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# Zeno's Paradox and Black Hole Information Loss


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- Island Rule and Replica Wormholes
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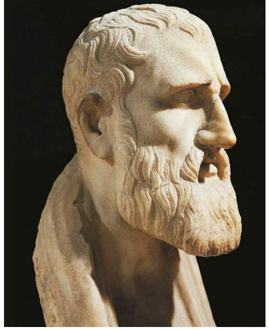
# Zeno's Paradox

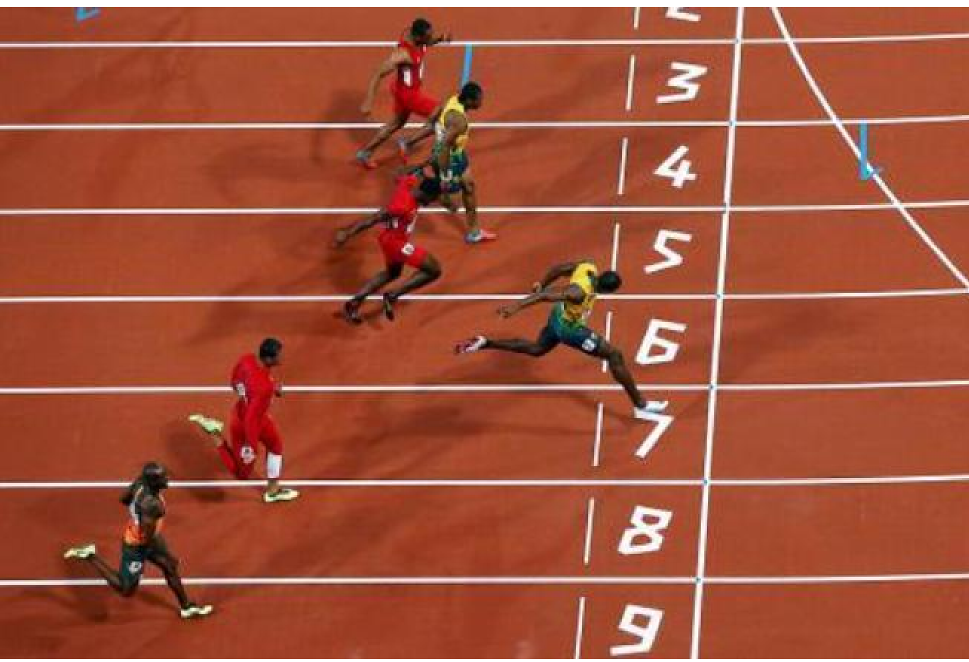


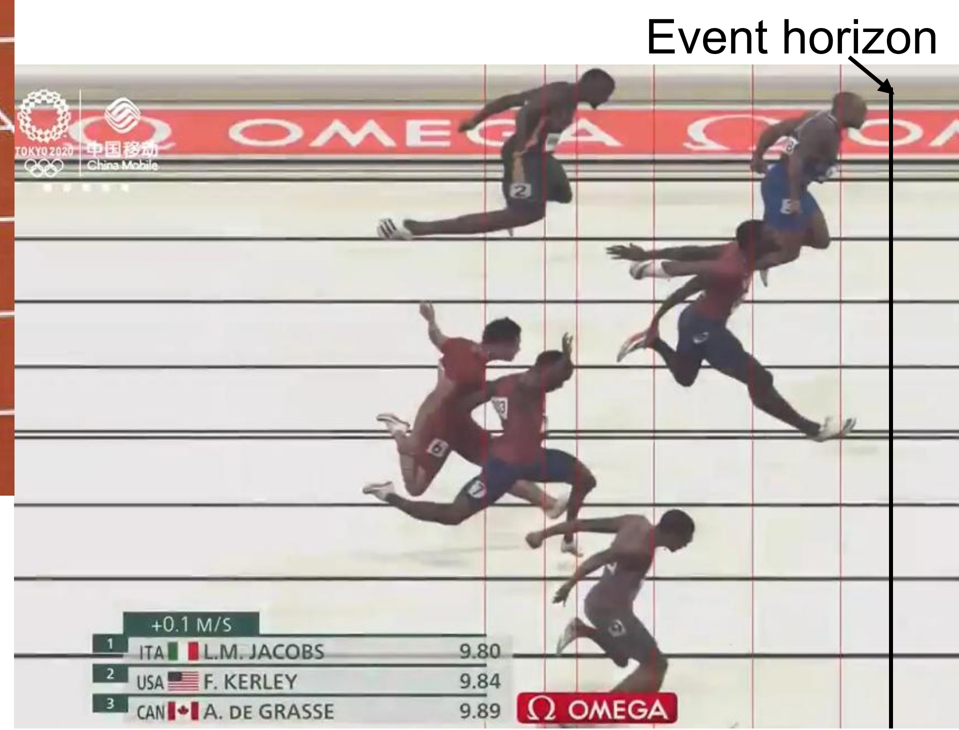
Zeno's "Arrow Paradox" is one of his famous paradoxes that challenge our understanding of motion. The paradox goes as follows:

