can be checked by computing the integral directly with regularization for specific Δ.

Non-conserved primary operator in CFT_4 (II)

- (3)-type, scalar primary operator

$$p_2[\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3]=C\int_{\mathbb{D}^2}d^2\mu_0\int_{\mathbb{D}^2}d^2\mu_0'\int_0^\pi d\theta\frac{\sin\theta}{(a+b\cos\theta)^{\frac{\Delta_{12,3}}{2}}},$$

where $C=-2^{4-\Delta_1-\Delta_2-\Delta_3}\pi^3 C_{123}$ and

$$\begin{aligned} d^2\mu_0 &= d\zeta d\bar{\zeta}(\zeta+\bar{\zeta})^2(1-\zeta^2)^{\frac{\Delta_1-4}{2}}(1-\bar{\zeta}^2)^{\frac{\Delta_1-4}{2}}, \\ d^2\mu_0' &= d\zeta' d\bar{\zeta}'(\zeta'+\bar{\zeta}')^2(1-\zeta'^2)^{\frac{\Delta_2-4}{2}}(1-\bar{\zeta}'^2)^{\frac{\Delta_2-4}{2}} \end{aligned} \tag{5.7}$$

$$a=\zeta\bar{\zeta}+\zeta'\bar{\zeta}'+\frac{1}{2}(\zeta-\bar{\zeta})(\zeta'-\bar{\zeta}'),\quad b=-\frac{1}{2}(\zeta+\bar{\zeta})(\zeta'+\bar{\zeta}') \tag{5.8}$$

$$\Delta_{12,3}=\Delta_1+\Delta_2-\Delta_3 \tag{5.9}$$

- cyclic identity

$$p_2[4,6,8]=p_2[4,8,6]=p_2[8,4,6]=-\frac{\pi^3}{1728}C_{123}. \tag{5.10}$$

Example 3: Conserved currents in CFT_4 (I)

- OPE block is type-J, degree $q = 1$.
- (2)-type

$$p_1[\mathcal{J},\mathcal{J}] = \begin{cases} -\frac{\pi^2}{3} C_J, & J = 1, \\ -\frac{\pi^2}{40} C_T, & J = 2, \\ \cdots . \end{cases} \tag{5.11}$$

  - regularize the integral
  - UV/IR relation
  - For $J = 2$, stress tensor, this is consistent with universal results of modular Hamiltonian

E.Perlmutter, 1308.1083

$$\langle H_A^2 \rangle_c = -\frac{1}{2\pi^2} S'_{q=1} = -\frac{\pi^2}{40} C_T. \tag{5.12}$$

Conserved currents in CFT$_4$ (II)

- (3)-type

$$p_1[\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3]=\begin{cases}-\frac{\pi^3}{2}C_{T\mathcal{J}\mathcal{J}}, & J_1=J_2=1, J_3=2.\\ \frac{\pi^3}{12}C_{TTT}, & J_1=J_2=J_3=2.\\ \cdots. \end{cases} \tag{5.13}$$

- Three point function for conserved currents
  H.Osborn & A.C.Petkou, 9307010 J.Erdmenger & H.Osborn,9605009
  - Spin 1-1-2. Only two independent structures, a, b

$$C_{T\mathcal{J}\mathcal{J}}=\frac{3b-4a}{8}. \tag{5.14}$$

  - Spin 2-2-2. Only three independent structures, $\mathcal{A},\mathcal{B},\mathcal{C}$

$$C_{TTT}=\frac{-2(4-5d+2d^2)\mathcal{A}+d\mathcal{B}+2(5d-4)\mathcal{C}}{4d^2}, \quad d=4. \tag{5.15}$$

- $J_1=J_2=J_3$, universal results of modular Hamiltonian
  J.Lee, A.Lewkowycz, E.Perlmutter & B.R.Safdi, 1407.7816

$\langle Q_A[\mathcal{O}] \cdots Q_A[\mathcal{J}] \rangle_c$

- In previous discussion, OPE blocks in CCF belong to the same type.
- Type-O & type-J
- A severe puzzle

$$\begin{aligned} \langle Q_A[\mathcal{O}] \cdots Q_A[\mathcal{J}] \rangle_c \to \langle Q_A[\mathcal{O}] Q_A[\tilde{\mathcal{O}}] Q_B[\mathcal{J}] \rangle_c &= D[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}] G_{\Delta,J}(z), \\ \langle Q_A[\mathcal{O}] \cdots Q_A[\mathcal{J}] \rangle_c \to \langle Q_A[\tilde{\mathcal{O}}] Q_A[\mathcal{J}] Q_B[\mathcal{O}] \rangle_c &= D[\tilde{\mathcal{O}}, \mathcal{O}, \mathcal{J}] G_{\Delta',J'}(z) \end{aligned}$$

- UV/IR relation, rather different degree q.

$$\begin{aligned} p_1[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}] &= E[\mathcal{J}] D[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}], \\ p_2[\tilde{\mathcal{O}}, \mathcal{O}, \mathcal{J}] &= E[\mathcal{O}] D[\tilde{\mathcal{O}}, \mathcal{O}, \mathcal{J}]. \end{aligned} \quad (6.1)$$

- $D[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}]$ is divergent.

