高自旋理论 Vasiliev 方程解

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arXiv:1712.02401， arXiv:1903.01399， arXiv:1909.12097

第一届全国场论与弦论学术研讨会
国家自然科学基金委员会理论物理专款彭桓武高能基础理论研究中心

2020 年 11 月 30 日
@　合肥　中国科学技术大学

Fronsdal eqs. (1978) – free higher-spin gauge theory

$$\Box T_{\mu_1\mu_2\cdots\mu_s} - \partial_{(\mu_1}\partial^\sigma T_{\mu_2\cdots\mu_s)\sigma} + \partial_{(\mu_1}\partial_{\mu_2} T_{\mu_3\cdots\mu_s)\sigma}{}^\sigma = 0$$

$$\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} T_{\mu_1\mu_2\cdots\mu_s} = 0$$

Gauge sym.: $\delta T_{\mu_1\mu_2\cdots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\cdots\mu_s)}$, $\eta^{\mu_2\mu_3}\xi_{\mu_2\cdots\mu_s} = 0$

$s = 0 \Rightarrow$ Klein-Gordon eq. $\Box T = 0$

$s = 1 \Rightarrow$ Maxwell eqs.

$s = 2 \Rightarrow$ Linearized Einstein field eqs.

$s = 3, 4, 5, \ldots$

- Can be extended to (A)dS background
- Historically difficult to construct a higher-spin gauge theory with interactions

（Weinberg (1964), Coleman-Mandula (1967), Weinberg-Witten (1980) … ）

- Vasiliev’s higher-spin gravity (1990) is known as an interacting theory of higher-spin gauge fields.

- General relativity has various solutions describing drastically different objects e.g. gravitational waves and black holes. Likewise, physical implications of Vasiliev's equations may be much richer than merely higher spins. That's why we are interested in studying its solutions.

- What I will present:
  A general method to solve Vasiliev’s equations (in 4D with only integer spins), such that at the linearized level the solution describes perturbative fields with desirable properties on an (A)dS background.

Outline

1. Notations for Vasiliev's equations

2. (A)dS backgrounds and perturbations

3. An example in dS: FRW

4. An example in AdS: generalized BTZ in 4D

5. Stargenfunctions with complex eigenvalues

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Vasiliev's equations

- Vasiliev's equations written in the most compact way:

$$\begin{aligned} dA + A \star A + \Phi \star J &= 0 \\ d\Phi + A \star \Phi - \Phi \star \pi(A) &= 0 \end{aligned}$$

- d : exterior derivative
  Φ : 0-form
  A : 1-form
  J : 2-form $\quad J \star f = \pi(f) \star J \ , \ dJ = 0$

- $\star$ - product : associative, non-commutative
  π : automorphism

- Gauge symmetry:

$$\begin{aligned} \delta A &= d\epsilon + [A, \epsilon]_\star \\ \delta \Phi &= \Phi \star \pi(\epsilon) - \epsilon \star \Phi \end{aligned}$$

Vasiliev's equations

$$\begin{aligned} dA + A \star A + \Phi \star J &= 0 \\ d\Phi + A \star \Phi - \Phi \star \pi(A) &= 0 \end{aligned}$$

- External Coordinates:

x^μ : 3+1 dimensional spacetime

$(z^\alpha, \ \bar{z}^{\dot{\alpha}})$: 4 dimensional symplectic manifold

$$(z^\alpha)^\dagger = -\bar{z}^{\dot{\alpha}}$$

- $\alpha, \dot{\alpha}$ are SL(2,ℂ) indices, raised or lowered by $\varepsilon^{\alpha\beta}, \varepsilon_{\alpha\beta}, \varepsilon^{\dot{\alpha}\dot{\beta}}, \varepsilon_{\dot{\alpha}\dot{\beta}}$ (NW-SE convention)

- Decomposition:

$$\begin{aligned} d &= dx^\mu \partial_\mu + dz^\alpha \partial_\alpha + d\bar{z}^{\dot{\alpha}} \partial_{\dot{\alpha}} \\ A &= U_\mu dx^\mu + V_\alpha \, dz^\alpha + \bar{V}_{\dot{\alpha}} \, d\bar{z}^{\dot{\alpha}} \end{aligned}$$

Vasiliev's equations

$$\begin{aligned} dA + A\star A + \Phi\star J &= 0 \\ d\Phi + A\star\Phi - \Phi\star\pi(A) &= 0 \end{aligned}$$

- Internal coordinates (auxiliary variables):

$$(y^{\alpha},\ \bar{y}^{\dot{\alpha}}) \quad , \quad (y^{\alpha})^{\dagger} = \bar{y}^{\dot{\alpha}}$$

another 4 dimensional symplectic manifold.

- Bilinear combinations:

$$\{\ y^{\alpha}\bar{y}^{\dot{\beta}}\ ,\quad y^{\alpha}y^{\beta}\ ,\quad \bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}\ \}$$

are the generators of (A)dS$_4$ isometry algebra.

- Corresponding components in A are

$$e_{\mu}{}^{\alpha\dot{\beta}} \ \leftrightarrow\ e_{\mu}{}^{a} \quad \text{and} \quad \omega_{\mu}{}^{(\alpha\beta)},\ \bar{\omega}_{\mu}{}^{(\dot{\alpha}\dot{\beta})} \ \leftrightarrow\ \omega_{\mu}{}^{[ab]}$$

Indices are converted by van der Waerden symbols:

$(\sigma^{a})_{\alpha\dot{\beta}} \ =$ (One identity matrix a = 0, Three Pauli matrices a = 1,2,3)

Vasiliev's equations

$$\begin{aligned} dA + A \star A + \Phi \star J &= 0 \\ d\Phi + A \star \Phi - \Phi \star \pi(A) &= 0 \end{aligned}$$

- All powers of $y^\alpha, \bar{y}^{\dot{\alpha}}$ give an infinite set of generators, constituting the higher-spin algebra.
- The $\star$ is the product operation between elements that gives the right commutation rules.
- A and Φ are "master fields" with infinitely many components, whose indices are contracted with $y^\alpha, \bar{y}^{\dot{\alpha}}$.
- An infinite tower of higher-spin gauge fields and their on-shell curvature tensors are included in the theory as components of A and Φ.
- If we choose the gauge $z^\alpha V_\alpha = 0$, do some field redefinitions, and perturbatively expand the equations around (A)dS_4, we obtain Fronsdal's equations for all spins.

Vasiliev's equations

$$\begin{aligned} dA + A \star A + \Phi \star J &= 0 \\ d\Phi + A \star \Phi - \Phi \star \pi(A) &= 0 \end{aligned}$$

Explicit definition of $\star$ and π

- $f_1(y, \bar{y}, z, \bar{z}) \star f_2(y, \bar{y}, z, \bar{z})$
$$= \int d^2u d^2\bar{u} d^2v d^2\bar{v} \; (2\pi)^{-4} \; e^{i\left(v^\alpha u_\alpha + \bar{v}^{\dot{\alpha}} \bar{u}_{\dot{\alpha}}\right)}$$
$$f_1(y+u, \bar{y}+\bar{u}; z+u, \bar{z}+\bar{u}) \, f_2(y+v, \bar{y}+\bar{v}; z-v, \bar{z}-\bar{v})$$

- $$\begin{aligned} \pi\left(x^\mu; y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}}\right) &= \left(x^\mu; -y^\alpha, \bar{y}^{\dot{\alpha}}; -z^\alpha, \bar{z}^{\dot{\alpha}}\right) \\ \bar{\pi}\left(x^\mu; y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}}\right) &= \left(x^\mu; y^\alpha, -\bar{y}^{\dot{\alpha}}; z^\alpha, -\bar{z}^{\dot{\alpha}}\right) \end{aligned}$$

- Klein operators: $\kappa_y = 2\pi\delta^2(y)$, $\kappa_z = 2\pi\delta^2(z)$, $\kappa = \kappa_y \star \kappa_z$

  satisfy $\kappa_y \star \kappa_y = \kappa_z \star \kappa_z = 1$, $\pi(f) = \kappa \star f \star \kappa$,

  and similarly $\bar{\pi}(f) = \bar{\kappa} \star f \star \bar{\kappa}$.

