5D Weyl Double Copy

Pujian Mao

Center for Joint Quantum Studies, Tianjin University

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Work in collaboration with Weicheng Zhao and Jun-Bao Wu

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Outline

1. Double copy
2. Classical double copy
3. Weyl double copy in 4d
4. Weyl double copy in 5d

Full four-gluon tree amplitude in Yang-Mills theory

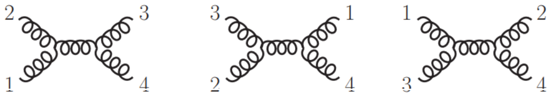

Figure 3: The three Feynman diagrams corresponding to the s, t and u channels.

$$i\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \right) \tag{1}$$

where

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2, \quad u = (p_3 + p_1)^2, \tag{2}$$

are kinematic invariants of a four-point scattering process.

- The s-channel color factor

$$c_s = -2 f^{a_1 a_2 b} f^{b a_3 a_4} \tag{3}$$

- The s-channel kinematic numerator

$$n_s = -\frac{1}{2} \left[(\epsilon_1 \cdot \epsilon_2) p_1^\mu + 2(\epsilon_1 \cdot p_2) \epsilon_2^\mu - (1 \leftrightarrow 2) \right] \cdot \Big[(\epsilon_3 \cdot \epsilon_4) p_{3\mu} + 2(\epsilon_3 \cdot p_4) \epsilon_{4\mu} - (3 \leftrightarrow 4) \Big] - \frac{1}{2} s \left[(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) - (\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3) \right] \tag{4}$$

- The color/kinematic duality

$$c_s + c_t + c_u = 0, \quad n_s + n_t + n_u = 0 \tag{5}$$

- The kinematic factors satisfy the same relations as the color factors suggesting that they are mutually exchangeable.

[Bern, Carrasco, Johansson, 2008; 2010]

Double copy from BCJ duality

- Swapping color factors for kinematic factors in the YM four-point amplitude, which gives a new gauge-invariant object that, is a four-graviton amplitude,

$$i\mathcal{A}_4^{\text{tree}}|_{g\to\frac{\kappa}{2},c_i\to n_i} = i\mathcal{M}_4^{\text{tree}} = (\frac{\kappa}{2})^2\left(\frac{n_s^2}{s}+\frac{n_t^2}{t}+\frac{n_u^2}{u}\right) \tag{6}$$

- The new amplitude has the following properties:
  - the external states are captured by symmetric polarization tensors
  - the interactions are of the two-derivative type
  - invariant under linearized diffeomorphism transformations
- This amplitude should describe the scattering of four gravitons of Einstein theory.

Double copy of classical solutions in the Kerr-Schild form

- The Kerr-Schild form is made by a null vector field k_μ and a scalar function ϕ in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \ . \tag{7}$$

- One can show that k^μ is null when contracted with both the full metric $g_{\mu\nu}$ and the Minkowski one $\eta_{\mu\nu}$.
- The inverse metric is just

$$g^{\mu\nu} = \eta^{\mu\nu} - \phi k^\mu k^\nu \ . \tag{8}$$

- We can raise the index of k using both $g^{\mu\nu}$ and $\eta^{\mu\nu}$.

Stationary phase space

- In the stationary case in which all time derivatives vanish and $k^0 = 1$ where all dynamics in the zeroth component contained in the scalar field, the Ricci tensor is reduced to

$$R^0{}_0 = \frac{1}{2}\nabla^2\phi \tag{9}$$

$$R^i{}_0 = -\frac{1}{2}\partial_j\left[\partial^i(\phi k^j) - \partial^j(\phi k^i)\right] \tag{10}$$

$$R^i{}_j = \frac{1}{2}\partial_l\left[\partial^i(\phi k^l k_j) + \partial_j(\phi k^l k^j) - \partial^l(\phi k^i k_l)\right] \tag{11}$$

