On Partial Entanglement Entropy

第一届全国“场论与弦论”学术研讨会

中国科学技术大学

2020/11/28

文强

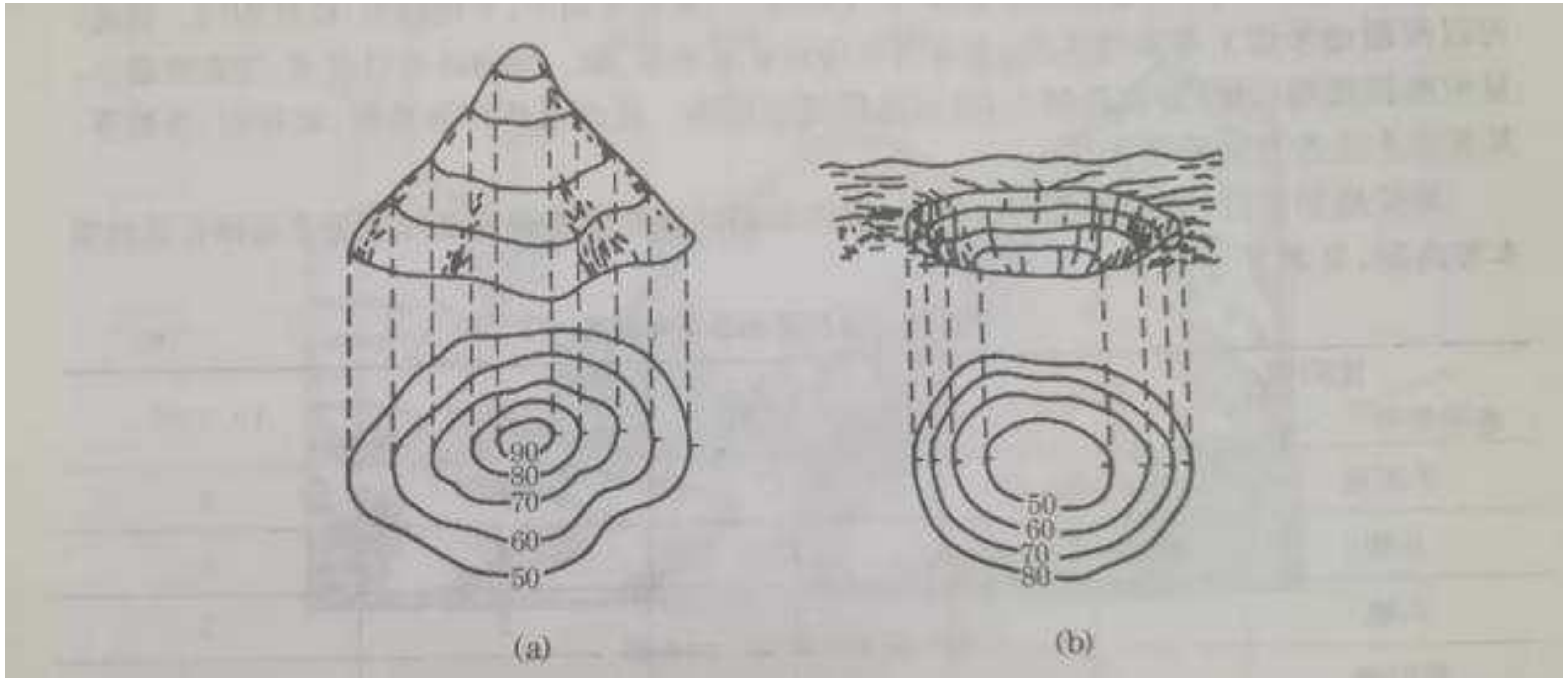
东南大学丘成桐中心

- **Based on my recent papers on Partial Entanglement entropy**
- Fine structure in holographic entanglement and entanglement contour
- **QW**, PRD, **1803.05552;**
- Entanglement contour and modular flow from subset entanglement entropies,
- **QW**, JHEP, **1902.06905;**
- Entanglement entropies from entanglement contour: annuli and spherical shells,
- **M. Han** and **QW, 1905.05522;**
- Formulas for Partial Entanglement Entropy,
- **QW**, Phys.Rev.Research, **1910.10978 .**