Vasiliev's equations

$$
\begin{aligned}
dA + A \star A + \Phi \star J &= 0 \\
d\Phi + A \star \Phi - \Phi \star \pi(A) &= 0
\end{aligned}
$$

- We only consider integer spins:

$$\pi\bar{\pi}(\Phi) = \Phi \; , \quad \pi\bar{\pi}(A) = A$$

$$\hookrightarrow \quad \pi\bar{\pi}(U_\mu) = U_\mu \; , \quad \pi\bar{\pi}(V_\alpha) = -V_\alpha$$

- We set

$$J_{\mu\nu} = 0 \quad , \quad J_{\mu\alpha} = 0 \quad , \quad J_{\alpha\beta} = \frac{i}{2}\varepsilon_{\alpha\beta}\kappa$$

- Reality conditions:

$$\Phi^\dagger = \begin{cases} \pi(\Phi) \\ \Phi \end{cases} \quad , \quad A^\dagger = \begin{cases} -A & \text{in AdS} \\ -\pi(A) & \text{in dS} \end{cases}$$

- crucial difference between AdS and dS, corresponding to different slices of so(5,$\mathbb{C}$)

Outline

Decomposition of Vasiliev's equations

Using

$$\begin{aligned} d &= dx^\mu \partial_\mu \; + \; dz^\alpha \partial_\alpha \; + \; d\bar{z}^{\dot\alpha} \partial_{\dot\alpha} \\ A &= U_\mu dx^\mu \; + V_\alpha \, dz^\alpha \; + \bar{V}_{\dot\alpha} \, dz^{\dot\alpha} \end{aligned}$$

to rewrite the equations:

$$\begin{aligned} \partial_{[\mu} U_{\nu]} + U_{[\mu} \star U_{\nu]} &= 0 \\ \partial_\mu \Phi + U_\mu \star \Phi - \Phi \star \pi\left(U_\mu\right) &= 0 \end{aligned}$$

$$\partial_\mu V_\alpha - \partial_\alpha U_\mu + \left[U_\mu, V_\alpha\right]_\star = 0 \quad \text{and} \quad \text{h.c.}$$

$$\begin{aligned} \partial_{[\alpha} V_{\beta]} + V_{[\alpha} \star V_{\beta]} + \tfrac{i}{4} \varepsilon_{\alpha\beta} \Phi \star \kappa &= 0 \quad \text{and} \quad \text{h.c.} \\ \partial_\alpha \Phi + V_\alpha \star \Phi - \Phi \star \bar{\pi}\left(V_\alpha\right) &= 0 \quad \text{and} \quad \text{h.c.} \\ \partial_\alpha \bar{V}_{\dot\alpha} - \partial_{\dot\alpha} V_\alpha + \left[V_\alpha, \bar{V}_{\dot\alpha}\right]_\star &= 0 \end{aligned}$$

(A)dS$_4$ background

The field U_μ is the spacetime component of the gauge field. To obtain a pure spacetime-background solution, we set all fields to zero, except U_μ.

$\Rightarrow \quad U_\mu = L^{-1}\star\partial_\mu L \quad ; \quad L(x;y,\bar{y})$

$$\begin{aligned}
\partial_{[\mu}U_{\nu]} + U_{[\mu}\star U_{\nu]} &= 0 \\
\cancel{\partial_\mu\Phi + U_\mu\star\Phi - \Phi\star\pi(U_\mu)} &= 0
\end{aligned}$$

$$\cancel{\partial_\mu V_\alpha} - \partial_\alpha U_\mu + \cancel{[U_\mu, V_\alpha]}_\star = 0 \quad \text{and} \quad \text{h.c.}$$

$$\cancel{\begin{aligned}
\partial_{[\alpha}V_{\beta]} + V_{[\alpha}\star V_{\beta]} + \tfrac{i}{4}\varepsilon_{\alpha\beta}\Phi\star\kappa &= 0 \quad \text{and} \quad \text{h.c.} \\
\partial_\alpha\Phi + V_\alpha\star\Phi - \Phi\star\bar{\pi}(V_\alpha) &= 0 \quad \text{and} \quad \text{h.c.} \\
\partial_\alpha\bar{V}_{\dot\alpha} - \partial_{\dot\alpha}V_\alpha + \left[V_\alpha, \bar{V}_{\dot\alpha}\right]_\star &= 0
\end{aligned}}$$

(A)dS$_4$ background

One way to produce the (A)dS$_4$ background, is to set

$$L\left(x;y,\bar{y}\right) \;=\; \frac{2h}{1+h}\exp\left(\frac{i\lambda}{1+h}x^a(\sigma_a)^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}}\right)$$

Denote $h = \sqrt{1-\lambda^2\eta_{ab}x^a x^b}$

and $\lambda = \ell^{-1}$ in AdS , $\lambda = i\ell^{-1}$ in dS ; ℓ : (A)dS radius

$$\Rightarrow U_\mu = L^{-1}\star\partial_\mu L = -\frac{i}{2}e_\mu^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}} - \frac{i}{4}\left(\omega_\mu^{\alpha\beta}y_\alpha y_\beta + \bar{\omega}_\mu^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}}\right)$$

Vierbein

$$e_\mu^{\alpha\dot{\alpha}} = -\lambda h^{-2}\delta_\mu^a\left(\sigma_a\right)^{\alpha\dot{\alpha}}$$

Spin-connection

$$\omega_\mu^{\alpha\beta} = -\lambda^2 h^{-2}\delta_\mu^a x^b\left(\sigma_{ab}\right)^{\alpha\beta}$$

corresponding to the stereographic coordinates of (A)dS

$$x^a \;=\; \frac{X^a}{1+|\lambda X^{0'}|} \quad , \qquad a = 0,1,2,3$$

where X 's are embedding coordinates:

$$-(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2-\mathrm{sign}(\lambda^2)(X^{0'})^2 = -\lambda^{-2}$$

Reality conditions

$$U_\mu = L^{-1} \star \partial_\mu L = -\frac{i}{2} e_\mu^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} - \frac{i}{4} \left(\omega_\mu^{\alpha\beta} y_\alpha y_\beta + \bar{\omega}_\mu^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} \right)$$

Vierbein

$$e_\mu^{\alpha\dot{\alpha}} = -\lambda h^{-2} \delta_\mu^a \left(\sigma_a\right)^{\alpha\dot{\alpha}}$$

Spin-connection

$$\omega_\mu^{\alpha\beta} = -\lambda^2 h^{-2} \delta_\mu^a x^b \left(\sigma_{ab}\right)^{\alpha\beta}$$

From the fact that $\lambda \in \mathbb{R}$ in AdS , $\lambda \in i\mathbb{R}$ in dS

the spin-connection term of U_μ is always imaginary; the vierbein term is imaginary in AdS but real in dS. On the other hand, the operation π flips the sign of the vierbein term only. Therefore we can see the difference in the reality condition:

$$U_\mu^\dagger = -U_\mu \ \text{ in AdS} , \quad U_\mu^\dagger = -\pi\left(U_\mu\right) \ \text{ in dS}$$

(By taking the h.c. of Vasiliev’s equations, we can further derive the reality conditions on other fields.)

Perturbation around (A)dS$_4$ background

$$
\begin{aligned}
\partial_{[\mu}U_{\nu]} + U_{[\mu}\star U_{\nu]} &= 0 \\
\cancel{\partial_\mu\Phi + U_\mu\star\Phi - \Phi\star\pi(U_\mu)} &= 0 \\
\cancel{\partial_\mu V_\alpha} - \partial_\alpha U_\mu + \left[\cancel{U_\mu, V_\alpha}\right]_\star &= 0 \quad \text{and} \quad \text{h.c.}
\end{aligned}
$$

$$
\cancel{
\begin{aligned}
\partial_{[\alpha}V_{\beta]} + V_{[\alpha}\star V_{\beta]} + \tfrac{i}{4}\varepsilon_{\alpha\beta}\Phi\star\kappa &= 0 \quad \text{and} \quad \text{h.c.} \\
\partial_\alpha\Phi + V_\alpha\star\Phi - \Phi\star\bar{\pi}(V_\alpha) &= 0 \quad \text{and} \quad \text{h.c.} \\
\partial_\alpha\bar{V}_{\dot\alpha} - \partial_{\dot\alpha}V_\alpha + \left[V_\alpha, \bar{V}_{\dot\alpha}\right]_\star &= 0
\end{aligned}
}
$$

We have set all fields to zero, except U_μ to get the (A)dS$_4$ background solution.