- Define a vector field $A_\mu = \phi k_\mu$, with the standard definition of field strength.
- The vacuum Einstein equation implies that the gauge field satisfies Maxwell equations

$$\partial_\mu F^{\mu\nu} = \partial_\mu\left[\partial^\mu(\phi k^\nu) - \partial^\nu(\phi k^\mu)\right] = 0. \tag{12}$$

[Monteiro, O'Connell, White, 2014]

pp-wave

- Using a pair of lightcone coordinates (u, v, x, y), we can express a pp-wave using

$$k_\mu \mathrm{d}x^\mu = \mathrm{d}u = \mathrm{d}z - \mathrm{d}t, \quad \phi = \phi(u, x, y) \tag{13}$$

- Einstein equation yields

$$\partial_\mu \partial^\mu \phi = 0. \tag{14}$$

- The gauge field $A_\mu = \phi k_\mu$ a solution of the Maxwell equations.

Weyl double copy for type D solutions

- The Weyl double copy (WDC) formula

$$C_{ABCD} = \frac{1}{S} f_{(AB} f_{CD)}. \tag{15}$$

- C_{ABCD} is the anti-delf-dual part of the Weyl tensor,

$$C_{ABCD} = \frac{1}{4} \sigma^{\mu\nu}_{AB} \sigma^{\alpha\beta}_{CD} W_{\mu\nu\alpha\beta}. \tag{16}$$

- f_{AB} is defined from the field strength of the Maxwell tensor

$$f_{AB} = \frac{1}{2} \sigma^{\mu\nu}_{AB} F_{\mu\nu}. \tag{17}$$

[Luna, Monteiro, NIcholson, O'Connell, 2018]

- Van der Waerden matrices

$$\sigma^{\mu\nu}_{AB} = \sigma^{[\mu}_{A\dot{A}} \tilde{\sigma}^{\nu]\dot{A}C} \epsilon_{CB}, \qquad \sigma^{\mu}_{A\dot{A}} = e^{\mu}_{a} \sigma^{a}_{A\dot{A}}, \qquad \sigma^{a}_{A\dot{A}} = \frac{1}{\sqrt{2}} (1, \sigma^{i}). \tag{18}$$

Plebanski-Demianski metric

- The line-element in (p, q, τ, σ) coordinates

$$\mathrm{d}s^2 = \frac{1}{(1-pq)^2}\Bigg[\frac{p^2+q^2}{P(p)}\mathrm{d}p^2 + \frac{p^2+q^2}{Q(q)}\mathrm{d}q^2 + \frac{P(p)}{p^2+q^2}(\mathrm{d}\tau + q^2\mathrm{d}\sigma)^2 - \frac{Q(q)}{p^2+q^2}(\mathrm{d}\tau - p^2\mathrm{d}\sigma)^2\Bigg], \quad (19)$$

where

$$P(p) = \gamma(1-p^4) + 2np - \epsilon p^2 + 2mp^3, \quad (20)$$

$$Q(q) = \gamma(1-q^4) - 2mq + \epsilon q^2 - 2nq^3. \quad (21)$$

The Weyl double copy

- The Weyl spinor

$$C_{ABCD}\xi^A\xi^B\xi^C\xi^D = -\frac{3i}{2}\frac{(m+in)(1-pq)^3[(\xi^1)^2-(\xi^2)^2]^2}{(p-iq)^3}, \quad (22)$$

where $\xi^A = (\xi^1, \xi^2)$ is an arbitrary spinor.

- The field strength spinor

$$f_{AB}\xi^A\xi^B = -\frac{1}{2}\frac{(m+in)(1-pq)^3[(\xi^1)^2-(\xi^2)^2]}{(p-iq)^2} \quad (23)$$

- The scalar field

$$S = \frac{i}{6}\frac{(m+in)(1-pq)^3}{(p-iq)} \quad (24)$$

satisfies the wave equation in the flat background.