- **And**
- Remarks on the entanglement entropy for disconnected regions,
- **Casini** and **Huerta**, JHEP, **0812.1773.**

Outline

- Concept of the entanglement contour and partial entanglement entropy (PEE) ;
- Physical requirements for PEE;
- Approaches to PEE or entanglement contour;
- Future directions



The concept of Contour

The entanglement contour gives the contribution to the entanglement entropy of A from each point in A

- In general entanglement entropies obey certain inequalities.

(a) *Subadditivity*: $S(A) + S(B) \geq S(AB)$,
(b) *Araki-Lieb*: $S(AB) \geq |S(A) - S(B)|$,
(c) *Strong subadditivity 1*: $S(AB) + S(BC) \geq S(ABC) + S(B)$,
(d) *Strong subadditivity 2*: $S(AB) + S(BC) \geq S(A) + S(C)$.

Lieb and Ruskai, 1973'

- Entanglement entropy is non-local

- The contour function is a functional of the region thus recover non-locality

Definition following a physical meaning

Entanglement contour as the density founction of the entanglement in a given region

$$S_{\mathcal{A}} = \int_{\mathcal{A}} f_{\mathcal{A}}(\mathbf{x}) d\sigma_{\mathbf{x}} \,.$$

Chen and Vidal
1406.1471

Partial Entanglement Entropy

$$s_{\mathcal{A}}(\mathcal{A}_i) = \int_{\mathcal{A}_i} f_{\mathcal{A}}(\mathbf{x}) d\sigma_{\mathbf{x}} = \mathcal{I}(\bar{\mathcal{A}}, \mathcal{A}_i)$$

Motivation

- The local properties of entanglement entropy.
- **Area law** obeyed by the ground state of gapped systems
- **Volume law** obeyed by more generic states
- Massive deformation from CFT_2

$$S = \frac{c}{3}\log\frac{l}{a} \quad \xrightarrow{\; l \gg \xi \;} \quad S = \frac{c}{3}\log\frac{\xi}{a}$$

- Degrees of freedom at different position contribute differently.
- **The entanglement contour?**

Chen and Vidal
J. Stat. 14'

WQ, PRResearch 19'

No definition so far, only Physical Requirements

1. **Additivity**: by definition we should have

$$s_{\mathcal{A}}(\mathcal{A}_i) = s_{\mathcal{A}}(\mathcal{A}_i^a) + s_{\mathcal{A}}(\mathcal{A}_i^b)\,, \quad \mathcal{A}_i^a \cup \mathcal{A}_i^b = \mathcal{A}_i\,, \quad \mathcal{A}_i^a \cap \mathcal{A}_i^b = \varnothing. \tag{1.4}$$

2. **Invariance under local unitary transformations**: $s_{\mathcal{A}}(\mathcal{A}_i)$ is invariant under any local unitary transformations act only inside $\mathcal{A}_i$ and $\bar{\mathcal{A}}$.

3. **Symmetry**: For any symmetry transformation $\mathcal{T}$ under which $\mathcal{T}\mathcal{A} = \mathcal{A}'$ and $\mathcal{T}\mathcal{A}_i = \mathcal{A}'_i$, we have $s_{\mathcal{A}}(\mathcal{A}_i) = s_{\mathcal{A}'}(\mathcal{A}'_i)$.

4. **Normalization**: $S_{\mathcal{A}} = s_{\mathcal{A}}(\mathcal{A}_i)|_{\mathcal{A}_i \to \mathcal{A}}\,.$

5. **Positivity**: $s_{\mathcal{A}}(\mathcal{A}_i) \geq 0$.

