Progress on Finite Feynman Integrals with Uniform Transcendentality

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This talk is based on the ongoing project

with

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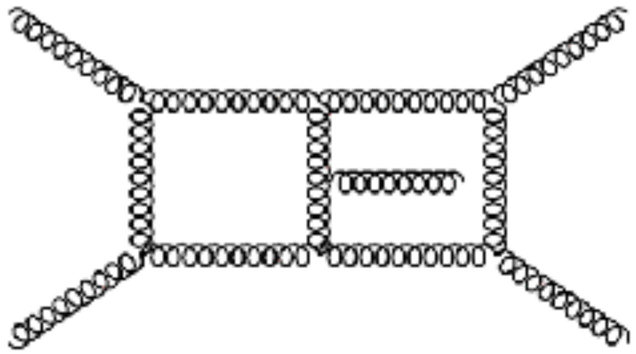
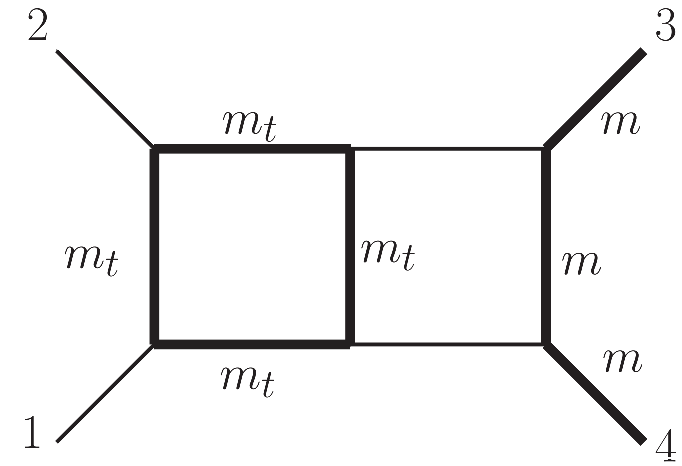
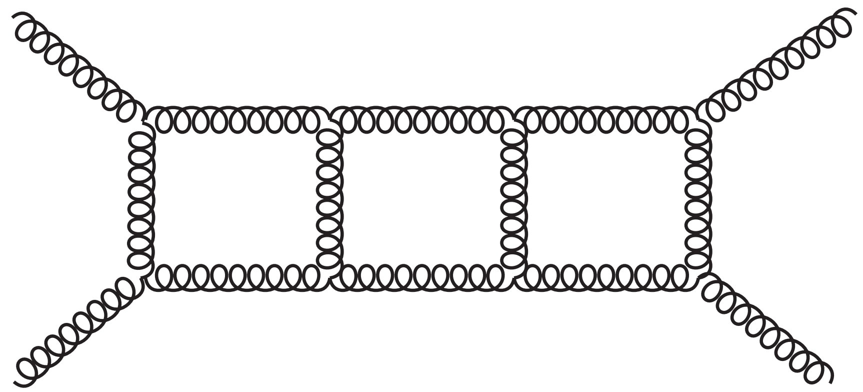
Feynman integral evaluation







Multi-loop Feynman integrals are the hardcore objects for a perturbative QFT computation

- Important for high-energy phenomenology
- Theoretically important

Feynman integral evaluation: the State of Art

recent analytic results

- 2-loop qqbar $\rightarrow$ ttbar nonplanar integrals
  (Di Vita, Gehrmann, Laporta, Mastrolia, Pierpaolo; Primo, Schubert, Ulrich, 2019)
- 4-loop form factor planar integrals
  (von Manteuffel, Schabinger, 2019)
- 2-loop Higgs + one jet production (with t quark mass dep.) nonplanar integrals (almost all)
  (Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, Moriello, Salvatori, Smirnov, 2019)
- 2-loop five-point massless planar and nonplanar integrals
  (Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, 2019)

....

and there would be many more in the near future

uncharted territory

Feynman integral with elliptic polylogarithms,
usually in multi-loop nonplanar diagrams with "large" massive internal loop
new transcendental functions (new, even for mathematicians)

see, eg, Broedel, Duhr, Dulat, Tancredi, 2018

Mainstream Analytic Methods

Differential equation for Feynman integrals

$d = 4 - 2\epsilon$ $\quad I(\bar{x}, \epsilon)$ master integrals as a column vector

$$\frac{\partial}{\partial x_i} I(\bar{x}, \epsilon) = A_i(\bar{x}, \epsilon) I(\bar{x}, \epsilon) \qquad \text{(Kotikov 1991)}$$

