Is a Graviton Detectable?[1]

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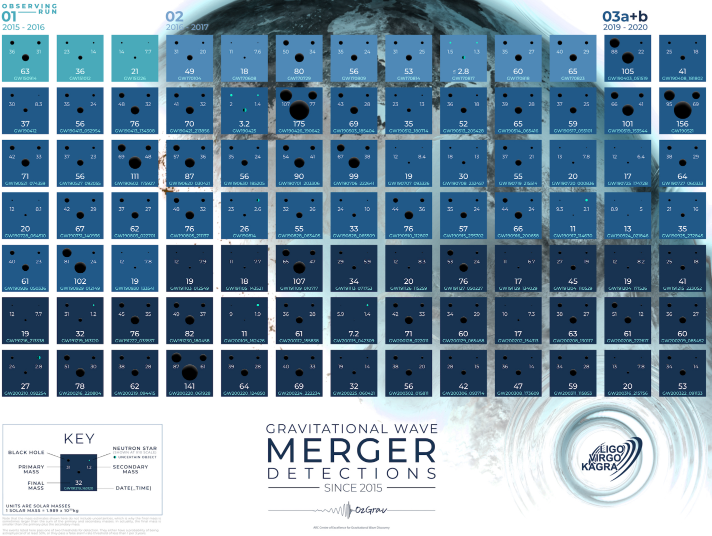
[1] F. Dyson, "Is a graviton detectable?," *Int. J. Mod. Phys. A* **28** (2013) 1330041.

Gravitational Waves



GRAVITON?

YES or NO!

F. Dyson: No!

The energy density in a gravitational wave is $\rho = \frac{1}{4} M_{\rm Pl}^2 \langle \partial_t h^{\mu\nu} \partial_t h_{\mu\nu} \rangle$, with $M_{\rm Pl}$ the reduced Planck mass.[6] For a wave of strain h and frequency ω, this corresponds to $\rho = \frac{1}{4} h^2 \omega^2 M_{\rm Pl}^2$. Dividing this energy up into gravitons of energy $E_\omega = \omega$, we find that the number of quanta per de Broglie volume $\lambda_{\rm dB}^3 = f^{-3}$ is

$$n\lambda_{\rm dB}^3 = \frac{\pi h^2 M_{\rm Pl}^2}{2 f^2} \simeq 2 \times 10^{35} \left(\frac{h}{10^{-22}}\right)^2 \left(\frac{1\ \rm kHz}{f}\right)^2, \qquad (1)$$

with $f = \omega / 2\pi$ is the linear frequency. Huge Number! But for $f \sim 10^{18} \rm Hz$ and $h \sim 10^{-27}$, one has

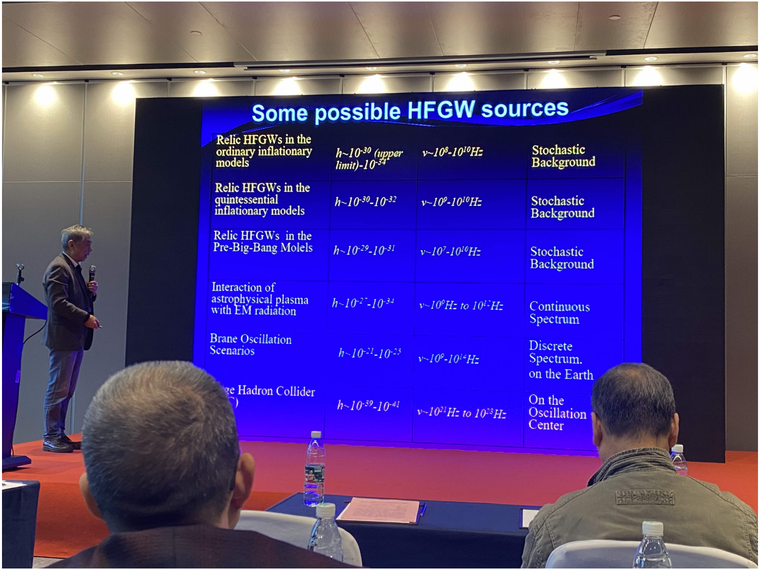
$$n\lambda_{\rm dB}^3 \sim 10^{-5}, \qquad (2)$$

which is highly dilute, does it mean individual graviton? For $n\lambda_{\rm dB}^3 \sim 1$, $h \simeq f / f_{\rm Pl}$ is required.

F. Dyson, "Is a graviton detectable?," *Int. J. Mod. Phys. A* **28** (2013) 1330041.

[6] $\hbar = c = 1$. $f_{\rm Pl} = 2.952 \times 10^{42} \rm Hz$

$f \simeq h f_{\rm Pl} \sim h \times 10^{42}{\rm Hz}$?



Taken from Prof. F.-Y. Li's talk.

S. Weinberg: No!

From the definition of energy-momentum tensor $T^{\mu\nu}$, one obtains

$$\delta S_{\rm m} = \frac{\delta S_{\rm m}}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu} + \cdots = -\frac{1}{2}\int d^4x\sqrt{-g}\,T^{\mu\nu}\,\delta g_{\mu\nu} + \cdots\,. \tag{3}$$

The Hamiltonian for the linear interaction is given

$$\mathcal{H}_{\rm int} = \frac{1}{2}h_{\mu\nu}T^{\mu\nu}, \tag{4}$$

Using perturbation theory, the rate at which gravitons are emitted by an atom using Fermi's Golden rule

$$\Gamma_{\rm spon} = \frac{2\pi}{\hbar}|\langle f|\,\hat{\mathcal{H}}_{\rm int}\,|i\rangle|^2\rho, \tag{5}$$

where $\rho = \frac{V\omega^2}{2\pi^2\hbar c^3}$ is the graviton density of states, at frequency ω, and a characteristic volume V. S. Weinberg calculated the quadrupole transition $3d \rightarrow 1s$, and got $\Gamma_{\rm spon} \approx 10^{-40}{\rm Hz}$.

