Positive Geometry and Amplitu-hedron
正几何与振幅多面体

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- Positive $d\log$ Forms ("正"对数微分形式)
- Geometrical Meaning
- Physical Correspondence: Planar $\mathcal{N}=4$ super Yang-Mills Amplitudes

1/7: Positive $d\log$ Forms

- A divergent integral in the positive region:

$$\int_0^\infty \frac{dx}{x} = \int d\log x \tag{1}$$

for $x > 0$. Its residue at 0 or ∞ is ± 1.

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- A projective length (投影长度)? More nontrivial cases with shifting:

$$\int \frac{dx}{x-a} \tag{2}$$

for $x > a$, and its complement

$$\int \frac{dx}{x} - \int \frac{dx}{x-a} = \int \frac{dx}{x} \frac{a}{a-x} \tag{3}$$

for $x < a$, with two boundaries at 0 and a (residue $= \pm 1$).

2/7: Positive $d\log$ Forms

- Completeness relation and dimensionless ratio (无量纲比):

$$\int \frac{dx}{x}\left[\frac{x}{x-a}+\frac{a}{a-x}=1\right], \tag{4}$$

naturally a is also treated as a positive variable.

2/7: Positive $d\log$ Forms

- Completeness relation and dimensionless ratio (无量纲比):

$$\int \frac{dx}{x}\left[\frac{x}{x-a}+\frac{a}{a-x}=1\right], \tag{4}$$

naturally a is also treated as a positive variable.

- Generalization by induction for $\sum_n x_i > a$ (!加法与乘法):

$$\frac{\sum_{n-1} x_i}{\sum_{n-1} x_i - a}\times 1+\frac{a}{a-\sum_{n-1} x_i}\times\frac{x_n}{x_n-\left(a-\sum_{n-1} x_i\right)}=\frac{\sum_n x_i}{\sum_n x_i - a}, \tag{5}$$

naturally a can be a positive sum:

$$\frac{\sum_n x_i}{\sum_n x_i - \sum_m y_j} \tag{6}$$

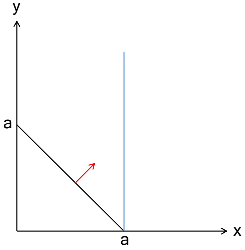
for $\sum_n x_i > \sum_m y_j$.

3/7: Geometrical Meaning

- A simplest nontrivial example: $x+y>a$ with

$$\frac{x}{x-a}\times 1+\frac{a}{a-x}\times\frac{y}{y-(a-x)}=\frac{x+y}{x+y-a}, \qquad (7)$$

internal boundaries or spurious poles cancel (内部边界相消).



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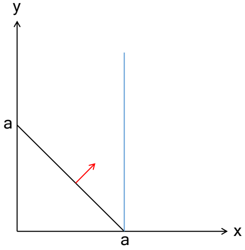

- Adding the a axis: a plane touching the origin (投影面积→体积).

4/7: Geometrical Meaning

- Most general ratio for a positive linear polynomial:

$$P(x_i, y_j) = P_0(y_j) + \sum x_i\, P_i(y_j) > 0, \tag{8}$$

$$\frac{P_0^+ + \sum x_i\, P_i^+}{P_0 + \sum x_i\, P_i} = \frac{P^+}{P}. \tag{9}$$

Examples:

$$P = 1-(x+y+z),\ Q = 1-x(1-y(1-z)),\ R = (1-x)(1-y)(1-z). \tag{10}$$

It is still a plane in arbitrarily high dimensions that carves out the corresponding volume.

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- Multiple positive linear polynomials: nontrivial interceptions.

5/7: Physical Correspondence: Planar $\mathcal{N}=4$ super Yang-Mills Amplitudes

- 4-particle (MHV or $k=2$) L-loop amplitude/integrand defined by:

$$D_{ij} = (x_j - x_i)(z_i - z_j) + (y_j - y_i)(w_i - w_j) > 0 \qquad (11)$$

for $i, j = 1, \ldots, L$.

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- 4-particle 2-loop integrand defined by:

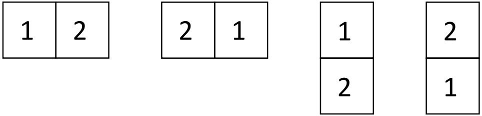
$$D_{12} = (x_2 - x_1)(z_1 - z_2) + (y_2 - y_1)(w_1 - w_2) > 0. \qquad (12)$$

6/7: Physical Correspondence

- Trivially we get

$$\frac{D_{12}^{+}}{D_{12}} = \frac{x_2 z_1 + x_1 z_2 + y_2 w_1 + y_1 w_2}{(x_2 - x_1)(z_1 - z_2) + (y_2 - y_1)(w_1 - w_2)} \tag{13}$$

which corresponds to 4 diagrams:



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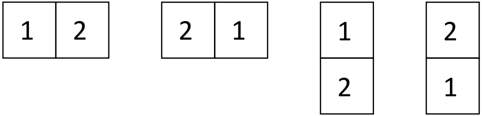
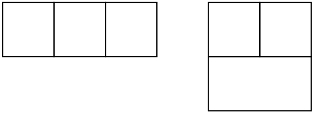


- At 3-loop order, $D_{12}, D_{13}, D_{23} > 0$ and we have 2 topologies:



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  $n > 4$ MHV $L > 2$ case with various combinations of 1-loop cells?

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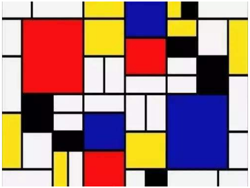

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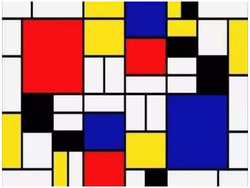

- 简单原理→复杂计算→复杂整理→简单结果

- Thank you!
- Some References:

  1312.2007

  1312.7878

  1712.09990

  1812.01822

  1910.14612

  2007.15650