Weyl double copy for type N solutions

- In the Newman-Penrose formalism, the type N WDC reads

$$\Psi_4 = \frac{1}{S}(\Phi_2)^2. \tag{25}$$

- The Bianchi identities

$$D\Psi_4 = (\rho - 4\epsilon)\Psi_4, \qquad \delta\Psi_4 = (\tau - 4\beta)\Psi_4. \tag{26}$$

- The Maxwell's equation for degenerate Maxwell fields

$$D\Phi_2 = (\rho - 2\epsilon)\Phi_2, \qquad \delta\Phi_2 = (\tau - 2\beta)\Phi_2. \tag{27}$$

- The scalar field

$$D\log S - \rho = 0, \qquad \delta\log S - \tau = 0, \tag{28}$$

  which are integrable $(\delta D - D\delta)S = \delta\rho - D\tau$.
- Wave equation on type N background

$$\Box S = 0. \tag{29}$$

[Godazgar^2, Monteiro, Veiga, Pope, 2020; 2021]

Two remarks

- Decomposition of the Weyl spinor and the Maxwell spinor

$$C_{ABCD} = a_{(A}b_B c_C d_{D)}, \qquad f_{AB} = a_{(A}b_{B)}. \tag{30}$$

- The Weyl double copy in the Newman-Penrose formalism

$$\begin{gathered}\Psi_4 = 3c\frac{(\Phi_2)^2}{S}, \qquad \Psi_3 = 3c\frac{\Phi_2\Phi_1}{S}, \\ \Psi_2 = c\frac{2(\Phi_1)^2 + \Phi_0\Phi_2}{S}, \\ \Psi_1 = 3c\frac{\Phi_0\Phi_1}{S}, \qquad \Psi_0 = 3c\frac{(\Phi_0)^2}{S}.\end{gathered} \tag{31}$$

[Godazgar^2, Monteiro, Veiga, Pope, 2021; Mao, Zhao, 2023]

5d vielbein formalism

The vielbein bases

$$l = e_0 = e^1, \quad n = e_1 = e^0, \quad m_i = e_i = m^i = e^i, \quad i = 2, 3, 4, \tag{32}$$

where l and n are null, and m_i are spacelike. The basis vectors satisfy the orthogonal and normalization conditions

$$l \cdot l = n \cdot n = l \cdot m^i = n \cdot m^i = 0, \tag{33}$$

$$l \cdot n = 1, \quad m^i \cdot m^j = \delta^{ij}. \tag{34}$$

Directional derivatives

$$D = l^\mu \partial_\mu, \qquad \Delta = n^\mu \partial_\mu, \qquad \delta_i = m_i^\mu \partial_\mu. \tag{35}$$

The following quantities are defined to denote the spin coefficients

$$L_{\mu\nu} = \nabla_\nu l_\mu, \quad N_{\mu\nu} = \nabla_\nu n_\mu, \quad M^k_{\mu\nu} = \nabla_\nu m^k_\mu. \tag{36}$$

- Decomposition of the Weyl tensor

$$C_{\mu\nu\rho\sigma} = \underbrace{4C_{0i0j}n_{\{\mu}m^i_\nu n_\rho m^j_{\sigma\}}}_{\text{boost weight }+2} + \underbrace{8C_{010i}n_{\{\mu}l_\nu n_\rho m^i_{\sigma\}} + 4C_{0ijk}n_{\{\mu}m^i_\nu m^j_\rho m^k_{\sigma\}}}_{+1}$$
$$+ \underbrace{4C_{0101}n_{\{\mu}l_\nu n_\rho l_{\sigma\}} + 4C_{01ij}n_{\{\mu}l_\nu m^i_\rho m^j_{\sigma\}} + 8C_{0i1j}n_{\{\mu}m^i_\nu l_\rho m^j_{\sigma\}} + C_{ijkl}m^i_{\{\mu}}_{0}$$
$$+ \underbrace{8C_{101i}l_{\{\mu}n_\nu l_\rho m^i_{\sigma\}} + 4C_{1ijk}l_{\{\mu}m^i_\nu m^j_\rho m^k_{\sigma\}}}_{-1} + \underbrace{4C_{1i1j}l_{\{\mu}m^i_\nu l_\rho m^j_{\sigma\}}}_{-2}.$$

