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A Correspondence
between
Ricci-flat Kerr
and
Charged Kaluza-Klein AdS Black Hole

and
Negative Dimensions?

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Introduction

Dualities are perhaps more important in string theory than any other branch of physics.

- Electromagnetic duality: within the same spacetime dimensions.
- T-duality and U-duality: manifest as global symmetries in Kaluza-Klein reduced low-energy EFTs. [BHO,CLPS,Cremmer;Julia;HT;...]
- The AdS/CFT correspondence: an alternative to compactification. [Maldacena; KGP; Witten; ...]
- AdS/Ricci-flat correspondence: another alternative to compactification and the focus of this talk. [Caldarelli, Camps, Gouteraux, Skenderis]

$$\sqrt{|\hat{g}|}\hat{R} \qquad \underset{???}{\leftrightarrow} \qquad \sqrt{|\tilde{g}|}(\tilde{R}-2\Lambda)\,.$$

The AdS/Ricci-flat correspondence

CCGS 1312.7874

Minkowski spacetime in $\hat{D} = p + 2 + n$ dimensions

$$ds^2_{\hat{D}} = -dt^2 + dx^i dx^i + dr^2 + r^2 d\Omega_n^2 = \frac{r^2}{\ell^2}\Big(ds^2_{p+2} + \ell^2 d\Omega_n^2\Big)\,,$$

which is conformal to a direct product of S^n and AdS$_{p+2}$

$$ds^2_{p+2} = \frac{\ell^2}{r^2}\Big(- dt^2 + dx^i dx^i + dr^2\Big),$$

This can be mapped to the AdS vacuum of Einstein gravity with a negative cosmological constant in $\widetilde{D} = p + 2 + q$ on torus $\mathcal{T}^q$

$$d\tilde{s}^2_{\widetilde{D}} = ds^2_{p+2} + \frac{\ell^2}{r^2} ds^2_{\mathcal{T}^q}\,,$$

provided with

$$q \leftrightarrow -(\hat{D} - 2)\,, \qquad \Leftrightarrow \qquad n \leftrightarrow -(\tilde{D} - 2)\,.$$

Negative dimensions involved!

Objection and resolution

The middle of Minkowski $r = 0$ corresponds to the AdS boundary $r \to \infty$, via $r \to \ell^2/r$.

In the (asymptotic Minkowski) Schwarzschild black hole, $r = 0$ is singular and could not possibly map to the AdS boundary.

The negative dimension saves the day:

$$f = 1 - \frac{\mu}{r^{n-1}} \to f = 1 - \mu r^{\tilde{D}-1}\,.$$

In negative dimensions, $r \to 0$ formally gives to the flat region, whilst $r \to \infty$ is singular:

$$\mathrm{Riem}^2 = \frac{\mu^2}{r^{2(n+1)}}\,.$$

The Ricci-flat Schwarzschild black hole can then map to the Schwarzschild-AdS black hole.

Including matter

We would like to extend the correspondence by including matter, Maxwell field A_μ in particular.

Adding Maxwell field naively to both theories will break the correspondence. New inspiration is needed:

- In string theory, duality typically involves interchange matter fields with geometrical ones.
- T-duality interchanges the winding modes (matter) to the Kaluza-Klein modes (geometry). Buscher, 1987]
- U-duality enhances the $GL(n, R)$ geometrical symmetry to $E_{n(+n)}$ group by including appropriate matter. [CJLP]
- The AdS/Ricci-flat correspondence also interchanges the Ricci scalar of the sphere with (matter) cosmological constant.

We thus expect the extended correspondence interchanges the Kaluza-Klein vector with a matter vector.

Which theory has the matter and which gives the Kuluza-Klein?

Something peculiar about odd-dimensional sphere

In odd $n = 2m + 1$ dimensions, the sphere can be expressed as a $U(1)$ bundle over $\mathbb{CP}^m$:

$$d\Omega_n^2 = \sigma^2 + d\Sigma_m^2 \,, \qquad \sigma = d\psi + B^{(m)} \,, \qquad n = 2m + 1 \,,$$

where $J = 2dB^{(m)}$ is the Kähler 2-form of the $\mathbb{CP}^m$, whose metric is $d\Sigma_m^2$, of $2m$ real dimensions.

The isometry is preserved with the following two operations:

- Squash the fibre direction.
- Include the Kaluza-Klein vector in the fibre $\sigma \to \sigma + A$ with $A = A_\mu dx^\mu$.