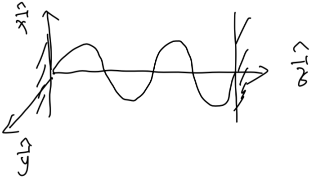
Classical: Waves

vs

Quantum: Gravitons

Hints from Quantum Optics

Electric field in a cavity resonator of length L



$$E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z), \;\; H_y(z,t) = \sum_j A_j \frac{\dot{q}_j \epsilon_0}{k_j} \cos(k_j z), \tag{6}$$

and the classical Hamiltonian,

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \int_V d\tau (\epsilon_0 E_x^2 + \mu_0 H_y^2), \\ &= \frac{1}{2} \sum_j (m_j \omega_j^2 q_j^2 + \frac{p_j^2}{m_j}), \end{aligned} \tag{7}$$

where $p_j = m_j \dot{q}_j$, $A_j = 2\omega_j^2 m_j / V\epsilon_0$ and $\omega_j = j\pi c/L$. Harmonic Oscillator!

Quantization: Promotion to Operators

Identifying q_j and p_j as operators which obey the commutation relations

$$[\hat{q}_j, \hat{p}_{j'}] = i\hbar\delta_{jj'}, \ [\hat{q}_j, \hat{q}_{j'}] = [\hat{p}_j, \hat{p}_{j'}] = 0, \tag{8}$$

A canonical transformation to operators $\hat{a}_j$ and $\hat{a}_j^\dagger$:

$$\mathcal{H} = \hbar \sum_j \omega_j (\hat{a}_j^\dagger \hat{a}_j + 1/2), \tag{9}$$

$$\hat{a}_j e^{-i\omega_j t} = (m_j\omega_j \hat{q}_j + i\hat{p}_j)/\sqrt{2m_j\hbar\omega_j}, \tag{10}$$

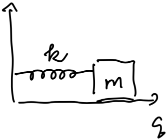
$$\hat{a}_j^\dagger e^{i\omega_j t} = (m_j\omega_j \hat{q}_j - i\hat{p}_j)/\sqrt{2m_j\hbar\omega_j}, \tag{11}$$

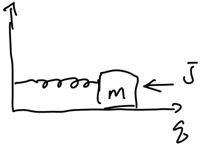
$$[\hat{a}_j, \hat{a}_{j'}^\dagger] = \delta_{jj'}, \ [\hat{a}_j, \hat{a}_{j'}] = [\hat{a}_j^\dagger, \hat{a}_{j'}^\dagger] = 0. \tag{12}$$

$$E_x(z,t) = \sum_j \left(\hbar w_j/\epsilon_0 V\right)^{1/2} \left(\hat{a}_j e^{-i\omega_j t} + \hat{a}_j^\dagger e^{i\omega_j t}\right) \sin(k_j z), \tag{13}$$

$$H_x(z,t) = \sum_j -ic\epsilon_0 \left(\hbar w_j/\epsilon_0 V\right)^{1/2} \left(\hat{a}_j e^{-i\omega_j t} - \hat{a}_j^\dagger e^{i\omega_j t}\right) \sin(k_j z), \tag{14}$$

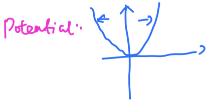
Harmonic Oscillators:

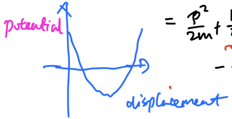




$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} k q^2$$

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} k q^2 - J q = \frac{p^2}{2m} + \frac{1}{2} k \left(q - \frac{J}{k}\right)^2 - \frac{1}{2} k \left(\frac{J}{k}\right)^2$$





$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} k q^2 - J(Gq - Hq^2)$$

$$= \frac{p^2}{2m} + \frac{1}{2}(k + 2JH) q^2 - JGq$$

$$k' = k + 2JH$$

Quantum States:

- Vacuum State:

$$\hat{a}|0\rangle = 0, \quad (15)$$

- Coherent State:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle = \hat{D}(\alpha)|0\rangle, \quad (16)$$

where $\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)$ is the displacement operator. Comparison to

$$\hat{\vec{E}}(\mathbf{x},t) = \sum_{\mathbf{k},\lambda} \mathcal{E}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}}^{(\lambda)} \hat{a}_{\mathbf{k},\lambda} e^{-i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})} + h.c, \vec{E}(\mathbf{x},t) = \sum_{\mathbf{k},\lambda} \mathcal{E}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}}^{(\lambda)} \alpha_{\mathbf{k},\lambda} e^{-i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})} \quad (17)$$

one finds out a coherent state is a nearly classical state.

- Squeezed State:

$$|\psi_s\rangle = \hat{S}(\xi)|\psi\rangle, \quad (18)$$

where $\hat{S}(\xi) = \exp\left[\frac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})\right]$ is the squeezed operator.

Bogoliubov transformation=Squeezing operation

The Bogoliubov transformation is given

$$[\hat{a}, \hat{a}^\dagger] = 1,\ \hat{b} = u\hat{a} + v\hat{a}^\dagger,\ \hat{b}^\dagger = u^*\hat{a}^\dagger + v^*\hat{a}, \tag{19}$$

where $|u|^2 - |v|^2 = 1$ for $[\hat{b}, \hat{b}^\dagger] = 1$.
From BCH lemma, one has

$$\hat{b} = \hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r, \tag{20}$$
$$\hat{b}^\dagger = \hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) = \hat{a}^\dagger\cosh r - \hat{a}e^{-i\theta}\sinh r. \tag{21}$$