Now in order to get solutions that represent perturbations around the background, things have to be done the other way around…

Perturbation around (A)dS$_4$ background

$$\begin{aligned}
\cancel{\partial_{[\mu}U_{\nu]} + U_{[\mu} \star U_{\nu]}} &= 0 \\
\partial_\mu \Phi + \cancel{U_\mu \star \Phi - \Phi \star \bar{\pi}\left(U_\mu\right)} &= 0 \\
\partial_\mu V_\alpha - \cancel{\partial_\alpha U_\mu + \left[U_\mu, V_\alpha\right]_\star} &= 0 \qquad \text{and} \quad \text{h.c.}
\end{aligned}$$

$$\begin{aligned}
\partial_{[\alpha}V_{\beta]} + V_{[\alpha} \star V_{\beta]} + \tfrac{i}{4}\varepsilon_{\alpha\beta}\Phi \star \kappa &= 0 \qquad \text{and} \quad \text{h.c.} \\
\partial_\alpha \Phi + V_\alpha \star \Phi - \Phi \star \bar{\pi}\left(V_\alpha\right) &= 0 \qquad \text{and} \quad \text{h.c.} \\
\partial_\alpha \bar{V}_{\dot\alpha} - \partial_{\dot\alpha} V_\alpha + \left[V_\alpha, \bar{V}_{\dot\alpha}\right]_\star &= 0
\end{aligned}$$

- U_μ is a pure gauge. We can simply gauge it away, and focus on getting a solution for the rest of the fields. After that we can restore the (A)dS$_4$ by a gauge transformation.
- A consequence of gauging away U_μ is that Φ and V_α become independent of spacetime, which simplifies the problem.

Perturbation around (A)dS$_4$ background

$$
\begin{aligned}
\partial_{[\mu} U_{\nu]} + U_{[\mu} \star U_{\nu]} &= 0 \\
\partial_\mu \Phi + U_\mu \star \Phi - \Phi \star \pi\left(U_\mu\right) &= 0 \\
\partial_\mu V_\alpha - \partial_\alpha U_\mu + \left[U_\mu, V_\alpha\right]_\star &= 0 \quad \text{and} \quad \text{h.c.}
\end{aligned}
$$

$$
\begin{aligned}
\partial_{[\alpha} V_{\beta]} + V_{[\alpha} \star V_{\beta]} + \tfrac{i}{4}\varepsilon_{\alpha\beta}\Phi \star \kappa &= 0 \quad \text{and} \quad \text{h.c.} \\
\partial_\alpha \Phi + V_\alpha \star \Phi - \Phi \star \bar{\pi}\left(V_\alpha\right) &= 0 \quad \text{and} \quad \text{h.c.} \\
\partial_\alpha \bar{V}_{\dot\alpha} - \partial_{\dot\alpha} V_\alpha + \left[V_\alpha, \bar{V}_{\dot\alpha}\right]_\star &= 0
\end{aligned}
$$

- Ansatz:
$$
\begin{aligned}
U_\mu &= L^{-1} \star \partial_\mu L \\
\Phi &= L^{-1} \star \Phi' \star \pi(L) \\
V_\alpha &= L^{-1} \star V'_\alpha \star L \quad \cancel{+ L^{-1} \star \partial_\alpha L}
\end{aligned}
$$
  - The primed fields are independent of spacetime coord.
  - directly solves the first three equations

Perturbation around (A)dS$_4$ background

$$
\begin{aligned}
\partial_{[\alpha} V'_{\beta]} + V'_{[\alpha} \star V'_{\beta]} + \tfrac{i}{4}\varepsilon_{\alpha\beta}\Phi' \star \kappa &= 0 \\
\partial_\alpha \Phi' + V'_\alpha \star \Phi' - \Phi' \star \bar{\pi}\left(V'_\alpha\right) &= 0 \\
\partial_\alpha \bar{V}'_{\dot{\alpha}} - \partial_{\dot{\alpha}} V'_\alpha + \left[V'_\alpha, \bar{V}'_{\dot{\alpha}}\right]_\star &= 0
\end{aligned}
$$

- In the next step, we only need to solve the last three equations for the primed fields.
- Ansatz:

$$
\begin{aligned}
\Phi' &= \Psi \star \kappa_y & \qquad \Psi(y,\bar{y}) \ \mathrm{adj.repr.} \\
V'_\alpha &= \sum_{n=1}^{\infty} a_\alpha^{(n)} \star \Psi^{\star n} & \qquad \kappa_z \star a_\alpha^{(n)}(z) \star \kappa_z = -a_\alpha^{(n)}(z)
\end{aligned}
$$

- The last two equations are directly solved;

- The only equation left becomes…

Perturbation around (A)dS$_4$ background

$$\sum_m \partial_{[\alpha} a^{(m)}_{\beta]} \star \Psi^{\star m} + \sum_{m,n} a^{(m)}_{[\alpha} \star a^{(n)}_{\beta]} \star \Psi^{\star(m+n)} = -\frac{i}{4}\varepsilon_{\alpha\beta}\kappa_z \star \Psi$$

- We can first solve the following equation instead:

$$\partial_{[\alpha}\mathring{a}_{\beta]} + \mathring{a}_{[\alpha} \star \mathring{a}_{\beta]} = -\frac{i}{4}\varepsilon_{\alpha\beta}\nu\kappa_z$$

$$\text{where} \quad \mathring{a}_\alpha(\nu) = \sum_m a^{(m)}_\alpha \nu^m$$

then expand the solution for $\mathring{a}_\alpha$ in power series of υ to obtain the solution for $a^{(n)}_\alpha$.

- The solution for $\mathring{a}_\alpha$ is given by…

Perturbation around (A)dS$_4$ background

$$\left(\mathring{a}_\sigma\right)_\alpha = 2iz_\alpha \int_{-1}^{1} d\tau \frac{j_\sigma(\tau)}{(\tau+1)^2} \exp\left[c(\tau)\, U^{\beta\gamma} z_\beta z_\gamma\right]$$

with

$$\begin{aligned} c(\tau) &= \varsigma i \frac{\tau-1}{\tau+1} \\ j_\sigma(\tau) &= -\frac{\varsigma\nu}{4}\, {}_1F_1\left[\frac{1}{2}; 2; \frac{\varsigma\nu}{2}\log\left(\tau^2\right)\right] \\ U^{\beta\gamma} &= \left(u^+\right)^{(\beta}\left(u^-\right)^{\gamma)} \end{aligned}$$

where u^+ and u^- are a set of spinor basis vectors obeying

$$\left(u^+\right)^\alpha\left(u^-\right)_\alpha = 1\ , \quad \left(u^+\right)^\alpha\left(u^+\right)_\alpha = \left(u^-\right)^\alpha\left(u^-\right)_\alpha = 0$$

and $\varsigma = \pm 1$ is free to choose.

Result from Iazeolla & Sundell arXiv:1107.1217 ,
based on Prokushkin & Vasiliev arXiv:hep-th/9806236 .

Perturbation around (A)dS$_4$ background

$$
\begin{aligned}
\partial_{[\mu}U_{\nu]} + U_{[\mu}\star U_{\nu]} &= 0 \\
\partial_\mu \Phi + U_\mu \star \Phi - \Phi \star \pi\left(U_\mu\right) &= 0 \\
\partial_\mu V_\alpha - \partial_\alpha U_\mu + \left[U_\mu, V_\alpha\right]_\star &= 0 \quad \text{and} \quad \text{h.c.} \\
\partial_{[\alpha}V_{\beta]} + V_{[\alpha}\star V_{\beta]} + \frac{i}{4}\varepsilon_{\alpha\beta}\Phi\star\kappa &= 0 \quad \text{and} \quad \text{h.c.} \\
\partial_\alpha \Phi + V_\alpha \star \Phi - \Phi \star \bar{\pi}\left(V_\alpha\right) &= 0 \quad \text{and} \quad \text{h.c.} \\
\partial_\alpha \bar{V}_{\dot{\alpha}} - \partial_{\dot{\alpha}} V_\alpha + \left[V_\alpha, \bar{V}_{\dot{\alpha}}\right]_\star &= 0
\end{aligned}
$$

- $$
\begin{aligned}
U_\mu &= L^{-1}\star\partial_\mu L \\
\Phi &= L^{-1}\star\Phi'\star\pi(L) \\
V_\alpha &= L^{-1}\star V'_\alpha\star L
\end{aligned}
\qquad
\begin{aligned}
\Phi' &= \Psi\star\kappa_y \\
V'_\alpha &= \sum_{n=1}^{\infty} a_\alpha^{(n)}\star\Psi^{\star n}
\end{aligned}
$$
- Φ' (or Ψ) is the like the initial data. Once we have an explicit expression of Φ' or Ψ, everything will be automatically decided.