Thus we consider theory-A as simply pure Einstein gravity, prepared to be reduced on the $U(1)$ bundle over $\mathbb{CP}^m$.

The other theory

The theory-B is the Kaluza-Klein AdS ("super") gravity

$$\tilde{\mathcal{L}} = \sqrt{|\tilde{g}|}\Big(\tilde{R} - \frac{1}{2}(\partial\phi)^2 - V(\phi) - \frac{1}{4}e^{a\phi}F^2\Big),$$

$$a = \sqrt{\frac{2(\widetilde{D}-1)}{\widetilde{D}-2}}, \qquad V = \Big(\frac{dW}{d\phi}\Big)^2 - \frac{\tilde{D}-1}{2(\tilde{D}-2)}W^2,$$

$$W = \frac{1}{\ell\sqrt{2}}\Big((\tilde{D}-3)e^{-\frac{(\tilde{D}-1)\phi}{\sqrt{(\tilde{D}-1)(\tilde{D}-2)}}} + (\tilde{D}-1)e^{\frac{(\tilde{D}-3)\phi}{\sqrt{2(\tilde{D}-1)(\tilde{D}-2)}}}\Big).$$

- For $V = 0$, it is simply the Kaluza-Klein theory of pure gravity on S^1.
- For $\widetilde{D} = 4, 5, 6, 7$, the theory can be embedded into maximal gauged supergravities that are consistent sphere reductions of M-theory and type IIB supergravity
- Notable examples include the dyonic AdS black hole in $\widetilde{D} = 4$ [LLP], general charged rotating black holes in all $\widetilde{D}$ [Wu].
- We can thus establish a link between Ricci-flat metrics to M-branes and D3-branes.

Prepare for Kaluza-Klein reduction

Theory-A:

$$\hat{\mathcal{L}} = \sqrt{|\hat{g}|}\hat{R}\,,$$

on

$$d\hat{s}^2_{\hat{D}} = X^2\Big(ds^2_{p+2} + \ell^2\big(U^{-2(n-1)}(\sigma + \ell^{-1}\mathcal{A})^2 + U^2 d\Sigma^2_m\big)\Big)\,.$$

Theory-B:

$$\begin{aligned}\tilde{\mathcal{L}} \;&=\; \sqrt{|\tilde{g}|}\Big(\tilde{R} - \tilde{n}(\tilde{n}-1)\Phi^{-2}(\partial\Phi)^2 - \frac{1}{4}\Phi^{-2(\tilde{n}-1)}F^2 \\ &\qquad -\frac{1}{\ell^2\Phi^2}\big((\tilde{n}-1)\Phi^{-2\tilde{n}} - (\tilde{n}^2-1)\big)\Big),\end{aligned}$$

where

$$\tilde{n} = -(\widetilde{D}-2), \qquad \Phi = \exp\left(\frac{\phi}{\sqrt{2(\widetilde{D}-1)(\widetilde{D}-2)}}\right),$$

on

$$d\tilde{s}^2_{\widetilde{D}} = d\tilde{s}^2_{p+2} + Y^2 ds^2_{\mathcal{T}^q}\,.$$

The Correspondence

We find that the two lower-dimensional theories are the same, up to the field redefinitions

$$U = \Phi\,, \qquad \mathcal{A} = A\,, \qquad X = Y^{-1}\,,$$

together with the mapping of the parameters of the dimensions

$$n \leftrightarrow \tilde{n}\,,$$

which implies the two parallel and equivalent maps:

$$q \leftrightarrow -(\hat{D}-2)\,, \qquad \Leftrightarrow \qquad n \leftrightarrow -(\tilde{D}-2)\,.$$

The T-duality allows one to map type IIA solutions to type IIB ones and vice versa, creating new solutions on the way.

We can do the same thing here.

It allows us to relate old known solutions and also to construct new ones in M-theory and type IIB supergravities.