6. **Upper bound**: $s_{\mathcal{A}}(\mathcal{A}_i) \leq S_{\mathcal{A}_i}\,.$

7. **Symmetry under the permutation**: $\mathcal{I}(\bar{\mathcal{A}}, \mathcal{A}_i) = \mathcal{I}(\mathcal{A}_i, \bar{\mathcal{A}})$, which implies $s_{\mathcal{A}}(\mathcal{A}_i) = s_{\bar{\mathcal{A}}_i}(\bar{\mathcal{A}})$.

Motivation

- Gives a finer description for the entanglement structure.
- Discriminate between gapped systems and gapless systems with a finite number of zero modes in $d = 3$;
- Characterizing the evolution of the entanglement structure;
- Generating the local modular flow;
- Useful probe of slowly scrambling and non-thermalizing dynamics for some interacting many-body systems;
- Finer correspondence between quantum entanglement and bulk geometry.

Vidal etc., Tonni, Sierra etc., Ryu etc. and QW

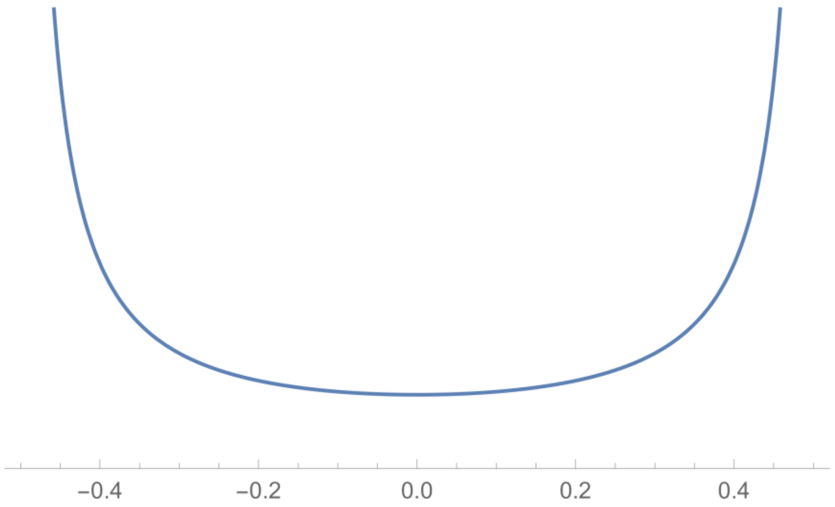
- **An explicit example of entanglement contour**

- Rindler transformation, which is a symmetry

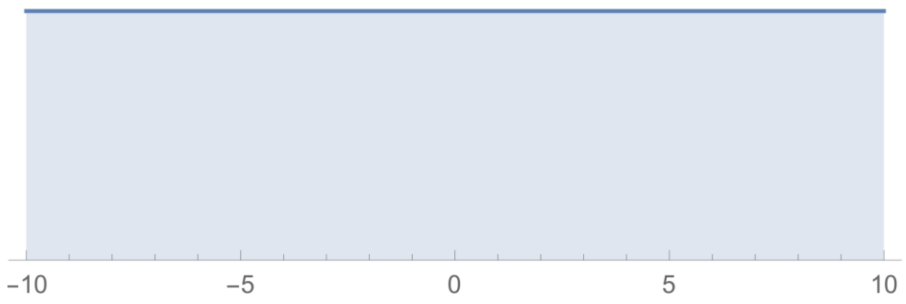
Entanglement wedge ⟺ "Rindler space"

Entanglement entropy ⟺ Thermal entropy

Entanglement contour ⟺ Entropy density (flat contour)