An example

- A simplest nontrivial CCF, spin 2-0-0.

$$\langle Q_A[T_{\mu\nu}]Q_A[\mathcal{O}]Q_A[\mathcal{O}]\rangle_c \tag{6.2}$$

- From $\langle Q_A[T_{\mu\nu}]Q_A[\mathcal{O}]Q_B[\mathcal{O}]\rangle_c$

$$D[T_{\mu\nu},\mathcal{O},\mathcal{O}] = -\frac{\pi^5 4^{3-2\Delta}\Gamma(\frac{\Delta}{2}-1)^4}{\Delta(\Delta-2)\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta-1}{2})^2\Gamma(\frac{\Delta+1}{2})}a.$$

$$\Rightarrow \quad p_2[T_{\mu\nu},\mathcal{O},\mathcal{O}] = \frac{2^{5-2\Delta}\pi^4\Gamma(\frac{\Delta}{2}-1)^2}{\Delta(\Delta-2)\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta-1}{2})}a. \tag{6.3}$$

- From $\langle Q_A[\mathcal{O}]Q_A[\mathcal{O}]Q_B[T_{\mu\nu}]\rangle_c$

$$D[\mathcal{O},\mathcal{O},T_{\mu\nu}] = -\frac{\pi^3}{3840}a \log\frac{R}{\epsilon} + \cdots \quad \text{for } \Delta = 4.$$

$$\Rightarrow \quad p_2 = \frac{\pi^3}{32}a \tag{6.4}$$

  - logarithmic divergence of D increases the degree by 1.

Divergent D

- The puzzle is from the assumption that D is always finite.
- The coefficient $D[\mathcal{O}, \mathcal{O}, T_{\mu\nu}]$ presents logarithmic divergence behaviour

$$\langle Q_A[\mathcal{O}]^2 Q_B[T_{\mu\nu}]\rangle_c \sim D_{\log}[\mathcal{O}, \mathcal{O}, T_{\mu\nu}] \log \frac{R}{\epsilon} G_{4,2}(z). \tag{6.5}$$

  - D is finite when the OPE blocks belong to the same type
  - $D[\mathcal{O}, \cdots, \mathcal{J}]$ should present logarithmic divergence behaviour to cure the puzzle.
- The degree $q = 2$ rather than 1.
- UV/IR relation becomes

$$p_2[\mathcal{O}, \mathcal{O}, T_{\mu\nu}] = E[T_{\mu\nu}] D_{\log}[\mathcal{O}, \mathcal{O}, T_{\mu\nu}]. \tag{6.6}$$

Conclusion

- We find new area law

$$\langle Q_A[\mathcal{O}_1]\cdots Q_A[\mathcal{O}_m]\rangle_c = \gamma \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + p_q[\mathcal{O}_1, \cdots, \mathcal{O}_m] \log^q \frac{R}{\epsilon} + \cdots . \tag{7.1}$$

- We obtain UV/IR relation

$$p = E \times D. \tag{7.2}$$

- We check cyclic identity for $m = 3$.

$$p[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3] = p[\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_1] = p[\mathcal{O}_3, \mathcal{O}_1, \mathcal{O}_2]. \tag{7.3}$$

Outlook

- Deformed reduced density matrix as a 'Wilson loop' in CFT

$$\rho_A = e^{-\mu Q_A} \tag{7.4}$$

  - Area law and rich information of CFT
  - Gravitational dual
- Connection to black hole physics

$$S = \frac{A}{4G} + C \log A + \cdots$$

  - Infinite number of area laws in black hole physics!
- Rich mathematical structure

$$S_n(\vec{\alpha}, \vec{\beta}; \gamma) = \prod_{i=1}^{n} \int_0^1 dz_i \prod_{j=1}^{n} z_j^{\alpha_j - 1}(1 - z_j)^{\beta_j - 1} \prod_{1 \le k < \ell \le n} |z_k - z_\ell|^{\gamma_{k\ell}}$$

Thanks for your attention!