Outline

1. Notations for Vasiliev's equations

2. (A)dS backgrounds and perturbations

3. <u>An example in dS: FRW</u>

4. An example in AdS: generalized BTZ in 4D

5. Stargenfunctions with complex eigenvalues

An example in dS

The first example has some motivation from cosmology.

We construct a set of solutions to Vasiliev's equations, such that at the linearized level they represent:

- the dS background

- the fields that are homogeneous and isotropic in space

Steps:

1. identify the symmetry algebra

2. obtain the initial data subject to the symmetry

3. do a gauge transformation to turn on the dS background

4. show e.g. the scalar field configuration over spacetime

Step1: identify the symmetry algebra

- The (A)dS$_4$ isometry algebra contains
  10 generators = 6 Lorentz + 4 transvections

- In the "star-product language":

  - Transvections $P_a = \frac{\lambda}{4}\,(\sigma_a)_{\alpha\dot\beta}\, y^\alpha \bar y^{\dot\beta}$

  - Lorentz $M_{ab} = -\frac{1}{8}[(\sigma_{ab})_{\alpha\beta}\, y^\alpha y^\beta + (\bar\sigma_{ab})_{\dot\alpha\dot\beta}\, \bar y^{\dot\alpha} \bar y^{\dot\beta}]$

- Star-product commutators:

$$
\begin{aligned}
[M_{ab}, M_{cd}]_\star &= i\eta_{ad}M_{bc} + i\eta_{bc}M_{ad} - i\eta_{ac}M_{bd} - i\eta_{bd}M_{ac} \\
[P_b, M_{cd}]_\star &= 2i\eta_{b[c}P_{d]} \\
[P_a, P_b]_\star &= i\lambda^2 M_{ab} \qquad (\lambda^2 < 0 \text{ for dS; } \cancel{\lambda^2 > 0 \text{ for AdS}})
\end{aligned}
$$

- simply the commutators of the $\mathrm{so}(1,4)$ ~~or so(2,3)~~ algebra
  explicitly realized by $y^\alpha, \bar y^{\dot\alpha}$ and their star-product.

Step1: identify the symmetry algebra

- On the other hand, isotropy and homogeneity of space contain only 6 generators. The symmetry of the perturbative fields is a 6-dimensional subalgebra of the 10-dimensional background isometry algebra.
- Embedding ($r,s,t,u\ldots$ are spatial indices):
  - Rotations: $\boldsymbol{M_{rs}}$ (spatial components of Lorentz)
  - "New" transvections: $\boldsymbol{T_r} = \alpha\, M_{r0} + \beta\, P_r \quad , \quad \alpha,\beta \in \mathbb{R}$
    (combination of boosts and the "old" transvections)
- Commutators of the subalgebra:

$$\begin{aligned}
[M_{rs}, M_{tu}]_\star &= i\eta_{ru}M_{st} + i\eta_{st}M_{ru} - i\eta_{rt}M_{su} - i\eta_{su}M_{rt} \\
[T_s, M_{tu}]_\star &= 2i\eta_{s[t}T_{u]} \\
[T_r, T_s]_\star &= -ikM_{rs} \quad , \quad k = -\alpha^2 - \lambda^2\beta^2 \\
& \quad k > 0 \ \ \mathrm{so}(4) \ ; \quad k = 0 \ \ \mathrm{iso}(3) \ ; \quad k < 0 \ \ \mathrm{so}(1,3) \ ;
\end{aligned}$$

Step2: obtain the initial data

- We explicitly have

$$
\begin{aligned}
T_r &= -\frac{\alpha}{8}\left[(\sigma_{r0})_{\alpha\beta}\, y^\alpha y^\beta + (\bar{\sigma}_{r0})_{\dot{\alpha}\dot{\beta}}\, \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}\right] + \frac{\lambda\beta}{4}\, (\sigma_r)_{\alpha\dot{\beta}}\, y^\alpha \bar{y}^{\dot{\beta}} \\
M_{rs} &= -\frac{1}{8}\left[(\sigma_{rs})_{\alpha\beta}\, y^\alpha y^\beta + (\bar{\sigma}_{rs})_{\dot{\alpha}\dot{\beta}}\, \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}\right]
\end{aligned}
$$

- Impose the FRW symmetry conditions on the initial data

$$T_r \star \Phi' - \Phi' \star \pi(T_r) = 0 \quad \text{(i)} \qquad , \qquad M_{rs} \star \Phi' - \Phi' \star \pi(M_{rs}) = 0 \quad \text{(ii)}$$

- Solve the above equations for Φ':

For (ii), assume that Φ' is a Taylor series of $P_0 = \frac{\lambda}{4}\, (\sigma_0)_{\alpha\dot{\beta}}\, y^\alpha \bar{y}^{\dot{\beta}}$

(i) $$\Rightarrow \left(\frac{\beta\lambda^2}{8} \frac{d^2}{(dP_0)^2} - i\alpha \frac{d}{dP_0} + 2\beta\right) \Phi'(P_0) = 0$$

Step2: obtain the initial data

$$\left(\frac{\beta\lambda^2}{8}\frac{d^2}{(dP_0)^2} - i\alpha\frac{d}{dP_0} + 2\beta\right)\Phi'(P_0) = 0$$

- closed space so(4), $k > 0$ (set $\beta > \alpha = 0$) :

$$\Phi' = \mu e^{-4i\lambda^{-1}P_0} + \bar{\mu} e^{4i\lambda^{-1}P_0} \quad , \quad \mu \in \mathbb{C}$$

- flat space iso(3), $k = 0$ (set $\alpha = |\lambda|\beta > 0$) :

$$\Phi' = (\nu - 4\tilde{\nu}\lambda^{-1}P_0)e^{4\lambda^{-1}P_0} \quad , \quad \nu, \tilde{\nu} \in \mathbb{R}$$

- open space so(1,3), $k < 0$ (set $\alpha > |\lambda|\beta > 0$) :

$$\Phi' = \nu_+ e^{-4\eta_+\lambda^{-1}P_0} + \nu_- e^{-4\eta_-\lambda^{-1}P_0} \quad , \quad \nu_+, \nu_- \in \mathbb{R}$$

$$\left(\eta_\pm = -\gamma \pm \sqrt{\gamma^2 - 1}\,, \quad \gamma = \frac{i\alpha}{\lambda\beta}\right)$$

The integration constants should be such chosen that Φ' satisfies the reality condition.

Step3: turn on the dS background

- Let us take the flat space iso(3) for example. (set $\lambda = i$ for simplicity)
- We would like a coordinate system with the FRW metric:

$$ds^2 = -dt^2 + e^{2t}\left(d\mathrm{x}_1^2 + d\mathrm{x}_2^2 + d\mathrm{x}_3^2\right)$$

- Two equivalent ways:

(1) Choose the gauge for the stereographic coordinates

$$L\left(x; y, \bar{y}\right) = e_\star^{4i\ \xi(x)\ x^a P_a}$$

then do a coordinate transformation

$$x^0 = \frac{\sinh t + \frac{1}{2}e^t(\mathrm{x}^i\mathrm{x}_i)}{1 + \cosh t - \frac{1}{2}e^t(\mathrm{x}^i\mathrm{x}_i)}\ , \quad x^i = \frac{e^t\ \mathrm{x}^i}{1 + \cosh t - \frac{1}{2}e^t(\mathrm{x}^i\mathrm{x}_i)}$$

(2) Find another gauge function that directly produces FRW

$$L\left(t, \mathrm{x}; y, \bar{y}\right) = e_\star^{\left(\frac{i}{\beta}\mathrm{x}^i T_i\right)} \star e_\star^{\left(itP_0\right)}$$

Step4: scalar field

- We have the initial data

$$\Phi'(y,\bar{y}) \;=\; (\nu - 4\tilde{\nu}\lambda^{-1}P_0)e^{4\lambda^{-1}P_0} \quad , \qquad \nu,\tilde{\nu}\in\mathbb{R}$$

and we have the gauge function for dS background (FRW)

$$L\,(t,\mathrm{x};y,\bar{y}) \;=\; e_\star^{(\frac{i}{\beta}\mathrm{x}^i T_i)} \star e_\star^{(itP_0)}$$

- Now we do a gauge transformation:

$$\Phi = L^{-1}\star\Phi'\star\pi(L)$$

- A fact of Vasiliev's equations is that the scalar field and the generalized spin-s Weyl tensor are components of Φ:

$$\phi \;=\; \Phi\big|_{y=\bar{y}=0} \quad , \qquad C_{\alpha_1\cdots\alpha_{2s}} = \left[\frac{\partial}{\partial y^{\alpha_1}}\cdots\frac{\partial}{\partial y^{\alpha_{2s}}}\Phi\right]_{y=\bar{y}=0}$$

- This statement is made in the $z^\alpha V_\alpha = 0$ gauge;
- In the L-gauge, it is true at the linearized order.