  For a type N solution, the non-zero Weyl scalars are $\psi_{ij} = C_{\mu\nu\rho\sigma}n^\mu m^{i\nu} n^\rho m^{j\sigma}$ in the null frame

- Correspondingly, $\phi_i = F_{\mu\nu}n^\mu m^{i\nu}$ are the non-zero Maxwell scalars of a degenerate Maxwell field
  [Coley, Milson, Pravda, Pravdova, 2004]

5d spinorial formalism

- Five γ matrices are chosen as the bases to connect the spacetime indices and spinor indices

$$\gamma^{\hat{\mu}}_{AB} = \begin{pmatrix} 0 & \sigma^{\hat{\mu}\alpha}{}_{\beta'} \\ \tilde{\sigma}^{\hat{\mu}}{}_{\alpha'}{}^{\beta} & 0 \end{pmatrix}, \quad \gamma^{4}_{AB} = -i \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\alpha'\beta'} \end{pmatrix}. \tag{37}$$

- $\hat{\mu} = 0, 1, 2, 3$ denotes the first four components of a 5d vector
- $A, B = 1, 2$ are spinor indices for 5d quantities
- α, β are now the spinor indices in the Van der Waerden matrices
- Any component of γ^{μ}_{AB}, for any given number of A, B, and $\mu = 0, 1, 2, 3, 4$, defines a (2×2) matrix

[Monteiro, Nicholson, O'Connell, 2018]

5d spinor bases

Correspondingly, the spinor bases in 5d are chosen as

$$
\begin{aligned}
k_1^A &= \begin{pmatrix} 0 \\ \bar{o}^{\alpha'} \end{pmatrix}, \quad k_2^A = \begin{pmatrix} o_\alpha \\ 0 \end{pmatrix}, \\
n_1^A &= \begin{pmatrix} \iota_\alpha \\ 0 \end{pmatrix}, \quad n_2^A = -\begin{pmatrix} 0 \\ \bar{\iota}^{\alpha'} \end{pmatrix}.
\end{aligned} \tag{38}
$$

One can simply package them as $k_a^A = (k_1^A, k_2^A)$ and $n_a^A = (n_1^A, n_2^A)$, where indices a and b are packaged from two basis spinors.

5d WDC relation for type N spacetime

- The algebraic classification in the spinorial formalism is equivalent to the CMPP classification.
[Monteiro, Nicholson, O'Connell, 2018]
- For the 5d type N spacetime, the non-zero little group bi-spinors for the Weyl tensor are

$$\psi^{(4)}_{abcd} = C_{\mu\nu\rho\sigma}\sigma^{\mu\nu}{}_{AB}\sigma^{\rho\sigma}{}_{CD}n^A{}_a n^B{}_b n^C{}_c n^D{}_d, \quad (39)$$

where $\sigma^{\mu\nu}{}_{AB} = \frac{1}{2}(\gamma^\mu{}_{AC}\gamma^{\nu C}{}_B - \gamma^\nu{}_{AC}\gamma^{\mu C}{}_B)$.

- A natural generalization of the WDC formula to 5d for the type N spacetime can be written as

$$\psi^{(4)}_{abcd} \propto \phi^{(2)}_{(ab}\phi^{(2)}_{cd)}, \quad (40)$$

where $\phi^{(2)}_{ab} = F_{\mu\nu}\sigma^{\mu\nu}{}_{AB}n^A{}_a n^B{}_b$ are the non-zero little group bi-spinors of a degenerate Maxwell tensor.

- More explicitly, we propose

$$\psi^{(4)}_{1111} = \frac{3c}{S_1}\phi^{(2)}_{11}\phi^{(2)}_{11}, \qquad \psi^{(4)}_{2222} = \frac{3c}{S'_1}\phi^{(2)}_{22}\phi^{(2)}_{22},$$
$$\psi^{(4)}_{1112} = \frac{3c}{S_2}\phi^{(2)}_{11}\phi^{(2)}_{12}, \qquad \psi^{(4)}_{2221} = \frac{3c}{S'_2}\phi^{(2)}_{22}\phi^{(2)}_{21}, \tag{41}$$
$$\psi^{(4)}_{1212} = \frac{c}{S_3}[\phi^{(2)}_{11}\phi^{(2)}_{22} + 2(\phi^{(2)}_{12})^2].$$

We have introduced two independent complex scalar fields S_1, S_2 and one real scalar field S_3.