Ricci-flat Kerr $\leftrightarrow$ Charged Kaluza-Klein-AdS black hole

Ricci-flat Kerr (equal J) in odd dimensions [Myers, Perry; FGL]

$$ds^2_{n+2} = \frac{dr^2}{f} - \frac{f}{H}dt^2 + r^2 H(\sigma+\omega)^2 + r^2 d\Sigma^2_m\,,$$

$$f = H - \frac{\mu}{r^{n-1}}\,,\qquad H = 1 + \frac{\nu}{r^{n+1}}\,,\qquad \omega = \frac{\sqrt{\mu\nu}}{r^{n+1}+\nu}dt\,.$$

The Ricci-flat metrics in $\widetilde{D} = p + 2 + n$:

$$d\hat{s}^2 = \frac{dr^2}{f} - \frac{f}{H}dt^2 + dx^i dx^i + r^2 H(\sigma+\omega)^2 + r^2 d\Sigma^2_m\,,$$

Performing the map leads to charged Kaluza-Klein AdS black hole

$$\begin{aligned}
ds^2_{\widetilde{D}} &= -\widetilde{H}^{-\frac{\widetilde{D}-3}{\widetilde{D}-2}}\bar{f}dt^2 + \widetilde{H}^{\frac{1}{\widetilde{D}-2}}\Big(\frac{d\tilde{r}^2}{\bar{f}} + \frac{\tilde{r}^2}{\ell^2}(dx^i dx^i + ds^2_{T^q})\Big)\,,\\
A &= \frac{\sqrt{\tilde{\mu}\tilde{\nu}}}{\tilde{r}^{\widetilde{D}-3}+\tilde{\nu}}dt\,,\qquad \Phi = \widetilde{H}^{\frac{1}{2(\widetilde{D}-2)}}\,,\qquad \widetilde{H} = 1 + \frac{\tilde{\nu}}{\tilde{r}^{\widetilde{D}-3}}\,,\\
\bar{f} &= \frac{\tilde{r}^2}{\ell^2}\widetilde{H} - \frac{\tilde{\mu}}{\tilde{r}^{\widetilde{D}-3}}\,,\qquad \tilde{\mu} = \mu\ell^{2(\widetilde{D}-3)}\,,\qquad \tilde{\nu} = \nu\ell^{2(\widetilde{D}-2)}\,.
\end{aligned}$$

Mass and angular momentum

$$M = \frac{n\Omega_n}{16\pi}\mu\,, \qquad J = \frac{\Omega_n}{8\pi}\sqrt{\mu\nu}\,,$$

maps to mass and electric charge

$$\widetilde{M} = \frac{(\widetilde{D}-2)\Omega_{\widetilde{D}-2}}{16\pi}\tilde{\mu}\,, \qquad \tilde{Q} = \frac{(\widetilde{D}-3)\Omega_{\widetilde{D}-2}}{16\pi}\sqrt{\tilde{\mu}\tilde{\nu}}\,.$$

For $\widetilde{D} = 4, 5, 6, 7$, the charged Kaluza-Klein AdS black holes have origins as singly rotating branes.
[10 authors, hep-th/9903214]

Multiply rotating (equal J) Ricci-flat Kerr $\leftrightarrow$ singly rotating fundamental branes.

Gravitational instantons/Domain walls

Gravitational instantons

$$ds_{n+1}^2 = d\rho^2 + a(\rho)^2\sigma^2 + b(\rho)^2 d\Sigma_m^2\,.$$

Bolts:

$$ds_{n+1}^2 = \frac{dr^2}{H} + r^2 H\sigma^2 + r^2 d\Sigma_m^2\,, \qquad H = 1 + \frac{\nu}{r^{n+1}}\,.$$

They are "BPS", with topology $\mathbb{R}^2 \times \mathbb{CP}^m$, asymptotic to $\mathbb{R}^{n+1}/\mathbb{Z}_2$. In four dimensions, it is the well-known Eguchi-Hanson instanton. [Egush,Hanson; Page,Pope;...]

NUTs:

$$\begin{aligned} ds_{n+1}^2 &= \frac{dr^2}{f} + fN^2\sigma^2 + (r^2 - N^2)d\Sigma_m^2\,, \qquad n = 2m+1\,,\\ f &= \Big(1 - \frac{N^2}{r^2}\Big)^{-\frac{1}{2}(n-1)}\Big(-\frac{c}{r^{n-2}}\\ &\quad + \frac{n+1}{n-2}\,{}_2F_1[\frac{1}{2}(1-n), \frac{1}{2}(2-n); \frac{1}{2}(4-n); \frac{N^2}{r^2}]\Big)\,. \end{aligned}$$

They are non-BPS, with topology $\mathbb{R}^{n+1}$, asymptotic to $S^1 \times \mathcal{M}_n$, where $\mathcal{M}_n$ is the Ricci-flat cone of $\mathbb{CP}^m$. The $D=4$ example is the well-known Taub-NUT. [Awad,Chamblin]