Step4: scalar field

- Then let's have a look at the linearized scalar field:

$$\phi \;=\; \Phi\big|_{y=\bar{y}=0} \;=\; (\nu+\tilde{\nu})\, e^{-t} - \tilde{\nu} e^{-2t}$$

- Obviously the scalar field is homogeneous and isotropic.
- It satisfies the Klein-Gordon equation

$$(\Box - 2)\,\phi \;=\; 0$$

as expected from the Vasiliev's equations.

- Summary for this example:
We have shown that the Vasiliev's equations contain a solution that represents spatially homogeneous and isotropic fields on dS background.

Outline

1. Notations for Vasiliev's equations
2. (A)dS backgrounds and perturbations
3. An example in dS: FRW
4. An example in AdS: generalized BTZ in 4D
5. Stargenfunctions with complex eigenvalues

B(H)TZ black hole

Banados, Henneaux, Teitelboim, Zanelli
arXiv:gr-qc/9302012

- The spacetime of 3D BTZ BH is locally the same as AdS$_3$. Starting from the embedding coordinates of AdS$_3$:

$$-(X^{0'})^2 - \left(X^0\right)^2 + \left(X^1\right)^2 + \left(X^2\right)^2 = -l^2$$

The BTZ metric can be achieved simply by a coordinate transformation (for simplicity, only outside the horizon) :

$$X^{0'} = \frac{r}{\sqrt{M}} \cosh\left(\sqrt{M}\phi\right) \;,\; X^0 = \sqrt{\frac{r^2}{M} - l^2} \sinh\left(\frac{\sqrt{M}}{l} t\right)$$

$$X^1 = \frac{r}{\sqrt{M}} \sinh\left(\sqrt{M}\phi\right) \;,\; X^2 = \sqrt{\frac{r^2}{M} - l^2} \cosh\left(\frac{\sqrt{M}}{l} t\right)$$

$$\Rightarrow\; ds^2 = -\left(l^{-2}r^2 - M\right) dt^2 + \left(l^{-2}r^2 - M\right)^{-1} dr^2 + r^2 d\phi^2$$

- For AdS, the coordinate ϕ runs from $-\infty$ to $+\infty$.
  For BTZ, $0 \le \phi < 2\pi$. The identification is imposed by hand.

4D BTZ generalization

Aminneborg, Bengtsson, Holst, Peldan
arXiv:gr-qc/9604005,gr-qc/9705067

- $-(X^{0'})^2-\left(X^0\right)^2+\left(X^1\right)^2+\left(X^2\right)^2+\left(X^3\right)^2=-l^2$
- The 4D generalization of BTZ is done by rotating between the last two spatial directions of the embedding coordinates:

$$X^{0'}=\frac{r}{\sqrt{M}}\cosh\left(\sqrt{M}\phi\right)\ ,\quad X^0=\sqrt{\frac{r^2}{M}-l^2}\,\sinh\left(\frac{\sqrt{M}}{l}t\right)$$

$$X^1=\frac{r}{\sqrt{M}}\sinh\left(\sqrt{M}\phi\right)\ ,\quad X^2=\sqrt{\frac{r^2}{M}-l^2}\,\cosh\left(\frac{\sqrt{M}}{l}t\right)\sin\theta$$

$$X^3=\sqrt{\frac{r^2}{M}-l^2}\,\cosh\left(\frac{\sqrt{M}}{l}t\right)\cos\theta$$

$$\Rightarrow\quad ds^2=-\left(l^{-2}r^2-M\right)dt^2+\left(l^{-2}r^2-M\right)^{-1}dr^2+r^2d\phi^2$$
$$+\left(r^2M^{-1}-l^2\right)\ \cosh^2\left(l^{-1}\sqrt{M}t\right)d\theta^2$$

Vasiliev's equations on (4D) BTZ background

- Q: Can we find such solutions to Vasiliev's equations that at the linearized level they represent perturbative fields on the (4D) BTZ background?
- A: Yes.
  - The background is trivial (locally AdS).
  - Fields must be periodic, i.e. identified at ϕ and $\phi + 2\pi$. (quantized angular momentum along the ϕ-direction)
- In 3D it was done in
  arXiv:hep-th/0612161 Didenko, Matveev, Vasiliev.
- We were doing it in 4D…
  - Before showing how we find the solutions, I first need to introduce the earlier work (Iazeolla & Sundell arXiv:1107.1217) to formulate in the star-product language a quantized system in AdS_4.

Isometry generators

- The AdS$_4$ isometry algebra has 4 types of generators:

$E = -\Gamma_0$ $\quad$ $J = -\Gamma_{12}$

$$-(X^{0'})^2 - (X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 = -l^2$$

$P = -\Gamma_1$ $\quad$ $B = -\Gamma_{03}$

$$(\Gamma^a)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{bmatrix} 0 & (\sigma^a)_\alpha{}^{\dot\beta} \\ (\bar\sigma^a)_{\dot\alpha}{}^{\beta} & 0 \end{bmatrix}, \quad (\Gamma^{ab})_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{bmatrix} (\sigma^{ab})_\alpha{}^{\beta} & 0 \\ 0 & (\bar\sigma^{ab})_{\dot\alpha}{}^{\dot\beta} \end{bmatrix}$$

- The underlined indices are contracted with the indices of $Y^{\underline{\alpha}} = (y^\alpha, \bar y^{\dot\alpha})$ and are raised or lowered by

$$C^{\underline{\alpha\beta}} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} \text{ and } C_{\underline{\alpha\beta}} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot\alpha\dot\beta} \end{pmatrix}.$$

Ladder operators and eigenstates

- To construct a quantum system, we need two Cartan generators, e.g. (E, J)
- In this case we can define two pairs of ladder operators:

$$a_1^+ = \frac{1}{2}\left(y^1 - i\bar{y}^2\right) \;,\; a_1^- = \frac{1}{2}\left(\bar{y}^1 + iy^2\right) \;;$$
$$a_2^+ = \frac{1}{2}\left(\bar{y}^1 - iy^2\right) \;,\; a_2^- = \frac{1}{2}\left(y^1 + i\bar{y}^2\right) \;.$$

- For each pair we have a number operator:

$$w_1 = a_1^+ \star a_1^- + \frac{1}{2} = a_1^+ a_1^- = \frac{1}{8}\left(E_{\underline{\alpha\beta}} - J_{\underline{\alpha\beta}}\right) Y^{\underline{\alpha}} Y^{\underline{\beta}}$$
$$w_2 = a_2^+ \star a_2^- + \frac{1}{2} = a_2^+ a_2^- = \frac{1}{8}\left(E_{\underline{\alpha\beta}} + J_{\underline{\alpha\beta}}\right) Y^{\underline{\alpha}} Y^{\underline{\beta}}$$

which are the difference and the sum of E and J .

Ladder operators and eigenstates

- The ground state (four eigenvalues all =½)

$$f_{\lambda_{1L}=\frac{1}{2},\ \lambda_{1R}=\frac{1}{2},\ \lambda_{2L}=\frac{1}{2},\ \lambda_{2R}=\frac{1}{2}} = 4e^{-\frac{1}{2}E_{\underline{\alpha\beta}}Y^{\underline{\alpha}}Y^{\underline{\beta}}},$$

left & right-eigenvalues of w_1 & w_2

- The excited states are obtained by doing $a_+ \star$ from the left and $\star\, a_-$ from the right, e.g.

$$a_1^+ \star f_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}} \star a_1^- \;=\; f_{\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}} \;=\; 4e^{-\frac{1}{2}E_{\underline{\alpha\beta}}Y^{\underline{\alpha}}Y^{\underline{\beta}}}(4w_1 - 1)$$

(If the left and right eigenvalues are the same, the excited states are always the ground state multiplied by polynomials of the number operators.)