Contour for an interval in CFT2



Contour for the thermal state in Rindler space

Approaches to PEE or entanglement contour

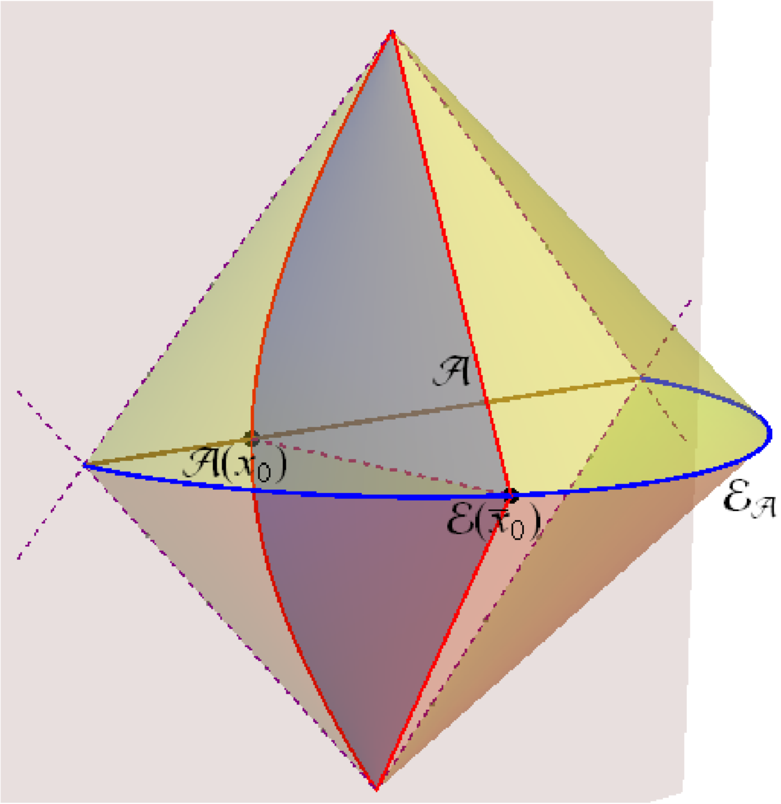
- 1: Gaussian formula — Chen and Vidal, 14'; Tonni etc. 16',17',19'; Ryu etc. 19',20';
- 2: Geometric construction in holography — WQ, PRD18'
- 3: PEE proposal — WQ, PRD18',JHEP19'
- 4: Solving all the requirements in Poincare invariant theories. — Casini and Huerta, JHEP08'; WQ, PRResearch19';

For cases with more than one approaches working
we find consistent results!