Therefore one has

$$u = \cosh r,\ v = e^{i\theta}\sinh r,\ \xi = re^{i\theta}. \tag{22}$$

In general, one defines the squeezed coherent state

$$|\alpha, \xi\rangle = \hat{S}(\xi)\hat{D}(\alpha)|0\rangle. \tag{23}$$

Expectation Values in a $|\alpha,\xi\rangle$ state:

$$
\begin{aligned}
\langle\hat{a}\rangle &= \langle\alpha,\xi|\hat{a}|\alpha,\xi\rangle = \alpha\cosh r - \alpha^* e^{i\theta}\sinh r, && (24)\\
\langle\hat{a}^2\rangle = \langle\hat{a}^{\dagger 2}\rangle^* &= \alpha^2\cosh^2 r + (\alpha^*)^2 e^{i2\theta}\sinh^2 r \\
&\quad -2|\alpha|^2 e^{i\theta}\sinh r\cosh r - e^{i\theta}\cosh r\sinh r, && (25)\\
\langle\hat{a}^\dagger\hat{a}\rangle &= |\alpha|^2(\cosh^2 r + \sinh^2 r) - (\alpha^*)^2 e^{i\theta}\sinh r\cosh r \\
&\quad -\alpha^2 e^{-i\theta}\sinh r\cosh r + \sinh^2 r, && (26)
\end{aligned}
$$

where $\xi = re^{i\theta}$. Using $\hat{X}_1 = (\hat{a}+\hat{a}^\dagger)/2$, $\hat{X}_2 = (\hat{a}-\hat{a}^\dagger)/2i$ and $\hat{Y}_1 + i\hat{Y}_2 = (\hat{X}_1 + i\hat{X}_2)e^{-i\theta/2}$, one has

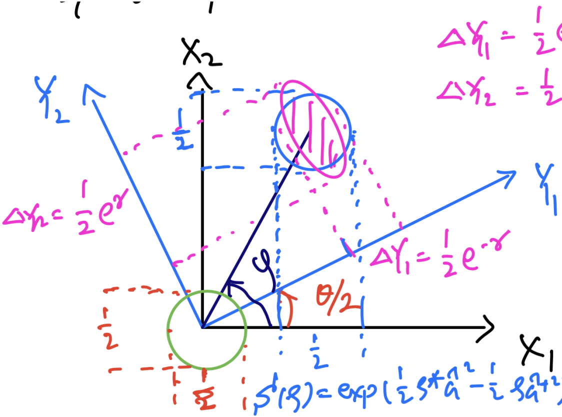
$$
\begin{aligned}
(\Delta\hat{Y}_1)^2 &= \langle\hat{Y}_1^2\rangle - \langle\hat{Y}_1\rangle^2 = e^{-2r}/4, && (27)\\
(\Delta\hat{Y}_2)^2 &= \langle\hat{Y}_2^2\rangle - \langle\hat{Y}_2\rangle^2 = e^{2r}/4. && (28)
\end{aligned}
$$

Thus, one has

$$(\Delta\hat{Y}_1)(\Delta\hat{Y}_2) = 1/4, \qquad (29)$$

the squeezed state is the minimum uncertainty state and is independent on α.

Error ellipse:



$\Delta Y_1 = \frac{1}{2}e^{-\gamma}$

$\Delta Y_2 = \frac{1}{2}e^{\gamma}$

$(\Delta Y_1)(\Delta Y_2) = \frac{1}{4}$

$|\alpha\rangle = D(\alpha)|0\rangle$

$\alpha = |\alpha| e^{i\varphi}$

$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$Y_1 + iY_2 = a e^{-i\theta/2}$

$S(\xi) = \exp\left(\frac{1}{2}\xi^* \hat{a}^2 - \frac{1}{2}\xi \hat{a}^{\dagger 2}\right)$,

$|\alpha, \xi\rangle = S(\xi) D(\alpha)|0\rangle$

$\xi = \gamma \exp(i\theta)$

Particles Number Distributions:

For a squeezed coherent state $|\alpha, \xi\rangle = \hat{S}(\xi)\hat{D}(\alpha)|0\rangle$,

$$\begin{aligned}
\langle \hat{n} \rangle &= \langle \alpha, \xi | \hat{a}^\dagger \hat{a} | \alpha, \xi \rangle \\
&= |\alpha|^2 (\cosh^2 r + \sinh^2 r) - (\alpha^*)^2 e^{i\theta} \sinh r \cosh r \\
&\quad -\alpha^2 e^{-i\theta} \sinh r \cosh r + \sinh^2 r, \qquad (30) \\
\langle \hat{n}^2 \rangle &= \langle \alpha, \xi | (\hat{a}^\dagger \hat{a})^2 | \alpha, \xi \rangle \\
&= (|\alpha|^2 + \sinh^2 r)^2 + 2 \sinh^2 r \cosh^2 r + |\alpha \cosh r - \alpha e^{i\theta} \sinh r|^2 \qquad (31)
\end{aligned}$$

thus

$$\begin{aligned}
(\Delta \hat{n})^2 &= \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \\
&= |\alpha|^2 \left[\cosh 2r - \cos(\theta - 2\phi) \sinh 2r\right] + 2 \sinh^2 r \cosh^2 r. \qquad (32)
\end{aligned}$$

where $\alpha = |\alpha| e^{i\phi}$ and $\xi = r e^{i\theta}$.

Fano Factor: $\mathcal{F} = (\Delta\hat{n})^2/\langle\hat{n}\rangle$

- For the squeezed vacuum state $\alpha = 0$:

$$(\Delta\hat{n})^2 = 2\sinh^2 r\cosh^2 r \geqslant \langle\hat{n}\rangle = \sinh^2 r,\ \mathcal{F} \geqslant 1, \text{Super-Poisson} \tag{33}$$

- For the coherent state $\xi = 0$ ('classical' quantum state):

$$(\Delta\hat{n})^2 = \alpha^2 = \langle\hat{n}\rangle = \alpha^2,\ \mathcal{F} = 1, \text{Poisson} \tag{34}$$

- For the squeezed coherent state $\alpha \neq 0$:
  - $\theta = 2\phi$, coherent and squeeze have the same direction ('non-classical' quantum state):

$$(\Delta\hat{n})^2 = |\alpha|^2 e^{-2r} + 2\sinh^2 r\cosh^2 r, \mathcal{F} \sim e^{-2r} < 1,\ |\alpha|^2 \gg e^{2r}, \text{Sub-Poiss} \tag{35}$$

  - $\theta = 2\phi + \pi$:

$$(\Delta\hat{n})^2 = |\alpha|^2 e^{2r} + 2\sinh^2 r\cosh^2 r, \mathcal{F} \sim e^{2r} > 1,\ |\alpha|^2 \gg e^{2r}, \text{Super-Poisso} \tag{36}$$

Subsummary: 'non-classical' states:

Measurement on a gravitational radiation having quantum properties requires:

- Sources have non-classical state of the radiation mode, e.g. squeezed coherent state;
- Detect some signatures of this non-classicality, e.g. correlation statistics, sub-poisson...