Can be used both numerically and analytically

eg. Gehrmann, Remiddi 1999, Papadopoulos 2014, Liu, Ma, Wang, 2018 …

Canonical Differential equation

Symbol letters

$$\frac{\partial}{\partial x_i} \tilde{I}(\bar{x}, \epsilon) = \epsilon \tilde{A}_i(\bar{x}) \tilde{I}(\bar{x}, \epsilon)$$

$$= \epsilon \left(\sum_{l=1} \frac{\partial \log(W_l)}{\partial x_i} m_l \right) \tilde{I}(\bar{x}, \epsilon)$$

Proportional to $\boldsymbol{\epsilon}$

rational number matrix

(Henn 2013)

Canonical Differential Equation

Can be solved order-by-order analytically

$$\tilde{I}(x) = P\exp\left(\epsilon \int_{\mathcal{C}} dA\right)\tilde{I}(x_0)$$

path-ordered

Chen's (陈国才) iterated integrals,
homotopically invariant

In term of poly-logarithm functions

$$G(\underbrace{0,\ldots,0}_{k};z) = \frac{1}{k!}\left(\log z\right)^k, \qquad G(a_1,\ldots,a_k;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_k;t)$$

implemented in the C++ library
GiNaC

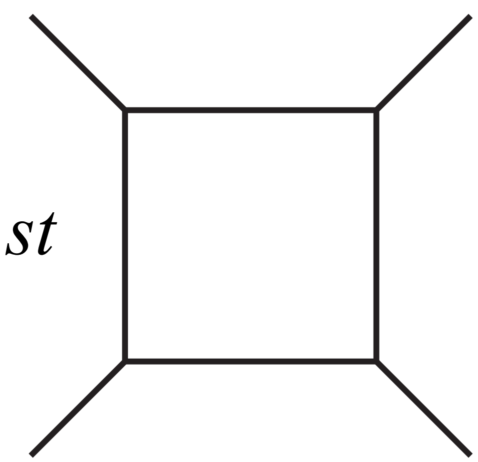
Uniformly transcendental (UT) integrals ⇒ Canonical DE

$$I(x) = \epsilon^{-2L} \sum_{i=0}^{\infty} f_i(x)\, \epsilon^i$$

$$\mathcal{T}(\log) = 1, \mathcal{T}(\pi) = 1, \mathcal{T}(\zeta_n) = n, \mathcal{T}(\mathrm{Li}_n) = n, \ldots, \mathcal{T}(f_1 f_2) = \mathcal{T}(f_1) + \mathcal{T}(f_2)$$

$$\boxed{\mathcal{T}(f_i) = i, \quad \mathcal{T}(\partial_x f_i) = i - 1}$$

$(-s)^\epsilon st$



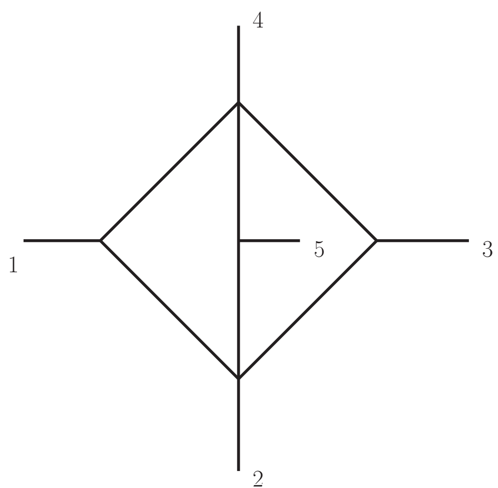
0 1 2 3

$$\frac{4}{\epsilon^2} - \frac{2\log(x)}{\epsilon} - \frac{4\pi^2}{3} - \frac{1}{6}\epsilon\Big(- 12\mathrm{Li}_3(-x) + 12\mathrm{Li}_2(-x)\log(x)$$
$$-2\log^3(x) + 6\log(x+1)\log^2(x) - 7\pi^2\log(x) + 6\pi^2\log(x+1) + 68\zeta(3)\Big)$$