2: Geometric construction in holography

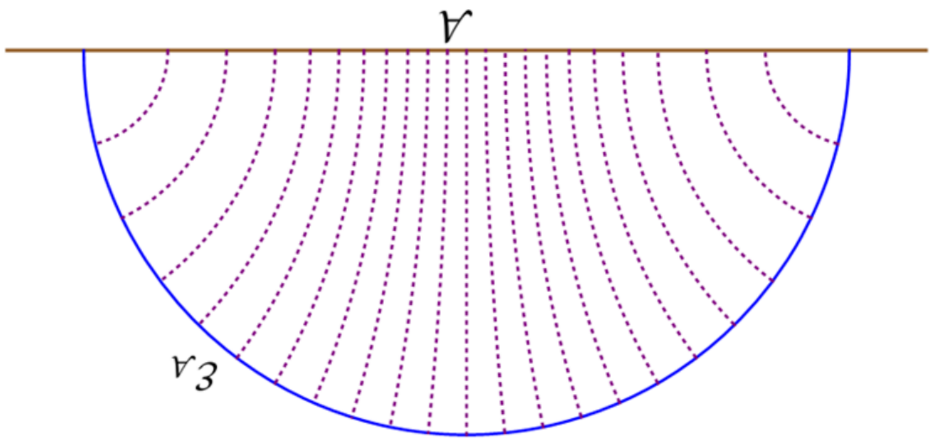
Modular slice

The orbit of the boundary modular flow line under the bulk modular flow

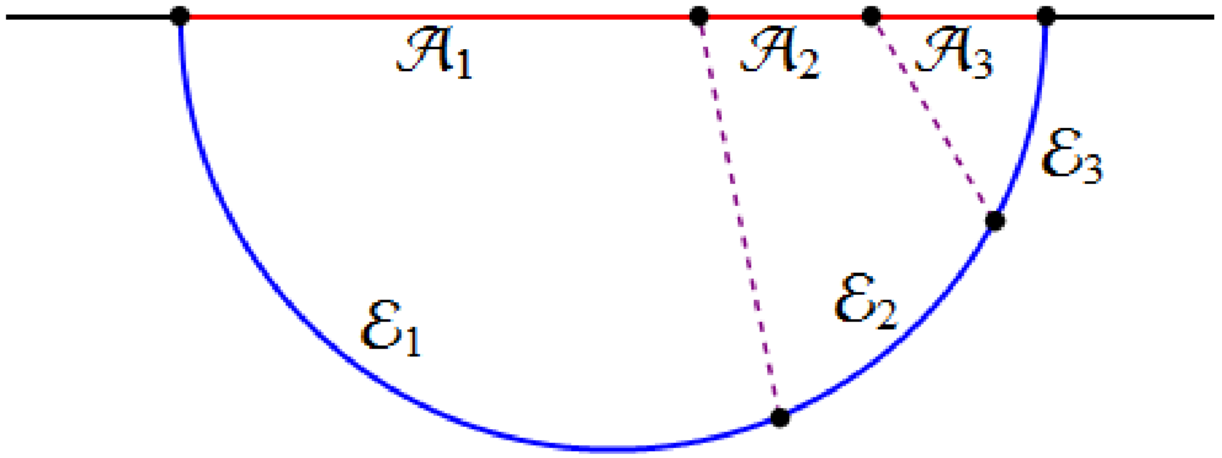


$$f_{\mathcal{A}}(r) = \frac{c}{6}\left(\frac{2R}{R^2 - r^2}\right)^{d-1}$$

Spheres in vacuum CFT_d



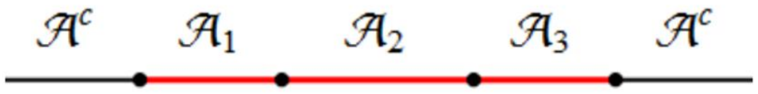
BTZ
Warped AdS
3d Flat space



$$s_{\mathcal{A}}(\mathcal{A}_i) = \frac{Length\,(\mathcal{E}_i)}{4G}$$

PEE ⟺ Geodesic chords

3:The partial entanglement entropy proposal



$$s_{\mathcal{A}}(\mathcal{A}_2) = \frac{1}{2}\left(S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_2 \cup \mathcal{A}_3} - S_{\mathcal{A}_1} - S_{\mathcal{A}_3}\right)$$

QW, 1803.05552 PRD
QW, 1902.06905 JHEP

Additivity for partial entanglement entropy

Consider a partition $\mathcal{A}_2 = \mathcal{A}_2^a \cup \mathcal{A}_2^b$

$$s_{\mathcal{A}}(\mathcal{A}_2^a) = \frac{1}{2}\left(S_{\mathcal{A}_1 \cup \mathcal{A}_2^a} + S_{\mathcal{A}_2 \cup \mathcal{A}_3} - S_{\mathcal{A}_1} - S_{\mathcal{A}_2^b \cup \mathcal{A}_3}\right)$$

$$s_{\mathcal{A}}(\mathcal{A}_2^b) = \frac{1}{2}\left(S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_2^b \cup \mathcal{A}_3} - S_{\mathcal{A}_1 \cup \mathcal{A}_2^a} - S_{\mathcal{A}_3}\right)$$

$$s_{\mathcal{A}}(\mathcal{A}_2) = s_{\mathcal{A}}(\mathcal{A}_2^a) + s_{\mathcal{A}}(\mathcal{A}_2^b)$$

- **Positivity**: The strong subadditivity $S_{\mathcal{A}_1\cup\mathcal{A}_2}+S_{\mathcal{A}_2\cup\mathcal{A}_3}-S_{\mathcal{A}_1}-S_{\mathcal{A}_3}\geq 0$ for any three regions indicates $s_{\mathcal{A}}(\mathcal{A}_2)\geq 0 \Rightarrow f_{\mathcal{A}}(x_1,\cdots x_{d-1})\geq 0$
- **Normalization**: $s_{\mathcal{A}}(\mathcal{A}_2)|_{\mathcal{A}_2\to\mathcal{A}}=S_{\mathcal{A}}$
- **Invariance under local transformations**: All the subset entanglement entropies are invariant under local transformations that only act on $\mathcal{A}_2$, so $s_{\mathcal{A}}(\mathcal{A}_2)$ is also invariant.
- **Upper bound**: subadditivity $S_{\mathcal{A}_1}+S_{\mathcal{A}_2}\geq S_{\mathcal{A}_1\cup\mathcal{A}_2}$ and $S_{\mathcal{A}_2}+S_{\mathcal{A}_3}\geq S_{\mathcal{A}_2\cup\mathcal{A}_3}$ indicates