Sources of 'Non-classical' Quantum States

Gravitational Waves

In quantum field theory, the metric field $h^{\lambda}_{\mathbf{k}}(\eta)$ is promoted to the operator. The operator $h^{\lambda}_{\mathbf{k}}$ satisfies

$$h''^{\lambda}_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right) h^{\lambda}_{\mathbf{k}} = 0\,. \qquad (37)$$

The desired solutions of the scale factors (squeezed term) are given by:

$$a(\tau) = \begin{cases} -\dfrac{1}{H_*\tau} & \textbf{De Sitter} \\ -\dfrac{1}{H_*\tau}\left[1 + \epsilon_* - \epsilon_* \ln\left(\dfrac{\tau}{\tau_*}\right) + \cdots\right] & \textbf{Inflation/Quasi De Sitter } (w \approx -1) \\ \left[\dfrac{(1+3w)}{3(1+w)}\tau\right]^{\frac{2}{(1+3w)}} & \textbf{Reheating } (0 < w < 1/3) \\ \dfrac{\tau}{2} & \textbf{Radiation } (w = 1/3) \\ \dfrac{\tau^2}{9} & \textbf{Matter } (w = 0) \\ a_* \left(-\gamma\dfrac{\tau}{\tau_*}\right)^{\alpha_*} & \textbf{Contraction (Pre-Bounce)} \\ a_* \left[1 + \left(\dfrac{\tau}{\tau_*}\right)^2\right] & \textbf{Matter Bounce} \\ a_* \,\mathrm{sech}\,(\alpha_*(\tau - \tau_*)) & \textbf{Sechyperbolic Bounce} \\ a_* \cosh(\alpha_*(\tau - \tau_*)) & \textbf{Coshyperbolic Bounce} \\ a_* \sinh(\alpha_*(\tau - \tau_*)) & \textbf{Sinhyperbolic Bounce} \\ a_* \,\mathrm{cosech}\,(\alpha_*(\tau - \tau_*)) & \textbf{Cosechyperbolic Bounce} \\ a_* \exp(\mathrm{InverseErf}\,(\alpha_*(\tau - \tau_*))) & \textbf{Exponential Bounce} \\ a_* \left(\dfrac{\tau}{\tau_*}\right)^{\frac{\alpha}{1-\alpha}} & \textbf{Power-Law Bounce } (0 < \alpha < 1) \\ a_* \sqrt{\gamma\left(\dfrac{\tau}{\tau_*}\right) + \delta\left(\dfrac{\tau}{\tau_*}\right)^2} & \textbf{Polynomial Bounce} \\ a_* \left(\gamma\dfrac{\tau}{\tau_*}\right)^{\alpha_*} & \textbf{Expansion (Post-Bounce)} \\ a_* \left[1 - \cos(\alpha_*(\tau - \tau_*))\right] & \textbf{Matter Cyclic} \\ a_* \sin(\alpha_*(\tau - \tau_*)) & \textbf{Radiation Cyclic} \\ a_*^{3/2} \exp\left(\dfrac{i\pi}{2}\right)\sqrt{\dfrac{2}{3}}\,\tau^{1/2} & \textbf{Black Hole Gas} \end{cases}$$

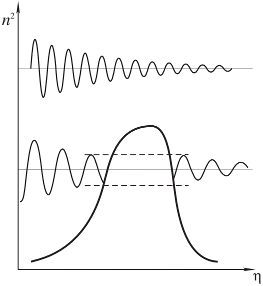


Fig. 3. Effective potential U(η).

L. P. Grishchuk, arXiv:gr-qc/0002035; K. Adhikari, *et al.*arXiv:2108.10334.

For the PGW, *Mukhanov-Sasaki equation* is:

$$h''^{\lambda}_{\mathbf{k}}(\eta) + \left(k^2 - \frac{1}{\eta^2}\left(\nu^2_{\mathrm{PGW}}(\eta) - \frac{1}{4}\right)\right) h^{\lambda}_{\mathbf{k}}(\eta) = 0. \tag{38}$$

where $\nu_{\mathrm{PGW}}(\eta) = \frac{1}{2}\sqrt{1 + 4(\eta\mathcal{H})^2\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)}$. The most general solution for rescaled PGW field variable for any quantum initial vacuum state can be written as:

$$\begin{aligned} h^{\lambda}_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2k}} 2^{\nu_{\mathrm{PGW}} - \frac{3}{2}} (-k\eta)^{\frac{3}{2} - \nu_{\mathrm{PGW}}} \left| \frac{\Gamma(\nu_{\mathrm{PGW}})}{\Gamma\left(\frac{3}{2}\right)} \right| \\ \times \left[\mathcal{C}_1 \left(1 - \frac{i}{k\eta}\right) \exp\left(-i\left\{k\eta + \frac{\pi}{2}\left(\nu_{\mathrm{PGW}} - \frac{3}{2}\right)\right\}\right) \right. \\ \left. + \mathcal{C}_2 \left(1 + \frac{i}{k\eta}\right) \exp\left(i\left\{k\eta + \frac{\pi}{2}\left(\nu_{\mathrm{PGW}} - \frac{3}{2}\right)\right\}\right) \right]. \end{aligned}$$

Here, $|\mathcal{C}_1|^2 - |\mathcal{C}_2|^2 = 1$, are arbitrary integration constants, fixed by the appropriate choice of quantum initial vacuum state.