Ladder operators and eigenstates

- We would like to construct this kind of system on the (4D) BTZ background.

$$X^{0'} = \frac{r}{M}\cosh\left(\sqrt{M}\phi\right) \;\;,\;\; X^{0} = \sqrt{\frac{r^2}{M} - l^2}\,\sinh\left(\frac{\sqrt{M}}{l}t\right)$$

$$X^{1} = \frac{r}{M}\sinh\left(\sqrt{M}\phi\right) \;\;,\;\; X^{2} = \sqrt{\frac{r^2}{M} - l^2}\,\cosh\left(\frac{\sqrt{M}}{l}t\right)\sin\theta$$

$$X^{3} = \sqrt{\frac{r^2}{M} - l^2}\,\cosh\left(\frac{\sqrt{M}}{l}t\right)\cos\theta$$

- On such a background, instead of the compact generators E and J, it is actually the non-compact generators B and P (boost and spatial transvection of AdS) that function as the generators of the time transvection and the spatial rotation of (4D) BTZ.

Ladder operators and eigenstates

- (B, P) are non-compact generators. To construct a similar quantum system as (E, J), we use (iB, iP) instead.
- In this case we can define two pairs of ladder operators:

$$
\begin{aligned}
a_1^+ &= \frac{1}{2}\left(y^1 + \bar{y}^1\right) \ , \ a_1^- = \frac{i}{2}\left(y^2 + \bar{y}^2\right) \ ; \\
a_2^+ &= \frac{i}{2}\left(y^1 - \bar{y}^1\right) \ , \ a_2^- = \frac{1}{2}\left(y^2 - \bar{y}^2\right) \ .
\end{aligned}
$$

- For each pair we have a number operator:

$$
\begin{aligned}
w_1 &= a_1^+ \star a_1^- + \frac{1}{2} = a_1^+ a_1^- = \frac{1}{8}\left(iB_{\underline{\alpha\beta}} - iP_{\underline{\alpha\beta}}\right) Y^{\underline{\alpha}} Y^{\underline{\beta}} \\
w_2 &= a_2^+ \star a_2^- + \frac{1}{2} = a_2^+ a_2^- = \frac{1}{8}\left(iB_{\underline{\alpha\beta}} + iP_{\underline{\alpha\beta}}\right) Y^{\underline{\alpha}} Y^{\underline{\beta}}
\end{aligned}
$$

which are the difference and the sum of iB and iP .

Ladder operators and eigenstates

- The ground state (four eigenvalues all =½)

$$f_{\lambda_{1L}=\frac{1}{2},\ \lambda_{1R}=\frac{1}{2},\ \lambda_{2L}=\frac{1}{2},\ \lambda_{2R}=\frac{1}{2}} = 4e^{-\frac{1}{2}iB_{\underline{\alpha\beta}}Y^{\underline{\alpha}}Y^{\underline{\beta}}}\ ,$$

left & right-eigenvalues of w_1 & w_2

- The excited states are obtained by doing $a^+\star$ from the left and $\star\, a^-$ from the right, e.g.

$$a_1^+ \star f_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}} \star a_1^- \ = \ f_{\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}} \ = \ 4e^{-\frac{1}{2}iB_{\underline{\alpha\beta}}Y^{\underline{\alpha}}Y^{\underline{\beta}}}(4w_1 - 1)$$

(The construction is more-or-less the same by replacing everywhere E with iB and J with iP.)

- All eigenvalues here are real, but for 4D BTZ we must generalize them to complex values. Reasons:
  - quasinormal mode, i.e. BH absorbs (weakens) the fields;
  - it is the imaginary part of the eigenvalues of $w_1 - w_2$ that matters for the periodicity in ϕ (sym. gen. P).

Eigenfunctions

- To obtain eigenfunctions with complex eigenvalues of (iB, iP), we first solve the prototype equations with only one pair of ladder operators:

$$\begin{aligned} \left(a^+a^-\right) \star f\left(a^+, a^-\right) &= \lambda_L f\left(a_+, a_-\right) \\ f\left(a^+, a^-\right) \star \left(a^+a^-\right) &= \lambda_R f\left(a^+, a^-\right) \end{aligned}$$

- We assume that the eigenfunction only depends on the ladder operators. After a straightforward calculation of the star-products, these equations can be converted into two partial differential equations:

$$\begin{aligned} \left(a^+\frac{\partial}{\partial a^+} - a^-\frac{\partial}{\partial a^-}\right) f\left(a^+, a^-\right) &= \left(\lambda_L - \lambda_R\right) f\left(a^+, a^-\right) \\ \left(2a^+a^- - \frac{1}{2}\frac{\partial^2}{\partial a^+\partial a^-}\right) f\left(a^+, a^-\right) &= \left(\lambda_L + \lambda_R\right) f\left(a^+, a^-\right) \end{aligned}$$

Eigenfunctions

- The simplest solution is

$$f_{\lambda_L,\lambda_R}\left(a^+,a^-\right) \;=\; \mathrm{C}\;\left(a^+\right)^{\lambda_L-\lambda_R} e^{-2a^+a^-} L^{\lambda_L-\lambda_R}_{\lambda_R-\frac{1}{2}}\left(4a^+a^-\right)$$

"generalized Laguerre function"

- We need to rewrite it for convenience of further star-product calculation:

$$f_{\lambda_L,\lambda_R}\left(a^+,a^-\right) \;=\; \mathrm{C}'\int_0^{+\infty} d\tau \frac{\tau^{\lambda_R-\lambda_L-1}}{\Gamma\left(\lambda_R-\lambda_L\right)} e^{-\tau a^+} \oint_{-1} \frac{ds}{2\pi i}\; e^{2sa^+a^-}\frac{\left(s-1\right)^{\lambda_L-\frac{1}{2}}}{\left(s+1\right)^{\lambda_R+\frac{1}{2}}}$$

- Prescription of analytic continuation is needed;
- Here only λ_L is generalized to complex values, we have to assume λ_R is a positive half-integer to make the contour integral well-defined.

Eigenfunctions

- Remember that we have two pairs of ladder operators, so we make a copy of the eigenfunction for each:

$$f^{(1)}_{\lambda_{1L},\lambda_{1R}}\left(a_1^+,a_1^-\right) \;\propto\; \int_0^{+\infty} d\tau \frac{\tau^{\lambda_{1R}-\lambda_{1L}-1}}{\Gamma\left(\lambda_{1R}-\lambda_{1L}\right)} e^{-\tau a_1^+} \oint_{-1} \frac{ds}{2\pi i}\; e^{2sa_1^+a_1^-} \frac{(s-1)^{\lambda_{1L}-\frac{1}{2}}}{(s+1)^{\lambda_{1R}+\frac{1}{2}}}$$

$$f^{(2)}_{\lambda_{2L},\lambda_{2R}}\left(a_2^+,a_2^-\right) \;\propto\; \int_0^{+\infty} d\tau \frac{\tau^{\lambda_{2R}-\lambda_{2L}-1}}{\Gamma\left(\lambda_{2R}-\lambda_{2L}\right)} e^{-\tau a_2^+} \oint_{-1} \frac{ds}{2\pi i}\; e^{2sa_2^+a_2^-} \frac{(s-2)^{\lambda_{2L}-\frac{1}{2}}}{(s+2)^{\lambda_{2R}+\frac{1}{2}}}$$

- Then we take a (star-)product of the two copies:

$$f_{\lambda_{1L},\lambda_{1R},\lambda_{2L},\lambda_{2R}} \;=\; f^{(1)}_{\lambda_{1L},\lambda_{1R}} \star f^{(2)}_{\lambda_{2L},\lambda_{2R}} \;=\; f^{(1)}_{\lambda_{1L},\lambda_{1R}} f^{(2)}_{\lambda_{2L},\lambda_{2R}}$$

- This eigenfunction will be used in the initial data for the Vasiliev's equations. By definition it satisfies

$$\left(-\tfrac{i}{4}P_{\underline{\alpha\beta}}Y^{\underline{\alpha}}Y^{\underline{\beta}}\right)\star f \;=\; \left(\lambda_{1L}-\lambda_{2L}\right) f$$

$$f\star\left(-\tfrac{i}{4}P_{\underline{\alpha\beta}}Y^{\underline{\alpha}}Y^{\underline{\beta}}\right) \;=\; \left(\lambda_{1R}-\lambda_{2R}\right) f$$

Finite transformation

- Q: What is the consequence of imposing the periodicity condition that identifies $\phi \rightarrow \phi + 2\pi$?