Domain Walls

The bolts and nuts are thus mapped to domain walls of Einstein-scalar theory. The bolts lead to

$$
\begin{aligned}
ds^2_{\widetilde{D}} &= \frac{\ell^2 dr^2}{\tilde{r}^2 \widetilde{H}^{\frac{\widetilde{D}-3}{\widetilde{D}-2}}} + \frac{\tilde{r}^2}{\ell^2} \widetilde{H}^{\frac{1}{\widetilde{D}-2}} dx^\mu dx^\nu \eta_{\mu\nu}\,,\\
\Phi &= \widetilde{H}^{\frac{1}{2(\widetilde{D}-2)}}\,, \qquad \widetilde{H} = 1 + \frac{\tilde{\nu}}{\tilde{r}^{\widetilde{D}-3}}\,.
\end{aligned}
$$

These are precisely the known domain walls that are related to distributed M-branes and D3-branes. [KLT; RS; CGLP]

The NUTs lead to

$$
\begin{aligned}
ds^2_{\widetilde{D}} &= \frac{\ell^2}{r^2 - N^2} \Big(\frac{N^2 \tilde{f}}{r^2 - N^2}\Big)^{\frac{1}{\widetilde{D}-2}} \Big(\frac{dr^2}{\tilde{f}} + \eta_{\mu\nu} dx^\mu dx^\nu\Big)\\
\Phi &= \Big(\frac{N^2 \tilde{f}}{r^2 - N^2}\Big)^{\frac{1}{2(\widetilde{D}-2)}}\,.
\end{aligned}
$$

These leads to new M-brane and D3-brane configurations.

Superpotential and domain walls

In Einstein-scalar theory with

$$V = \Big(\frac{dW}{d\phi}\Big)^2 - \frac{\tilde{D}-1}{2(\tilde{D}-2)} W^2 \,,$$

The "BPS" demain wall solution

$$ds^2_{\widetilde{D}} = e^{2B(r)} dr^2 + e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} \,, \qquad \phi = \phi(r) \,.$$

can be solved simply by

$$\phi' = \mp \sqrt{2} e^B \frac{dW}{d\phi} \,, \qquad A' = \pm \frac{e^B W}{\sqrt{2(\widetilde{D}-2)}} \,.$$

Indeed, the bolt walls are associated with the first-order equation, but the nut walls are not.

This is largely correct since the NUTs are not supersymmetric, except for $n+1=4$ dimensions.

But we do not like exceptions. We should except the scalar potential should admit more than just one superpotential.

This is what this correspondence inspires us to do, but would it work?

Two superpotentials in $\widetilde{D}=4$

The scalar potential in Kaluza-Klein AdS gravity is

$$V = -6\ell^{-2}\cosh\left(\frac{1}{\sqrt{3}}\phi\right),$$

we already knew one superpotential via susy transformation rule:

$$W = \frac{1}{\sqrt{2}\ell}\left(3e^{\frac{1}{2\sqrt{3}}\phi} + e^{-\frac{\sqrt{3}}{2}\phi}\right).$$

This gives the bolts wall, and can be lifted to become distributes M2-branes.

The correspondence inspired us to find new one

$$W = \frac{2\sqrt{2}}{\ell}\cosh^{\frac{3}{2}}\left(\frac{1}{\sqrt{3}}\phi\right).$$

This leads to the nuts wall, and can be lifted to become a new distortion of the M2-brane vacua.

It should be pointed out that without susy transformation rule, there is no obvious motivation to search new superpotentials, and hence it was not known.

Conclusions

We establish an explicit map between Einstein gravity on the $U(1)$ bundle over $\mathbb{CP}^m$ and Kaluza-Klein AdS gravity on some tori.

- Ricci-flat Kerr with equal J $\leftrightarrow$ charged Kaluza-Klein AdS black hole. $(M, J) \leftrightarrow (M, Q) \leftrightarrow (M, J_1)$.
- Ricci-flat bolts $\leftrightarrow$ domain walls associated with distributed fundamental branes.
- Ricci-flat bolts $\leftrightarrow$ new domain walls associated with new distortions of the branes
- The correspondence inspired us to find a new superpotential in addition to the known one.

This robustness of the correspondence suggests that we might not simply dismiss the idea of negative spacetime dimensions.

This is not a "hot" correspondence, and not much was done in literature. Our demonstration of its power makes it interesting to investigate it further.

Many unresolved issues remain ...