- **Motta-Allen (α, γ) vacua:** This is considered as a excited $SO(1,4)$ isometric vacuum state having oscillating feature. One can explicitly show that his type of quantum excited vacua is CPT symmetry breaking in nature and it is characterized by the real parameter γ.

$$\mathcal{C}_1 = \cosh\alpha, \quad \mathcal{C}_2 = \exp(i\gamma)\ \sinh\alpha \quad \forall \alpha, \gamma \in \mathbb{R}, \\ \textbf{subject to} \ \ |\mathcal{C}_1|^2 - |\mathcal{C}_2|^2 = 1. \tag{40}$$

$$f_{\lambda,\mathbf{k}}(\tau) = \frac{1}{\sqrt{2k}} 2^{\nu_{\rm PGW}-\frac{3}{2}} (-k\tau)^{\frac{3}{2}-\nu_{\rm PGW}} \left|\frac{\Gamma(\nu_{\rm PGW})}{\Gamma\left(\frac{3}{2}\right)}\right| \\ \times \left[\cosh\alpha\ \left(1 - \frac{i}{k\eta}\right)\ \exp\left(-i\left\{k\eta + \frac{\pi}{2}\left(\nu_{\rm PGW} - \frac{3}{2}\right)\right\}\right)\right. \\ \left. + \sinh\alpha\ \left(1 + \frac{i}{k\eta}\right)\ \exp\left(i\left\{\gamma + k\eta + \frac{\pi}{2}\left(\nu_{\rm PGW} - \frac{3}{2}\right)\right\}\right)\right]. \tag{41}$$

- α **vacua:** This is considered as a excited $SO(1,4)$ isometric vacuum state having oscillating feature. One can explicitly show that his type of quantum excited vacua is CPT symmetry preserving in nature.

$$\mathcal{C}_1 = \cosh\alpha, \quad \mathcal{C}_2 = \sinh\alpha \quad \forall\alpha \in \mathbb{R}, \quad \textbf{subject to} \;\; |\mathcal{C}_1|^2 - |\mathcal{C}_2|^2 = 1. \tag{42}$$

$$h^{\lambda}_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2k}} 2^{\nu_{\rm PGW}-\frac{3}{2}} (-k\eta)^{\frac{3}{2}-\nu_{\rm PGW}} \left|\frac{\Gamma(\nu_{\rm PGW})}{\Gamma\left(\frac{3}{2}\right)}\right| \times \Bigg[\cosh\alpha \left(1-\frac{i}{k\eta}\right) \exp\left(-i\left\{k\eta+\frac{\pi}{2}\left(\nu_{\rm PGW}-\frac{3}{2}\right)\right\}\right) + \sinh\alpha \left(1+\frac{i}{k\eta}\right) \exp\left(i\left\{k\eta+\frac{\pi}{2}\left(\nu_{\rm PGW}-\frac{3}{2}\right)\right\}\right)\Bigg]. \tag{43}$$

- **Bunch-Davies vacuum:** This is considered as a $SO(1,4)$ isometric ground state of the initial vacuum used in primordial cosmology. In literature *Bunch-Davies* vacuum state is identified as *Cherenkov vacuum* or *Hartle-Hawking vacuum* state, which is actually an *Euclidean* state.

$$\mathcal{C}_1 = 1, \quad \mathcal{C}_2 = 0 \quad \textbf{subject to} \quad |\mathcal{C}_1|^2 - |\mathcal{C}_2|^2 = 1. \tag{44}$$

$$h_{\mathbf{k}}^{\lambda}(\eta) = \frac{1}{\sqrt{2k}} 2^{\nu_{\rm PGW} - \frac{3}{2}} (-k\eta)^{\frac{3}{2} - \nu_{\rm PGW}} \left| \frac{\Gamma(\nu_{\rm PGW})}{\Gamma\left(\frac{3}{2}\right)} \right| \times \left(1 - \frac{i}{k\eta}\right) \exp\left(-i\left\{k\eta + \frac{\pi}{2}\left(\nu_{\rm PGW} - \frac{3}{2}\right)\right\}\right). \tag{45}$$

Relations between States at Different Stages:

In the inflationary era:

$$h_{\mathbf{k}}^{\lambda}(\eta) = b_{\mathbf{k}}^{\lambda}\, v_{\mathbf{k}}^{\mathrm{I}}(\eta) + b_{-\mathbf{k}}^{\lambda\dagger}\, v_{\mathbf{k}}^{\mathrm{I}*}(\eta)\,, \quad \left[b_{\mathbf{k}}^{\lambda}, b_{\mathbf{p}}^{\lambda'\dagger}\right] = \delta^{\lambda\lambda'}\delta_{\mathbf{k},\mathbf{p}} \tag{46}$$

$$v_{\mathbf{k}}^{\mathrm{I}}(\eta) \equiv \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}\,. \tag{47}$$

In the radiation-dominated era,

$$h_{\mathbf{k}}^{\lambda}(\eta) = c_{\mathbf{k}}^{\lambda}\, v_{\mathbf{k}}^{\mathrm{R}}(\eta) + c_{-\mathbf{k}}^{\lambda\dagger}\, v_{\mathbf{k}}^{\mathrm{R}*}(\eta)\,, \quad \left[c_{\mathbf{k}}^{\lambda}, c_{\mathbf{p}}^{\lambda'\dagger}\right] = \delta^{\lambda\lambda'}\delta_{\mathbf{k},\mathbf{p}} \tag{48}$$

$$v_{\mathbf{k}}^{\mathrm{R}}(\eta) \equiv \frac{1}{\sqrt{2k}}\,e^{-ik\eta}\,. \tag{49}$$