UT

Uniformly transcendental (UT) integrals: more complicated example

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia (2019)



$$I = \frac{1}{\epsilon^2} f^{(2)} + \frac{1}{\epsilon} f^{(3)} + f^{(4)} + \mathcal{O}(\epsilon)\,.$$

$$f^{(2)} = -3\Bigg[\mathrm{Li}_2\left(\frac{1}{W_{27}}\right) - \mathrm{Li}_2\left(W_{27}\right) + \mathrm{Li}_2\left(\frac{1}{W_{28}}\right) - \mathrm{Li}_2\left(\frac{1}{W_{27}W_{28}}\right) - \mathrm{Li}_2\left(W_{28}\right) + \mathrm{Li}_2\left(W_{27}W_{28}\right)\Bigg].$$

$$\begin{aligned}
&W_1 = v_1\,, && W_6 = v_3 + v_4\,, && W_{11} = v_1 - v_4\,, && W_{16} = v_1 + v_2 - v_4\,,\\
&W_2 = v_2\,, && W_7 = v_4 + v_5\,, && W_{12} = v_2 - v_5\,, && W_{17} = v_2 + v_3 - v_5\,,\\
&W_3 = v_3\,, && W_8 = v_5 + v_1\,, && W_{13} = v_3 - v_1\,, && W_{18} = v_3 + v_4 - v_1\,,\\
&W_4 = v_4\,, && W_9 = v_1 + v_2\,, && W_{14} = v_4 - v_2\,, && W_{19} = v_4 + v_5 - v_2\,,\\
&W_5 = v_5\,, && W_{10} = v_2 + v_3\,, && W_{15} = v_5 - v_3\,, && W_{20} = v_5 + v_1 - v_3\,,
\end{aligned}$$

$$\begin{aligned}
&W_{21} = v_3 + v_4 - v_1 - v_2\,, && W_{26} = \frac{v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 - \sqrt{\Delta}}{v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 + \sqrt{\Delta}}\,,\\
&W_{22} = v_4 + v_5 - v_2 - v_3\,, && W_{27} = \frac{-v_1v_2 + v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 - \sqrt{\Delta}}{-v_1v_2 + v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 + \sqrt{\Delta}}\,,\\
&W_{23} = v_5 + v_1 - v_3 - v_4\,, && W_{28} = \frac{-v_1v_2 - v_2v_3 + v_3v_4 + v_1v_5 - v_4v_5 - \sqrt{\Delta}}{-v_1v_2 - v_2v_3 + v_3v_4 + v_1v_5 - v_4v_5 + \sqrt{\Delta}}\,,\\
&W_{24} = v_1 + v_2 - v_4 - v_5\,, && W_{29} = \frac{v_1v_2 - v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 - \sqrt{\Delta}}{v_1v_2 - v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 + \sqrt{\Delta}}\,,\\
&W_{25} = v_2 + v_3 - v_5 - v_1\,, && W_{30} = \frac{-v_1v_2 + v_2v_3 - v_3v_4 + v_1v_5 - v_4v_5 - \sqrt{\Delta}}{-v_1v_2 + v_2v_3 - v_3v_4 + v_1v_5 - v_4v_5 + \sqrt{\Delta}}\,,
\end{aligned}$$

However, for multi-loop Feynman integrals,
it is not easy to find a complete UT basis

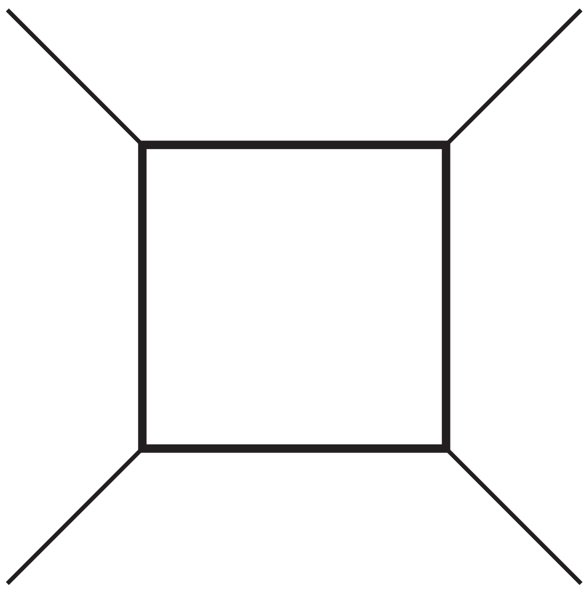
(4D Leading Singularity/dlog) Arkani-Hamed, Bourjaily, Cachazo, Trnka (2010)
(Matrix transformation) Meyer (2015)
(Morse theory for DE) Lee (2015)
(4D dlog algorithm) Wasser (2017)
(D-dim Leading Singularity) Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, (2019)
(D-dim Intersection) Chen, Xu, Yang, (2020) … …