$$s_{\mathcal{A}}(\mathcal{A}_2)\leq S_{\mathcal{A}_2}\,.$$

- **Symmetry**: Since $\mathcal{T}$ is a symmetry, the subsets $\mathcal{A}_i$ and $\mathcal{A}'_i$ should play the equivalent role, in other words we have $S_{\mathcal{A}_i}=S_{\mathcal{A}'_i}, S_{\mathcal{A}_i\cup\mathcal{A}_j}=S_{\mathcal{A}'_i\cup\mathcal{A}'_j}$. This means

$$\begin{aligned}S_{\mathcal{A}_1\cup\mathcal{A}_2}+S_{\mathcal{A}_2\cup\mathcal{A}_3}-S_{\mathcal{A}_1}-S_{\mathcal{A}_3}&=S_{\mathcal{A}'_1\cup\mathcal{A}'_2}+S_{\mathcal{A}'_2\cup\mathcal{A}'_3}-S_{\mathcal{A}'_1}-S_{\mathcal{A}'_3}\\ &\Rightarrow s_{\mathcal{A}}(\mathcal{A}_2)=s_{\mathcal{A}}(\mathcal{A}'_2)\,.\end{aligned}$$

- Invariance under permutation

$$
\begin{aligned}
\mathcal{I}(\bar{\mathcal{A}}, \mathcal{A}_2) &= s_{\mathcal{A}}(\mathcal{A}_2) \\
&= \frac{1}{2}\left(S_{\bar{\mathcal{A}}\cup\mathcal{A}_3} + S_{\bar{\mathcal{A}}\cup\mathcal{A}_1} - S_{\mathcal{A}_1} - S_{\mathcal{A}_3}\right) \\
&= s_{\bar{\mathcal{A}}_2}(\bar{\mathcal{A}}) = \mathcal{I}(\mathcal{A}_2, \bar{\mathcal{A}})\,,
\end{aligned}
$$

- The PEE proposal is a solution to all the requirements!

4:PEE from solving the requirements

Additivity

Symmetry under permutation

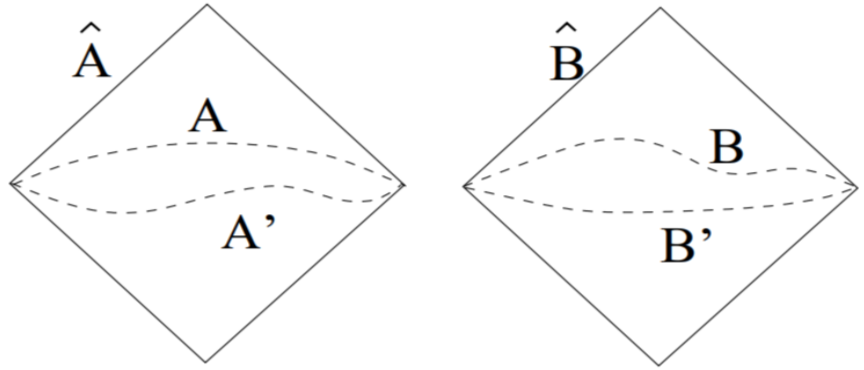
$$\mathcal{I}(\bar{A}, A_i) = \int_{\bar{A}} d\sigma_{\mathbf{x}} \int_{A_i} d\sigma_{\mathbf{y}}\, J(\mathbf{x}, \mathbf{y})$$

$$\mathcal{I}(\bar{A}, A_i) = \sum_{i \in \bar{A}} \sum_{j \in A_i} J_{ij}$$

Casini and Huerta 0812.1773

Derivation for extensive mutual information

Requirement 2, or causality



$$\mathcal{I}(A', B') = \mathcal{I}(A, B).$$

A' and B' are any space-like regions that share boundary with A and B. Then we can write:

$$\mathcal{I}(A, B) = \int_A d\sigma_{\mathbf{x}}{}^{\mu} \int_B d\sigma_{\mathbf{y}}{}^{\nu} J_{\mu\nu}(\mathbf{x}, \mathbf{y}),$$