- The finite transformation of the sym. gen. P for $\phi \rightarrow \phi + \varphi$

$$\gamma \;=\; e_\star^{-\frac{i}{8}\sqrt{M}\varphi P_{\underline{\alpha\beta}} Y^{\underline{\alpha}} Y^{\underline{\beta}}} \;=\; e_\star^{\frac{1}{2}\sqrt{M}\varphi(w_1 - w_2)}$$

- If $f_{\lambda_{1L},\lambda_{1R},\lambda_{2L},\lambda_{2R}}$ lives in the adjoint representation

$$f \;\rightarrow\; \gamma^{-1} \star f \star \gamma \quad = \quad e^{\frac{1}{2}\sqrt{M}[-(\lambda_{1L}-\lambda_{2L})+(\lambda_{1R}-\lambda_{2R})]\varphi} f$$

then obviously to identify $\phi \rightarrow \phi + 2\pi$ we shall impose

$$\frac{1}{2}\sqrt{M}\,[-(\lambda_{1L} - \lambda_{2L}) + (\lambda_{1R} - \lambda_{2R})] \;\in\; i\mathbb{Z}$$

Initial data

- The initial data can be constructed as

$$\Phi' = \sum_{\lambda_{1L},\lambda_{1R},\lambda_{2L},\lambda_{2R}} \nu_{\lambda_{1L},\lambda_{1R},\lambda_{2L},\lambda_{2R}}\ f_{\lambda_{1L},\lambda_{1R},\lambda_{2L},\lambda_{2R}} \star \kappa_y$$
$$+ \quad \text{conjugate terms}$$

- The sums are done over the λ's that are subject to the periodicity condition and the bosonic condition.

- The conjugate terms are $\pi\,(\cdot^{\dagger})$ of all the terms above, to make sure that Φ' satisfies the reality condition.

Scalar field from particular terms

- To get a bit more intuition, we have examined some particular terms in the sums where

$$\lambda_{1L} \;=\; \frac{1}{2} + \frac{n}{\sqrt{M}} i \;=\; \lambda_{2L}^{*} \quad (n \in \mathbb{Z}) \;, \quad \lambda_{1R} = \lambda_{2R} = \frac{1}{2} \;.$$

- We switch on the gauge function for AdS background, and set the Y-coordinates to zero to extract the scalar field:

$$C \;=\; L^{-1} \star f_{\frac{1}{2}+\frac{n}{\sqrt{M}}i,\; \frac{1}{2}-\frac{n}{\sqrt{M}}i,\; \frac{1}{2},\; \frac{1}{2}} \star \kappa_y \star \pi(L) \Big|_{y^\alpha = \bar{y}^{\dot{\alpha}} = 0}$$

- Then we do a coordinate transformation for the 4D BTZ:

$$C \propto e^{-in\phi} \frac{\cosh\left\{ \frac{n}{\sqrt{M}} \arccos\left[\sqrt{1 - \frac{l^2 M}{r^2}} \cosh\left(l^{-1}\sqrt{M} t \right) \sin(\theta) \right] \right\}}{\sqrt{\frac{r^2}{l^2 M} - \left(\frac{r^2}{l^2 M} - 1 \right) \cosh^2\left(l^{-1}\sqrt{M} t \right) \sin^2(\theta)}}$$

(The K-G eq $(\Box + 2l^{-2})C = 0$ is satisfied.)

$$\lambda_{1L} = \frac{1}{2} + \frac{n}{\sqrt{M}} i = \lambda_{2L}^* \quad (n \in \mathbb{Z}) \ , \quad \lambda_{1R} = \lambda_{2R} = \frac{1}{2}$$

$$\Rightarrow \quad C \propto e^{-in\phi} \cdots$$

Further discussion

- Key point of this model:

  The excitation of the (angular) momentum along the compactified direction corresponds to adding quantized imaginary numbers to the eigenvalues in the initial data.

- Further questions about the initial data:

  Q1: Can we set both of the left and right eigenvalues to complex numbers?

  Q2: With complex eigenvalues, is it possible to relate all eigenfunctions by star-multiplying creation and annihilation operators? e.g. $f_{\frac{1}{2}+\frac{n}{\sqrt{M}}i\ ,\frac{1}{2}} = \left(a^{+}\right)^{\frac{n}{\sqrt{M}}i} \star f_{\frac{1}{2},\frac{1}{2}}$?

Outline

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3. An example in dS: FRW

4. An example in AdS: generalized BTZ in 4D

5. <u>Stargenfunctions with complex eigenvalues</u>

More eigenfunctions

$$\begin{aligned}\left(a^+a^-\right)\star f\left(a^+,a^-\right) &= \lambda_L f\left(a_+,a_-\right)\\ f\left(a^+,a^-\right)\star\left(a^+a^-\right) &= \lambda_R f\left(a^+,a^-\right)\end{aligned}$$

$$\begin{aligned}\left(a^+\frac{\partial}{\partial a^+}-a^-\frac{\partial}{\partial a^-}\right)f\left(a^+,a^-\right) &= \left(\lambda_L-\lambda_R\right)f\left(a^+,a^-\right)\\ \left(2a^+a^- - \frac{1}{2}\frac{\partial^2}{\partial a^+\partial a^-}\right)f\left(a^+,a^-\right) &= \left(\lambda_L+\lambda_R\right)f\left(a^+,a^-\right)\end{aligned}$$

- Remember that our solution for the eigenfunction was:

$$f_{\lambda_L|\lambda_R}\left(a^+,a^-\right) = \mathrm{C}_1\ \left(a^+\right)^{\lambda_L-\lambda_R}e^{-2a^+a^-}L^{\lambda_L-\lambda_R}_{\lambda_R-\frac{1}{2}}\left(4a^+a^-\right)$$

but be aware that the above set of equations is invariant, if we simultaneously exchange

$$a^+\leftrightarrow a^- \quad , \quad \lambda_L\leftrightarrow\lambda_R \quad .$$

Therefore, we can create a new branch of solutions from the old by doing the same exchange…

More eigenfunctions

$$
\begin{aligned}
\left(a^+a^-\right) \star f\left(a^+, a^-\right) &= \lambda_L f\left(a_+, a_-\right) \\
f\left(a^+, a^-\right) \star \left(a^+a^-\right) &= \lambda_R f\left(a^+, a^-\right)
\end{aligned}
$$

$$
\begin{aligned}
\left(a^+\frac{\partial}{\partial a^+} - a^-\frac{\partial}{\partial a^-}\right) f\left(a^+, a^-\right) &= \left(\lambda_L - \lambda_R\right) f\left(a^+, a^-\right) \\
\left(2a^+a^- - \frac{1}{2}\frac{\partial^2}{\partial a^+ \partial a^-}\right) f\left(a^+, a^-\right) &= \left(\lambda_L + \lambda_R\right) f\left(a^+, a^-\right)
\end{aligned}
$$

- The "plus" branch and the "minus" branch:

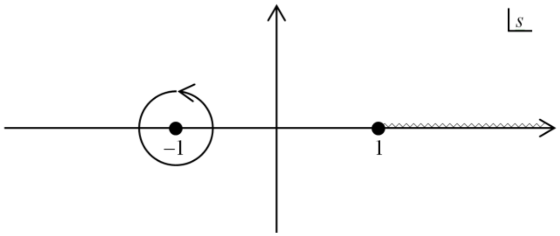
$$
\begin{aligned}
f_{\lambda_L|\lambda_R}\left(a^+, a^-\right) = \; & \mathrm{C}_1 \left(a^+\right)^{\lambda_L-\lambda_R} e^{-2a^+a^-} L^{\lambda_L-\lambda_R}_{\lambda_R-\frac{1}{2}}\left(4a^+a^-\right) \\
& + \mathrm{C}_2 \left(a^-\right)^{\lambda_R-\lambda_L} e^{-2a^+a^-} L^{\lambda_R-\lambda_L}_{\lambda_L-\frac{1}{2}}\left(4a^+a^-\right)
\end{aligned}
$$

- The two branches are different, except for $\lambda_R - \lambda_L \in \mathbb{Z}$.
- From the perspective of solving the above differential equations, there is no restriction at all for the eigenvalues – they can be any complex numbers.