1. **Instantaneous Position**: At any given instant in time, an arrow in flight occupies a specific position in space.
2. **Motion and Time**: If the arrow is at a specific position at a particular instant, it is not moving during that instant. Motion requires change in position over time.
3. **Sum of Instants**: Since time is composed of an infinite number of instants, and the arrow is not moving in any of these instants, it follows that the arrow is never in motion.

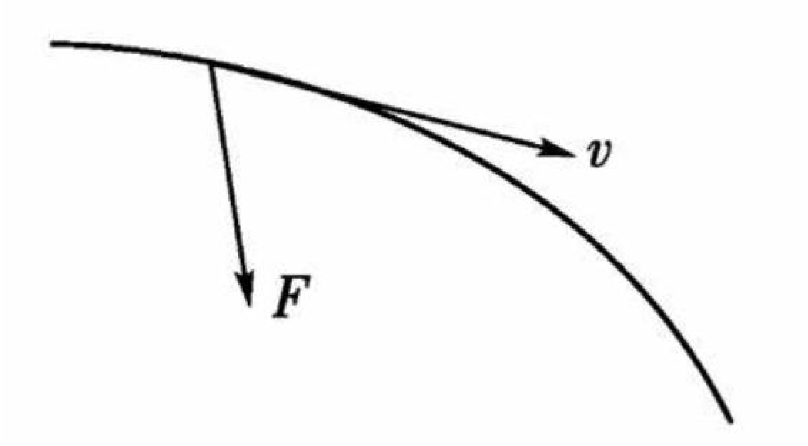
**Resolution:** Introduce the concepts of limit and instantaneous velocity. The instantaneous velocity is defined in "velocity space" and does not exist in the physical space where the moving object resides. It leads to Newtonian (classical) mechanics.

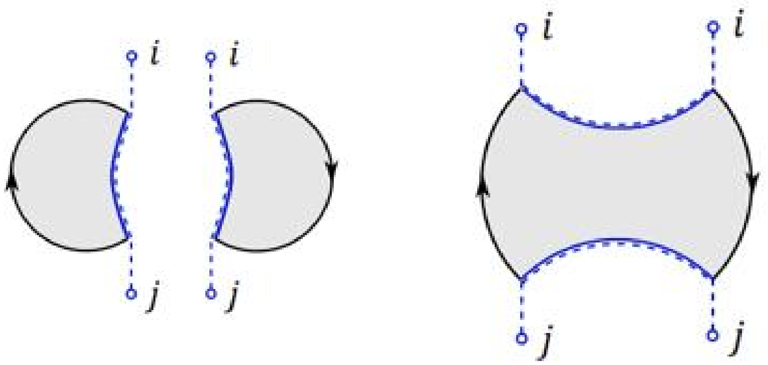
# The flying arrow is motionless, so does the flying person





# Main Points




$$\vec{v} = \frac{\mathrm{d}s}{\mathrm{d}t}\vec{\tau} = v\,\vec{\tau}$$

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

$$S_{\mathrm{vN}} = \lim_{n\to 1}\frac{1}{1-n}\log\left[\mathrm{Tr}(\rho_R^n)\right].$$

?