With

$$\partial_\mu J^{\mu\nu}(\mathbf{x}, \mathbf{y}) = 0$$

Poincare invariant theories

- Invariance under Poincare symmetry:

$$J^{\mu\nu}(\mathbf{x},\mathbf{y}) = \frac{(\mathbf{x}-\mathbf{y})^{\mu}(\mathbf{x}-\mathbf{y})^{\nu}}{(\mathbf{x}-\mathbf{y})^{2d}} G(l) - \frac{g^{\mu\nu}}{(\mathbf{x}-\mathbf{y})^{2(d-1)}} F(l)$$

- F and G are two dimensionless functions of l.

- Conservation: ➡

$$[G(l) - F(l)]' = -(d-1)\frac{2F(l) - G(l)}{l}$$

- **Positivity** → $${\sigma_{\mathbf{x}}}^{\mu}{\sigma_{\mathbf{y}}}^{\nu} J_{\mu\nu}(\mathbf{x},\mathbf{y}) \geq 0$$ for any time-like vectors $\vec{\sigma}_{\mathbf{x}}$ and $\vec{\sigma}_{\mathbf{y}}$

- Furthermore implies: $$2F(l) \geq G(l) \geq 0$$

- It is convenient to define $C(l) = G(l) - F(l)$, thus $C'(l) \leq 0$

- Which implies C(l) deceases under the RG flow, it is a c-function.

- Then we have

$$F(l) = -\frac{lC'(l)}{d-1} + C(l)\,, \qquad G(l) = -\frac{lC'(l)}{d-1} + 2C(l)$$

Then it is convenient to define another function $H(l)$ by

$$C(l) = (d-1)l^{2d-3}H'(l)\,.$$

Thus

$$J_{\mu\nu}(l) = -\partial_\mu \partial_\nu H(l) + g_{\mu\nu}\partial_\alpha \partial^\alpha H(l)\,.$$

At last, after we applied the Stokes' theorem we arrive at the following formula for PEE

$$\mathcal{I}(A,B) = \int_{\partial A}\int_{\partial B} d\vec{\eta}_{\mathbf{x}} \cdot d\vec{\eta}_{\mathbf{y}}\, H(|\mathbf{x}-\mathbf{y}|)\,,$$

General Formula!

where $\vec{\eta}_{\mathbf{x}}$ and $\vec{\eta}_{\mathbf{y}}$ are the infinitesimal subsets on the boundaries ∂A and ∂B with an outward pointing direction in the system and normal to ∂A and ∂B.

PEE in conformal field theories

Things become much more determined in the case of conformal field theories. Since $C(l)$ is a c-function, it should be a constant in CFTs. Let us define $C(l) = 2C_d(d-1)(d-2)$, then we have

$$H(|\mathbf{x}-\mathbf{y}|) = -\frac{C_d}{|\mathbf{x}-\mathbf{y}|^{2d-4}}\,, \qquad d > 2\,, \tag{4.16}$$

Cd is a constant that depend on the theory and dimension

When d=2, H(l)=#Log l/a, is just the entanglement entropy for a single interval with length l.