Complex eigenvalues

$$f_{\lambda_L|\lambda_R}\left(a^+,a^-\right) = \mathrm{C}_1\ \left(a^+\right)^{\lambda_L-\lambda_R} e^{-2a^+a^-} L^{\lambda_L-\lambda_R}_{\lambda_R-\frac{1}{2}}\left(4a^+a^-\right) + \mathrm{C}_2\ \left(a^-\right)^{\lambda_R-\lambda_L} e^{-2a^+a^-} L^{\lambda_R-\lambda_L}_{\lambda_L-\frac{1}{2}}\left(4a^+a^-\right)$$

- However, the restriction happens when we have to express the Laguerre functions in terms of integrals.
- For example, previously we used the contour integral:



$$\oint_{-1}\frac{ds}{2\pi i}\ e^{2sa^+a^-}\frac{(s-1)^{\lambda_L-\frac{1}{2}}}{(s+1)^{\lambda_R+\frac{1}{2}}}$$

We could not generalized λ_R , because otherwise there would be a branch cut intersecting the contour.

- Therefore, for generic complex eigenvalues, we have to use different integrals.

New integral representations

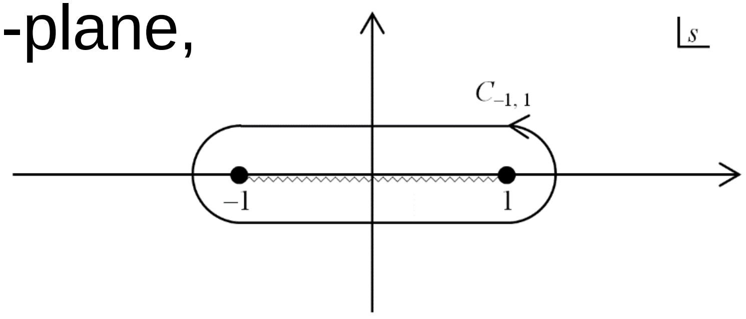
- Fortunately, to express formulas like $e^{-2a^+a^-}L^{\lambda_L-\lambda_R}_{\lambda_R-\frac{1}{2}}(4a^+a^-)$ we have alternative integrals (with similar integrands).
- Two options:

  - integrate on the real axis: $$\int_{-1}^{1} ds\ e^{2sa^+a^-}\frac{(1-s)^{\lambda_L-\frac{1}{2}}}{(1+s)^{\lambda_R+\frac{1}{2}}}$$

  (Conditions e.g. $\mathrm{Re}(\lambda_L) > -\frac{1}{2},\ \mathrm{Re}(\lambda_R) < \frac{1}{2}$ are needed for convergence – but with analytic continuation, we formally use the integrals to represent Laguerre functions)

  - or equivalently take contour integrals that are compatible with branch cuts on the complex s-plane, for example: $\lambda_L = \lambda_R = \lambda \in \mathbb{C}$

$$\oint_{C_{-1,1}} ds\ e^{2sa^+a^-}\frac{(s-1)^{\lambda-\frac{1}{2}}}{(s+1)^{\lambda+\frac{1}{2}}}$$



Star-products with creation / annihilation operators

- Now with the new integral representations, we can do star-products with generic complex eigenvalues.

- One interesting result:

  Take a diagonal $$f_{\lambda|\lambda}\left(a^+, a^-\right) = \int_{-1}^{1} e^{2ws} \frac{(1-s)^{\lambda-\frac{1}{2}}}{(1+s)^{\lambda+\frac{1}{2}}} ds$$

  star-multiplied with $$\left(a^{\pm}\right)^p = \int_0^{+\infty} \frac{\tau^{-p-1}}{\Gamma(-p)} e^{-\tau a^{\pm}} d\tau$$

  we obtain:

  $$\begin{aligned} f^{+}_{\lambda_L|\lambda_R} &= \left(a^{+}\right)^{\lambda_L-\lambda_R} \star f_{\lambda_R|\lambda_R} = f_{\lambda_L|\lambda_L} \star \left(a^{+}\right)^{\lambda_L-\lambda_R} \\ f^{-}_{\lambda_L|\lambda_R} &= \left(a^{-}\right)^{\lambda_R-\lambda_L} \star f_{\lambda_R|\lambda_R} = f_{\lambda_L|\lambda_L} \star \left(a^{-}\right)^{\lambda_R-\lambda_L} \end{aligned}$$

  i.e. creation $\rightarrow$ plus branch

  annihilation $\rightarrow$ minus branch (left or right doesn't matter)

- Q: What if both a^+ and a^- to complex powers are multiplied?

Star-products with creation / annihilation operators

- Q: What if both a^+ and a^- to complex powers are multiplied,

  like $\left(a^+\right)^{\lambda^+} \star \left(a^-\right)^{\lambda^-} \star f_{\lambda|\lambda}$?

- It remains to be an eigenfunction of the number operator.

$$\int_0^{+\infty} d\tau^+ \int_0^{+\infty} d\tau^- \int_{-1}^{1} ds \; \frac{\left(\tau^+\right)^{-\lambda^+-1}}{\Gamma\left(-\lambda^+\right)} \frac{\left(\tau^-\right)^{-\lambda^--1}}{\Gamma\left(-\lambda^-\right)} \frac{(1-s)^{\lambda-\frac{1}{2}}}{(1+s)^{\lambda+\frac{1}{2}}} \; e^{-\tau^+a^+} \star e^{-\tau^-a^-} \star e^{2sw}$$

$w \star$ gives $\lambda + \lambda^+ - \lambda^-$

$\star\, w$ gives λ

(Associativity: Important to do all star-products before integrations.)

- However, in general it is a combination of different branches (may involve the missing branch in the case of degeneracy).

Further problems at the non-linear level

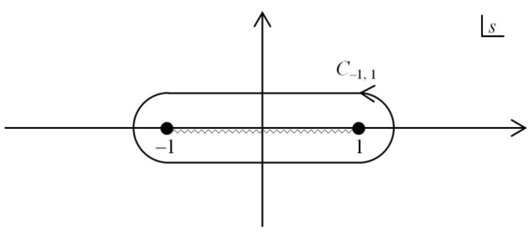
- More problems come when we take the star-product of two eigenfunctions.
- For example: $$f_{\lambda|\lambda}\left(a^{+},a^{-}\right)=\int_{-1}^{1}e^{2ws}\frac{(1-s)^{\lambda-\frac{1}{2}}}{(1+s)^{\lambda+\frac{1}{2}}}ds$$

  What we expect: $f_{\lambda|\lambda}\star f_{\lambda|\lambda}\propto f_{\lambda|\lambda}$ (true for half-integer λ)

  What we actually get:

  $$f_{\lambda|\lambda}\star f_{\lambda|\lambda}\ =\ f_{\lambda|\lambda}\int_{-1}^{1}ds\frac{1}{(1+s)\,(1-s)}\,,$$ divergent!

- However, if we replace the real-axis integral by the contour:



$$\oint_{C_{-1,1}}ds\frac{1}{(s+1)\,(s-1)}=0\quad\Rightarrow f_{\lambda|\lambda}\star f_{\lambda|\lambda}=0$$

(How to interpret?)

Summary

- Vasiliev's equations have a large variety of solutions that at the linearized level resemble simple systems on (A)dS background that we are familiar with.
- In my talk, I have shown an example in dS and another in AdS.
  Physically I have not shown anything new. At the linearized level, they are just simple well-known models reformulated in a different mathematical language – the star-product in Vasiliev's equations.
  However, new insights are expected to come at higher-orders of the perturbation, as we transform the solutions from the "L" gauge to Vasiliev's gauge (future work).
- In the last part of my talk, I have also shown some recent attempt to enlarge the solution space by allowing generic complex eigenvalues for the eigenfunctions in the initial data. Within certain subsets of the eigenfunctions, creation and/or annihilation operators act nicely.
  Much more work has to be done (e.g. normalization, closure of the algebra) to organize these eigenfunctions into quantum systems and to use them to build solutions to Vasiliev's equations.

Thank you !