In practice, sometimes
it is not necessary to use a complete UT basis

Finite UT integrals

Example

Caron-Huot and Henn 2014



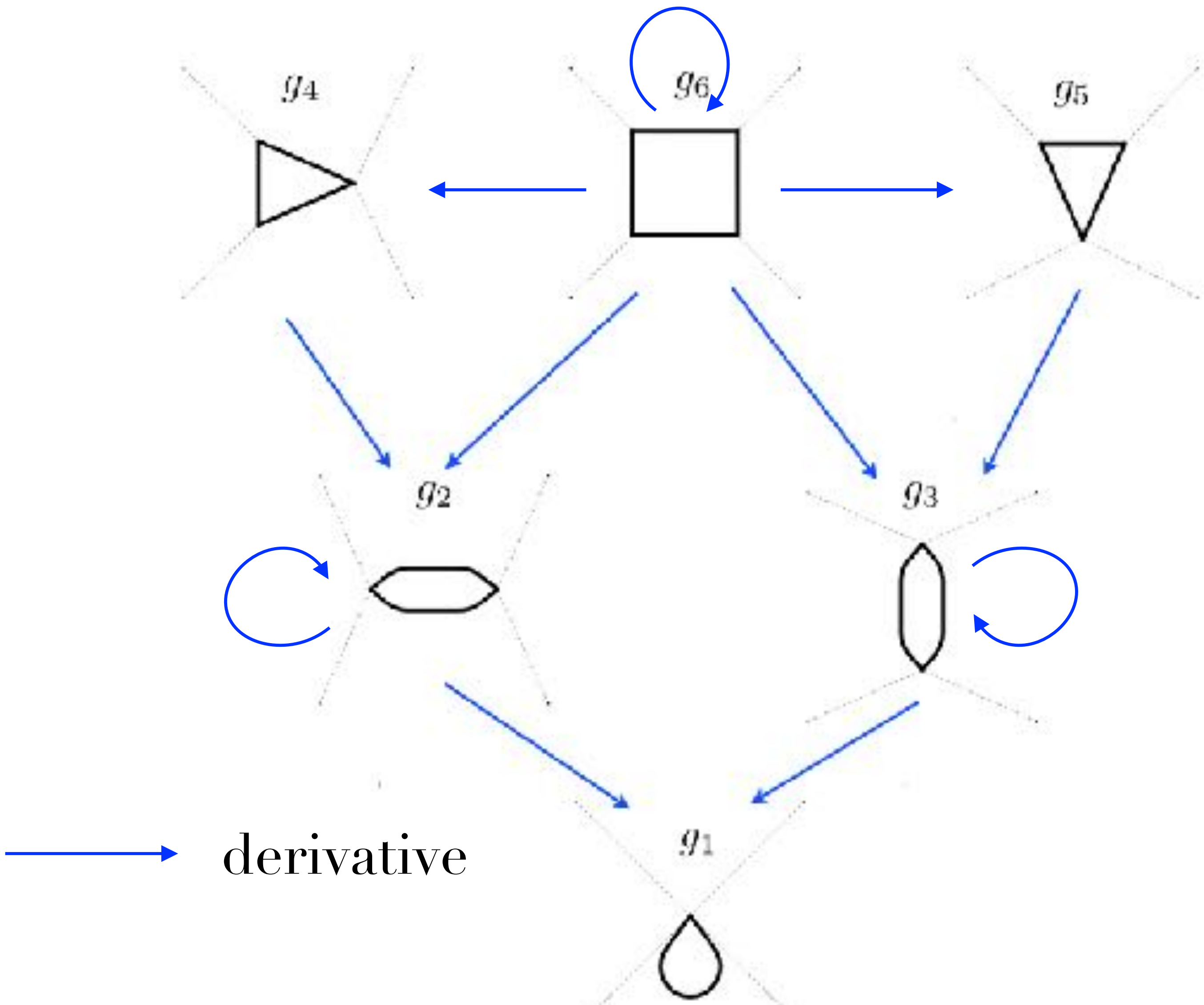
internal lines massive

$$g_1 = 2m^2 G[0,0,0,3]/\epsilon^2$$
$$g_2 = -\sqrt{s(s-4m^2)}G[1,0,2,0]/\epsilon$$
$$g_3 = -\sqrt{t(t-4m^2)}G[0,1,0,2]/\epsilon$$
$$g_4 = sG[1,1,1,0]$$
$$g_5 = tG[1,1,0,1]$$
$$g_6 = \frac{1}{2}\sqrt{st(st-4m^2(s+t))}G[1,1,1,1]$$