They are related by the *Bogoliubov transformation*

$$b_{\mathbf{k}} = \alpha_{\mathbf{k}}^{*}\, c_{\mathbf{k}} - \beta_{\mathbf{k}}\, c_{-\mathbf{k}}^{\dagger}\,, \alpha_{\mathbf{k}} = \left(1 - \frac{1}{2k^2\eta_T^2} - \frac{i}{k\eta_T}\right)e^{-2ik\eta_T}, \beta_{\mathbf{k}} = \frac{1}{2k^2\eta_T^2}\,, \tag{50}$$

$$\hat{S}(\zeta) = \exp\left[\zeta^{*} c_{\mathbf{k}}\, c_{-\mathbf{k}} - \zeta\, c_{\mathbf{k}}^{\dagger}\, c_{-\mathbf{k}}^{\dagger}\right], \quad \zeta = r_{\mathbf{k}} e^{i\theta}, \quad \tanh r_{\mathbf{k}} = \left|\frac{\beta_{\mathbf{k}}}{\alpha_{\mathbf{k}}^{*}}\right|. \tag{51}$$

BD Vacuum as a Coherent State:

From the definition of energy-momentum tensor $T^{\mu\nu}$, one obtains

$$\delta S_{\rm m} = \int d^4x \, \frac{\delta S_{\rm m}}{\delta g_{\mu\nu}} \, \delta g_{\mu\nu} + \cdots = -\frac{1}{2} \int d^4x \sqrt{-g} \, T^{\mu\nu} \, \delta g_{\mu\nu} + \cdots . \quad (52)$$

The interaction Hamiltonian becomes

$$\begin{aligned} i \int d\eta H_{\rm int} &= \frac{i}{2} \int d\eta \int d^3x \, a^2(\eta) h_{ij}(\mathbf{x}, \eta) \, T_{ij}(\mathbf{x}, \eta) \\ &= \sum_{\mathbf{k}} \left[\xi_{\mathbf{k}} \, b^\dagger_{\mathbf{k}} - \xi^*_{\mathbf{k}} \, b_{\mathbf{k}} \right] , \end{aligned} \quad (53)$$

where the coefficients $\xi_{\mathbf{k}}$ is given by

$$\xi_{\mathbf{k}} = -\frac{i}{2M_{Pl}} \int d\eta \, a(\eta) \, v^{\rm I *}_{\mathbf{k}}(\eta) \, \mathcal{T}(-\mathbf{k}, \eta) . \quad (54)$$

This interaction generates a coherent state such as

$$\begin{aligned} |\xi_{\mathbf{k}}\rangle_{\rm I} &= \exp\left[-i \int d\eta H_{\rm int} \right] |0\rangle_{\rm I} \\ &= \prod_{\mathbf{k}} \exp\left[\xi_{\mathbf{k}} \, b^\dagger_{\mathbf{k}} - \xi^*_{\mathbf{k}} \, b_{\mathbf{k}} \right] |0\rangle_{\rm I} . \end{aligned} \quad (55)$$

Non-classical Quantum State Conditions:

The non-classical quantum state requires the Fano factor $\mathcal{F} < 1$, *i.e.*

$$|\xi_{\mathbf{k}}|^2(1 - e^{-2r_{\mathbf{k}}}) > \sinh^2 r_{\mathbf{k}} + 2\sinh^4 r_{\mathbf{k}}, \tag{56}$$

Considering PGWs on super-horizon scales $r_{\mathbf{k}} \gg 1$, one has

$$|\xi_{\mathbf{k}}|^2 > e^{4r_{\mathbf{k}}}/8. \tag{57}$$

Using

$$\sinh r_{\mathbf{k}} = \frac{1}{2k^2\eta_T^2} = \frac{1}{2}\left(\frac{f_T}{f}\right)^2, \tag{58}$$

one has

$$e^{r_{\mathbf{k}}} \simeq \left(\frac{f_T}{f}\right)^2, \qquad |\xi_{\mathbf{k}}|^2 > \frac{1}{8}\left(\frac{f_T}{f}\right)^8 \geqslant \frac{1}{8}. \tag{59}$$

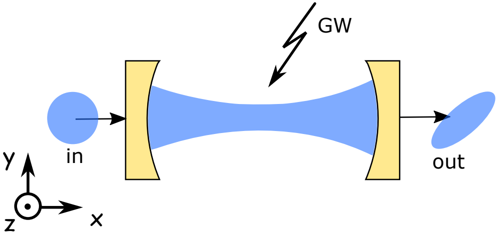
or

$$f > \left(\frac{1}{8}\right)^{\frac{1}{8}} 10^9\, |\xi_k|^{-\frac{1}{4}} \sqrt{\frac{H}{10^{-4}M_{\rm pl}}}\quad [\mathrm{Hz}]\,. \tag{60}$$

S. Kanno, J. Soda, arXiv:1810.07604.

Detection of 'Non-classical' Quantum States

Fabry-Pérot Cavity:



The resonance frequency of the cavity fundamental mode will then be

$$\omega = \frac{n\pi}{\ell_0(1+\frac{1}{2}h)} = \omega_0\left(1 - \frac{1}{2}h + \mathcal{O}(h^2)\right) \tag{61}$$

where $\omega_0 = n\pi/\ell_0$ and n is an integer. The total Hamiltonian is

$$\hat{H} = \hat{H}_q^{(0)} + \hat{H}_{EM}^{(0)} + \hat{H}_{GW}^{(0)} + \hat{H}_{OM} + \hat{H}_{GW}^{\text{int}}. \tag{62}$$

T. Guerreiro, arXiv:arXiv:1911.11593.

Interaction Hamiltonian:

The interaction Hamiltonian,

$$\hat{H}_{GW}^{\text{int}} = -\frac{\omega_0}{4}\hat{a}^\dagger\hat{a}\sum_k\left(\sqrt{\frac{8\pi G}{kV}}\hat{b}_k + \text{h.c.}\right), \tag{63}$$