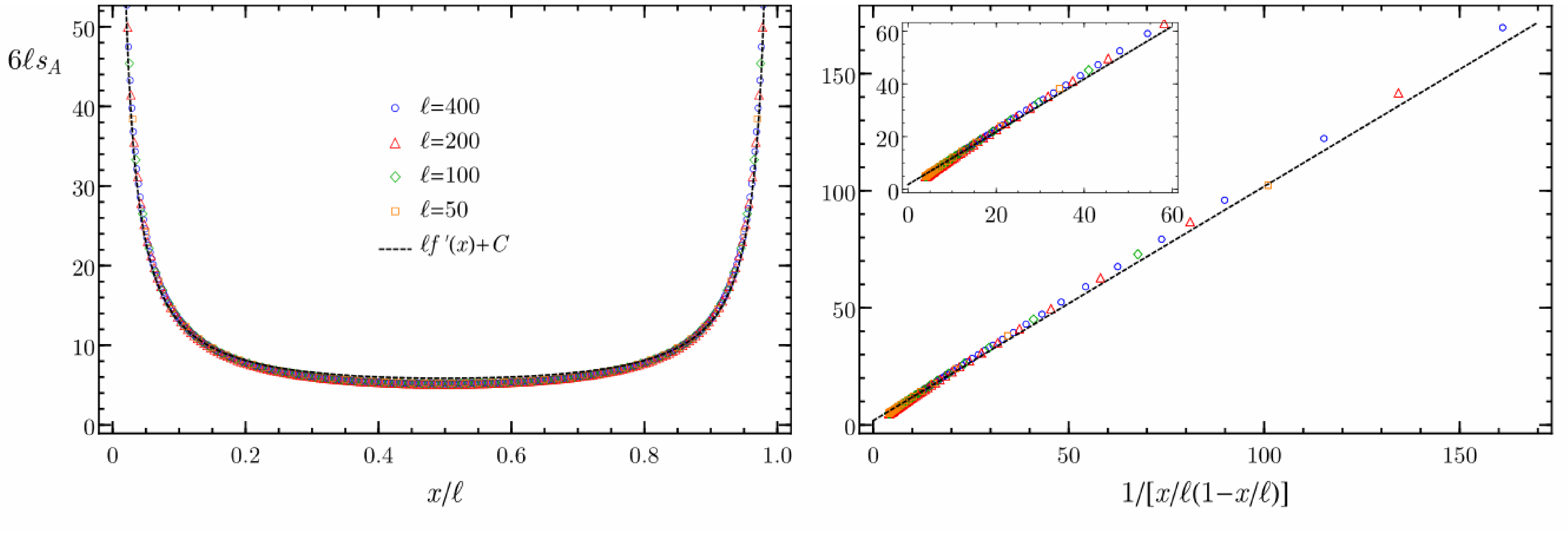
Consistency check

$$\mathcal{I}(\bar{\mathcal{A}}, \mathcal{A}_2) = -\int_{\partial A}\int_{\partial B} d\vec{\eta}_{\mathbf{x}} \cdot d\vec{\eta}_{\mathbf{y}} \frac{C_d}{|\mathbf{x}-\mathbf{y}|^{2d-4}}$$
$$= C_d \Omega_{d-2} \Omega_{d-3} \int_0^{\pi} d\theta \frac{z^{d-2}(\sin\theta)^{d-3}\cos\theta}{(1+z^2-2z\cos\theta)^{d-2}} .$$

For d=2,3,4

$$s_{\mathcal{A}}(\mathcal{A}_2) \sim \mathcal{I}(\bar{\mathcal{A}}, \mathcal{A}_2) \sim \left\{ \frac{z^2}{1-z^2}, \frac{z^3+z}{(z^2-1)^2} - \frac{1}{2}\tanh^{-1}\left(\frac{2z}{z^2+1}\right), \frac{z^4\left(z^2-3\right)}{(z^2-1)^3} \right\}.$$

Compare with numerical results from the Gaussian formula



Coser , Nobili and Tonni, J.Stat.Mech. 2018

“Entanglement Hamiltonian and entanglement contour in inhomogeneous 1D critical systems”

Future Problems

- PEE as an intrinsic measurement?
- Quantum correction to holographic entanglement contour
- PEE, minimal cross section and new way to evaluate entanglement of purification?
- Potential in condensed matter and quantum information?
- … …

Thanks