UT basis (6 integrals)

$$d\vec{g} = \epsilon(dA)\vec{g}$$

Finite UT integrals

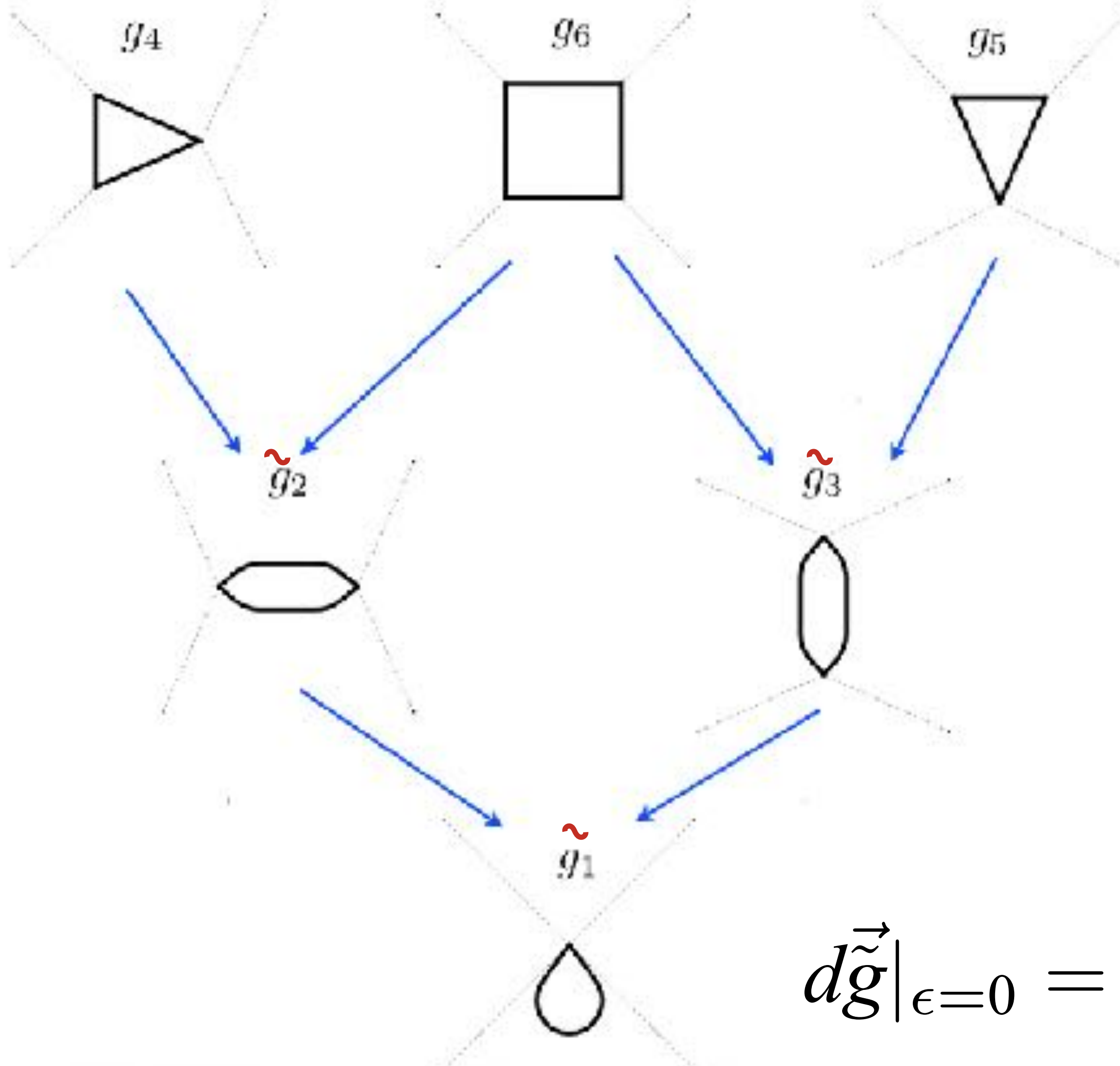


$$g_1 = 2m^2 G[0,0,0,3]/\epsilon^2$$
$$g_2 = -\sqrt{s(s-4m^2)}\,G[1,0,2,0]/\epsilon$$
$$g_3 = -\sqrt{t(t-4m^2)}\,G[0,1,0,2]/\epsilon$$
$$g_4 = sG[1,1,1,0]$$
$$g_5 = tG[1,1,0,1]$$
$$g_6 = \frac{1}{2}\sqrt{st(st-4m^2(s+t))}\,G[1,1,1,1]$$

$$d\vec{g} = \epsilon(dA)\vec{g}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
2\,\mathrm{ArcTanh}\left[\sqrt{1-\frac{4m^2}{s}}\right] & \mathrm{Log}[m^2]-\mathrm{Log}[4m^2-s] & 0 & 0 & 0 & 0 \\
2\,\mathrm{ArcTanh}\left[\sqrt{1-\frac{4m^2}{t}}\right] & 0 & \mathrm{Log}[m^2]-\mathrm{Log}[4m^2-t] & 0 & 0 & 0 \\
0 & -2\,\mathrm{ArcTanh}\left[\sqrt{1-\frac{4m^2}{s}}\right] & 0 & 0 & 0 & 0 \\
0 & 0 & -2\,\mathrm{ArcTanh}\left[\sqrt{1-\frac{4m^2}{t}}\right] & 0 & 0 & 0 \\
0 & 4\,\mathrm{ArcTanh}\left[\frac{\sqrt{(-4m^2+s)\,t}}{\sqrt{st-4m^2(s+t)}}\right] & 4\,\mathrm{ArcTanh}\left[\frac{\sqrt{s(-4m^2+t)}}{\sqrt{st-4m^2(s+t)}}\right] & -4\,\mathrm{ArcTanh}\left[\sqrt{1-\frac{4m^2(s+t)}{st}}\right] & -4\,\mathrm{ArcTanh}\left[\sqrt{1-\frac{4m^2(s+t)}{st}}\right] & \mathrm{Log}[m^2]+\mathrm{Log}[s+t]-\mathrm{Log}[-st+4m^2(s+t)]
\end{pmatrix}$$

Finite UT integrals with truncation