The effective interaction between the electromagnetic and gravitational fields is then

$$\hat{H} = \omega_0\hat{a}^\dagger\hat{a} + \sum_k\Omega_k\hat{b}_k^\dagger\hat{b}_k - \frac{\omega_0}{4}\hat{a}^\dagger\hat{a}\sum_k\left(\sqrt{\frac{8\pi G}{kV}}\hat{b}_k + \text{h.c.}\right),$$

Here, $a^\dagger a$ counts the number of excitations in the electromagnetic field coming from the free electromagnetic Hamiltonian; similarly for $b_k^\dagger b_k$ which counts the number of excitations in the gravitational field. Using $h_k = \sqrt{8\pi G/kV}$ the gravity strain for mode k, $g_k = \omega_0 h_k/4$ and the dimensionless quantities $q_k = g_k/\Omega_k$, the Hamiltonian becomes

$$\hat{H} = \omega_0\hat{a}^\dagger\hat{a} + \sum_k\Omega_k\hat{b}_k^\dagger\hat{b}_k - \sum_k g_k\hat{a}^\dagger\hat{a}\ (\hat{b}_k + \hat{b}_k^\dagger). \tag{64}$$

Evolution Operator:

The evolution operator is of the form

$$\hat{U}(t) = e^{-i\hat{H}t} = \prod_k \hat{U}_k(t) \tag{65}$$

where each unitary $\hat{U}_k(t)$ acts jointly *only* on the cavity mode and the gravitational wave mode k, having the form

$$\hat{U}_k(t) = e^{iA_k(t)(\hat{a}^\dagger \hat{a})^2} e^{q_k \hat{a}^\dagger \hat{a}(\gamma_k \hat{b}_k^\dagger - \gamma_k^* \hat{b}_k)} \tag{66}$$

where $\gamma_k = 1 - e^{-i\Omega_k t}$ and $A_k(t) = q_k^2 \left(\Omega_k t - \sin \Omega_k t\right)$. The Heisenberg equation for the oscillator's annihilation operator reads

$$\begin{aligned}
\hat{a}(t) &= \left(\prod_k \hat{U}_k(t)\right)^\dagger \hat{a} \left(\prod_k \hat{U}_k(t)\right) = && (67)\\
&= \prod_k e^{iA_k(t)} e^{i2A_k(t)\hat{a}^\dagger \hat{a}} e^{q_k(\gamma_k \hat{b}_k^\dagger - \gamma_k^* \hat{b}_k)} \hat{a} && (68)\\
&= \prod_k e^{iA_k(t)} e^{i2A_k(t)\hat{a}^\dagger \hat{a}} \hat{\mathcal{D}}(-q_k\gamma_k)\ \hat{a} && (69)
\end{aligned}$$

T. Guerreiro, arXiv:arXiv:1911.11593

Meaning of $U_k(t)$:

$$\hat{U}_k(t) = e^{iA_k(t)(\hat{a}^\dagger\hat{a})^2} e^{q_k\hat{a}^\dagger\hat{a}(\gamma_k\hat{b}_k^\dagger - \gamma_k^*\hat{b}_k)}, \tag{70}$$
$$A_k(t) = q_k^2\left(\Omega_k t - \sin\Omega_k t\right), \tag{71}$$
$$\gamma_k = 1 - e^{-i\Omega_k t} \tag{72}$$

- $e^{iA_k(t)(\hat{a}^\dagger\hat{a})^2}$ is Kerr like squeezed on EM mode.
- $e^{q_k\hat{a}^\dagger\hat{a}(\gamma_k\hat{b}_k^\dagger - \gamma_k^*\hat{b}_k)}$:
  - Acting on the EM mode: GW-induced phase operator acting on EM, the amounts proportional to the GW quadrature,

$$e^{q_k\hat{a}^\dagger\hat{a}(\gamma_k\hat{b}_k^\dagger - \gamma_k^*\hat{b}_k)}, \tag{73}$$

  - Acting on the GW mode: A displacement operator proportional to the number of photons contained in EM field,

$$e^{q_k\hat{a}^\dagger\hat{a}(\gamma_k\hat{b}_k^\dagger - \gamma_k^*\hat{b}_k)} = \mathcal{D}(-q_k\gamma_k(\hat{a}^\dagger\hat{a})) = \mathcal{D}(-q_k\gamma_k\bar{n}). \tag{74}$$

T. Guerreiro, arXiv:arXiv:1911.11593.

Mean for EM field:

It is reasonable to consider the input state is a tensor product: $|\Psi\rangle = |\alpha\rangle_{\rm EM} \otimes |\Psi\rangle_{\rm GW}$, here $|\Psi\rangle_{\rm GW}$ quantum states will take vacuum, coherent and squeezed coherent respectively. The field quadrature mean value $\langle \hat{\tilde{\mathcal{E}}}_+ \rangle \equiv \langle \hat{\mathcal{E}}_+ \rangle / \mathcal{E}_s = \langle \hat{a}^\dagger \rangle + \langle \hat{a} \rangle$, $\mathcal{E}_s = \sqrt{\omega_0 / 2\ell_0^3}$.

- Vacuum:

$$\langle \tilde{\mathcal{E}}_+ \rangle = 2\alpha D(t) e^{-\alpha^2(1-\cos 2F(t))} \cos\left(-F(t) + \alpha^2 \sin 2F(t)\right) \quad (75)$$

$$D(t) = \prod_k \langle 0_k | \hat{\mathcal{D}}(-q_k\gamma_k) | 0_k \rangle = \exp\left(-\frac{1}{2}\sum_k q_k^2 |\gamma_k|^2\right) \simeq 0, \quad (76)$$

$$F(t) = \sum_k A_k(t) \simeq \left(\frac{\omega_0}{E_{\rm Pl}}\right)\omega_0 t \ \propto (\omega_0/E_{\rm Pl})^2. \quad (77)$$

$$\begin{aligned}\Delta\hat{\tilde{\mathcal{E}}}_+^2 &= \langle \hat{\tilde{\mathcal{E}}}_+^2 \rangle - \langle \hat{\tilde{\mathcal{E}}}_+ \rangle^2 \\ &= 2\alpha^2 D^2 e^{-\alpha^2(1-\cos 4F(t))} \cos\left(-4F(t) + \alpha^2 \sin 4F(t)\right) \\ &\quad -2\alpha^2 D^2 e^{-2\alpha^2(1-\cos 2F(t))} \cos\left(-4F(t) + 2\alpha^2 \sin 2F(t)\right) \\ &\quad +2\alpha^2\left(1 - D^2 e^{-2\alpha^2(1-\cos 2F(t))}\right) + 1, \end{aligned} \quad (78)$$

Fano Factor $\mathcal{F} > 1$!