5D Weyl double copy relation in vielbein formalism

- Converting the WDC formulas (41) to the vielbein formalism

$$
\begin{aligned}
\psi_{22} - \psi_{33} - 2i\psi_{23} &= \frac{3c}{S_1}[(\phi_2)^2 - (\phi_3)^2 - 2i\phi_2\phi_3], \\
\psi_{22} - \psi_{33} + 2i\psi_{23} &= \frac{3c}{S_1'}[(\phi_2)^2 - (\phi_3)^2 + 2i\phi_2\phi_3], \\
\psi_{34} + i\psi_{24} &= \frac{3c}{S_2}(\phi_3\phi_4 + i\phi_2\phi_4), \\
\psi_{34} - i\psi_{24} &= \frac{3c}{S_2'}(\phi_3\phi_4 - i\phi_2\phi_4), \\
\psi_{44} &= \frac{c}{S_3}[(\phi_2)^2 + (\phi_3)^2 + 4(\phi_4)^2].
\end{aligned}
\tag{42}
$$

The above equations present a natural extension of the WDC formulas to 5d for the type N spacetime.

Reductions of the type N spacetime

- We now consider a 4d-like reduction of the 5d type N spacetime where we impose the restrictions $\psi_{4i} = 0$ for the Weyl scalars.
- The traceless property of the Weyl tensor then yields $\psi_{22} = -\psi_{33}$. Correspondingly, the Maxwell scalars and zeroth copies should be $\phi_4 = 0$, $S_3 \to \infty$, where the divergent scalar field S_3 is chosen to turn off the extra component from the doubling in five dimensions.
- We introduce the following definitions

$$\begin{aligned}
&\Psi_4 = -(\psi_{22} + i\psi_{23}), \quad \bar{\Psi}_4 = -(\psi_{22} - i\psi_{23}),\\
&\varphi_2 = -\frac{1}{\sqrt{2}}(\phi_2 + i\phi_3), \quad \bar{\varphi}_2 = -\frac{1}{\sqrt{2}}(\phi_2 - i\phi_3),\\
&\delta = \frac{1}{\sqrt{2}}(\delta_2 - i\delta_3), \quad \bar{\delta} = \frac{1}{\sqrt{2}}(\delta_2 + i\delta_3),\\
&\hat{\delta} = i\delta_4, \quad S = \frac{S_1}{c}.
\end{aligned} \tag{43}$$

Bianchi identity and Maxwell's equation

$$
\begin{aligned}
D\Psi_4 &= -(L_{22} + iL_{23})\Psi_4, \\
\delta\Psi_4 &= -\frac{1}{\sqrt{2}}\Big[(2L_{12} + 2M^2_{33} - L_{21}) \\
&\qquad\qquad + i(L_{31} - 2L_{13} + 2M^2_{32})\Big]\Psi_4, \\
\hat{\delta}\Psi_4 &= [(M^3_{42} - 2M^3_{24}) + i(L_{41} - 2L_{14} + M^2_{42})]\Psi_4,
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
D\varphi_2 &= -(L_{22} + iL_{23})\varphi_2, \\
\delta\varphi_2 &= -\frac{1}{\sqrt{2}}\Big[(L_{12} + M^2_{33} - L_{21}) \\
&\qquad\qquad + i(L_{31} - L_{13} + M^2_{32})\Big]\varphi_2, \\
\hat{\delta}\varphi_2 &= [(M^3_{42} - M^3_{24}) + i(L_{41} - L_{14} + M^2_{42})]\varphi_2.
\end{aligned} \tag{45}
$$

5d Weyl double copy

- The WDC formula in the reduced case is simply given by

$$\Psi_4 = \frac{1}{S}(\varphi_2)^2. \tag{46}$$

- Now it is clear that the previously chosen divergent scalar field S_3 is to prevent the mixing term constructed from $\varphi_2\bar{\varphi}_2$ in the Weyl doubling.
- Such a term is not involved at all in the 4d WDC relation, and it is reasonable to have a special treatment in 5d by simply imposing that there is no Weyl scalar associated to the $\varphi_2\bar{\varphi}_2$ term.