$$\tilde{g}_1 = 2m^2 G[0,0,0,3]/\epsilon^2$$
$$\tilde{g}_2 = -\sqrt{s(s-4m^2)}G[1,0,2,0]/\epsilon$$
$$\tilde{g}_3 = -\sqrt{t(t-4m^2)}G[0,1,0,2]/\epsilon$$
$$g_4 = sG[1,1,1,0]$$
$$g_5 = tG[1,1,0,1]$$
$$g_6 = \frac{1}{2}\sqrt{st(st-4m^2(s+t))}G[1,1,1,1]$$

$$d\vec{\tilde{g}}|_{\epsilon=0} = (d\tilde{A})\vec{\tilde{g}}|_{\epsilon=0}$$

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2\,\text{ArcTanh}\left[\sqrt{1-\frac{4\,mm}{s}}\right] & 0 & 0 & 0 & 0 & 0 \\ 2\,\text{ArcTanh}\left[\sqrt{1-\frac{4\,mm}{t}}\right] & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\,\text{ArcTanh}\left[\sqrt{1-\frac{4\,mm}{s}}\right] & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\,\text{ArcTanh}\left[\sqrt{1-\frac{4\,mm}{t}}\right] & 0 & 0 & 0 \\ 0 & 4\,\text{ArcTanh}\left[\frac{\sqrt{(-4\,mm+s)\,t}}{\sqrt{s\,t-4\,mm\,(s+t)}}\right] & 4\,\text{ArcTanh}\left[\frac{\sqrt{s\,(-4\,mm+t)}}{\sqrt{s\,t-4\,mm\,(s+t)}}\right] & 0 & 0 & 0 \end{pmatrix}$$

All diagonal elements vanish, and the DE can be solved recursively.

Summary

- (quasi)-finite integrals provides a simple truncated DE system
- Towards a systematical way of finding (quasi)-finite UT integrals
- graded syzygy to get a small IBP system for (quasi)-finite UT integrals
- working in progress …

Vielen Dank