Mean for EM field:

- Coherent:

$$\langle \hat{a}(t) \rangle = \langle \alpha | \hat{a} | \alpha \rangle_{\text{EM}} \langle \Psi | \hat{\mathcal{D}}(-q_{k_{GW}} \gamma_{\text{GW}}) | \Psi \rangle_{\text{GW}} \prod_{k \neq k_{GW}} \langle 0_k | \hat{\mathcal{D}}(-q_k \gamma_k) | 0_k \rangle$$

$$\langle \hat{\hat{\mathcal{E}}}_+ \rangle = \alpha \left(\langle \lambda e^{i\Omega_{GW} t} | \hat{\mathcal{D}}(-q\gamma) | \lambda e^{i\Omega_{GW} t} \rangle + c.c. \right) \quad (79)$$

$$= \alpha \left[e^{\left(-\frac{1}{2} q^2 |\gamma|^2 - i(\lambda q) \sin \Omega_{GW} t\right)} + c.c. \right] \quad (80)$$

GW induces a phase $e^{i\Delta\phi}$ with $\Delta\phi = (q\lambda) \sin \Omega_{GW} t$. Using $E = (1/32\pi G)\Omega_{GW}^2 h^2$, $E_g = \Omega_{GW}/V$, then for GW in coherent sate

$$\lambda = \sqrt{\langle N \rangle} = \sqrt{E/E_g} = \sqrt{V\Omega_{GW}/32\pi G} h, h = \sqrt{8\pi G/kV} \quad (81)$$

$$\Delta\phi = \frac{\omega_0}{8\Omega_{GW}} h \sin \Omega_{GW} t \quad (82)$$

The classical result is recovered: $\frac{\omega_0}{8\Omega_{GW}} h \sim 10^{-9}$ for $\Omega_{GW} = 100\text{Hz}$, $\omega_0 = 10^{10}\text{Hz}$, $h \sim 10^{-21}$, and $\Delta\phi = b \times (2\pi \ell_0/\lambda) h \approx 10^{-9}$, where $b \approx 200$ the number of bounces of the photon within the interferometer.

Mean for EM field:

- Squeezed: $|\psi\rangle = |\xi_0 e^{i\theta}\rangle$, where ξ_0 are chosen real numbers for simplicity and $\theta = 2\Omega_{GW} t$ is the time dependency of the squeezed state. The squeezed state as a squeezing operator acting on the vacuum state $|\xi_0 e^{i\theta}\rangle = \hat{\mathcal{S}}(\xi)|0\rangle$. The mean electric field is

$$\langle \hat{\mathcal{E}}_+ \rangle = \alpha \langle 0|\hat{\mathcal{S}}^\dagger(\xi)\hat{\mathcal{D}}(-q\gamma)\hat{\mathcal{S}}(\xi)|0\rangle + c.c. \quad (83)$$

Using $\hat{\mathcal{D}}(-q\gamma)\hat{\mathcal{S}}(\xi) = \hat{\mathcal{S}}(\xi)\hat{\mathcal{D}}\left[(-q\gamma)\cosh\xi_0 + (-q\gamma)^* e^{i\theta}\sinh\xi_0\right]$, one has ($\xi_0 \gg 1$ super-horizon scales)

$$\langle \hat{\mathcal{E}}_+ \rangle = \alpha \exp\left(-\frac{1}{2}|-qe^{\xi_0}(\gamma + \gamma^* e^{i\theta})|^2\right) + c.c. \quad (84)$$

$$\simeq 2\alpha\left(1 - 8\left(\frac{\omega_0}{E_{\rm Pl}}\right)^2\left(\frac{f_T}{\Omega_{GW}}\right)^4 \sin^4\left(\frac{\Omega_{GW} t}{2}\right)\right) \quad (85)$$

$e^{\xi_0} \approx \sinh\xi_0 = (f_T/\Omega_{GW})^2/2$, $f_T = 10^9\sqrt{H/10^{-4}M_{\rm Pl}}$ $[\rm Hz]$. For $H = 10^{-4}M_{\rm pl}$ and $\Omega_{GW} = 0.1[\rm Hz]$ the pre-factor is approximately 10^{-15}.

Summary

1. Non-classical Sources of GW are Required.

2. Non-classical Correlation Statistics are Required.

3. Then, ... MIGHT BE!

Thank You!

Decoherence

Following [40], note that a thermal background multiplies a damping factor at most $e^{-4q_{\max}^2(2\bar{n}+1)}$ where $\bar{n} = 1/(e^{\hbar\Omega_k/k_BT} - 1)$ and $q_{\max} \sim \omega_0/E_{Pl}$. For a gravitational wave mode of 10[Hz] (the expected peak of the cosmic spectrum) at the predicted temperature of the primordial gravitational wave background of $T \sim 1[\mathrm{K}]$ [41], this damping factor is approximately $\exp\left(-10^{-47}\right) \simeq 1$. This is essentially saying that the decoherence due to the stochastic gravitational wave background is a negligible effect.

[40] Y. Ma, F. Armata, K. E. Khosla and M. S. Kim, *Optical squeezing for an optomechanical system without quantising the mechanical motion*, Phys. Rev. Research 2, 023208 (2020).

[41] B. Allen, *The stochastic gravity-wave background: sources and detection*, arXiv:gr-qc/9604033 (1996).