Integrability conditions and wave equation

- Substituting the WDC relation (46) into the Bianchi identity and simplifying them with the Maxwell's equation on the type N background, we obtain the equations for the scalar field.

$$\begin{aligned} D\log S &= -(L_{22}+iL_{23}), \\ \delta\log S &= \frac{1}{\sqrt{2}}(L_{21}-iL_{31}), \\ \hat{\delta}\log S &= M^3_{42}+i(M^2_{42}+L_{41}). \end{aligned} \tag{47}$$

- Following the treatment in 4d, we have proven the integrability conditions for the above differential equations.
- It is straightforward to verify that any solution of (47) solves the Klein-Gordon equation $\Box S = 0$ on the type N background.

New realization of 4d type D WDC

- We extend the type N analysis in 4d to the type D case.
- Any type D vacuum solution admits an algebraically general Maxwell scalar on the curved background that squares to give the Weyl scalar.
- Bianchi identities

$$D\Psi_2 = 3\rho\Psi_2, \quad \Delta\Psi_2 = -3\mu\Psi_2, \quad \delta\Psi_2 = 3\tau\Psi_2, \quad \bar{\delta}\Psi_2 = -3\pi\Psi_2. \tag{48}$$

- Maxwell's equations

$$D\Phi_1 = 2\rho\Phi_1, \quad \Delta\Phi_1 = -2\mu\Phi_1, \quad \delta\Phi_1 = 2\tau\Phi_1, \quad \bar{\delta}\Phi_1 = -2\pi\Phi_1. \tag{49}$$

- The type D WDC relation

$$\Psi_2 = \frac{1}{S}\Phi_1^2. \tag{50}$$

New realization of 4d type D WDC

- Equations for the scalar field S

$$D\log S=\rho,\quad \Delta\log S=-\mu,\quad \delta\log S=\tau,\quad \bar{\delta}\log S=-\pi. \tag{51}$$

- Integrability conditions

$$\begin{aligned}
&(\delta D-D\delta)\log S=\delta\rho-D\tau, \quad (\bar{\delta}\Delta-\Delta\bar{\delta})\log S=\Delta\pi-\bar{\delta}\mu,\\
&(\Delta D-D\Delta)\log S=\Delta\rho+D\mu, \quad (\bar{\delta}\delta-\delta\bar{\delta})\log S=\bar{\delta}\tau+\delta\pi,\\
&(\bar{\delta}D-D\bar{\delta})\log S=\bar{\delta}\rho+D\pi, \quad (\delta\Delta-\Delta\delta)\log S=-\delta\mu-\Delta\tau.
\end{aligned} \tag{52}$$

- The WDC relation defines a scalar field

$$(\Box+2\Psi_2)S=0. \tag{53}$$

5d type D WDC

- We extend the type D WDC to 5d for a special class of vacuum solutions satisfying $C_{\hat{a}\hat{b}\hat{c}4} = 0$, where the integrability conditions consist of 10 equations.
- The scalar equaiton

$$\left[\Box - 2\psi_2 + 2(L_{41})^2\right] S = 0. \tag{54}$$

- Specially chosen fifth basis m^4 from $C_{\hat{a}\hat{b}\hat{c}4} = 0$ with the deviation tensor

$$\nabla_\mu m^4_\nu = L_{41}(m^2_\mu m^2_\nu + m^3_\mu m^3_\nu + l_\mu n_\nu + n_\mu l_\nu). \tag{55}$$

- The expansion L_{41} is the only non-zero component in the deviation that affects the evolution of the scalar S along the fifth direction m^4.

Thanks for your attention!