# Black Hole Information Loss Paradox

The black hole information paradox originates from the Hawking radiation [1]. The particle spectra exhibit a Planck distribution with a (Hawking) temperature $T_H$:

$$\langle \mathrm{in} | N^{\mathrm{out}} | \mathrm{in} \rangle = \frac{1}{e^{\frac{\hbar\omega}{k_B T_H}} - 1},$$

where the Hakwing temperature (Schwarzschild black holes as the example after here):

$$T_H = \frac{\hbar}{8\pi k_B M}. \tag{2}$$

A black hole gradually evaporate via Hawking radiation until it vanish completely. If a black hole formed by the pure state, then it will eventually transform into fully thermal (a mixed state) Hawking radiation at the end of evaporation. Namely, the evolution of a pure state into a mixed state violates the unitary principle of QM, which leads to the Information Loss [2].


1. S. W. Hawking, "Particle creation by black holes," Commun. Math. Phys. **43**, 199 (1975) [Erratum: Commun. Math. Phys. **46**, 206 (1976)].
2. S. Hawking, "Breakdown of Predictability in Gravitational Collapse," Phys. Rev. D **14**, 2460-2473 (1976).

# Resolution: Page Curve

**Page Theorem** [3]: For a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, if the dimensionality of the subsystem $A$ is much smaller than the dimensionality of the subsystem B, $\frac{|A|}{|B|} \ll 1$. Then the smaller subsystem approximates a thermal state.

Consider a (Schwarzschild) black hole: $\mathcal{H}_{BH} = \mathcal{H}_B \otimes \mathcal{H}_R$, where $B$ represents the (remaining ) black hole; $R$ represents the radiation. We first evaluate the time-dependent evolution of its mass $M$ and the thermal entropy $S_{\mathrm{BH}}$:

Stefan-Boltzmann Law:
$$\frac{dE}{dAdt} = \sigma T_H^4. \tag{3}$$

$$T_H = \left(\frac{1}{8\pi M}\right)^4 \sim M^{-4}, \qquad A = 16\pi M^2 \sim M^2. \tag{4}$$

The numerical calculation shows that [4]:

$$\frac{dM}{dt} =- \frac{\hbar^4 c^4}{G_N^2}\frac{\alpha_0}{M^2}, \qquad \frac{dS_{\mathrm{BH}}}{dt} =- \frac{8\pi\alpha_0}{M}, \tag{5}$$

where $\alpha_0$ is a dimensionless constant. Then:

$$M(t) \sim (M_0^3 - t)^{\frac{1}{3}} \sim M_0\left(1 - \frac{t}{t_{\mathrm{life}}}\right)^{\frac{1}{3}}. \tag{6}$$

Here $t_{\mathrm{life}}$ is denoted as the life time of black holes.

3. D. Page, “Average entropy of a subsystem,” Phys. Rev. Lett. **77**, 1291-1294 (1993).
4. D. N. Page, “Particle Emission Rates from a Black Hole: Massless Partiles from an Unchanged, Nonrotating Hole,” Phys. Rev. D **13**, 198 (1976)

Apply this theorem to the whole process of black hole evaproation:

- At early times ($t \ll t_{\text{Page}}$) : The black holes had just formed and started emitting Hawking radiation. So the radition is small, $|R| \ll |B|$. According to the Page theorem, the smaller subsystem $R$ is a thermal state. The entanglement entropy has the following relaiton:

$$S_R \approx \log|R| \equiv S_R^{\text{coarse}} \sim T_H t. \tag{7a}$$

$$S_R^{\text{coarse}} \ll S_B^{\text{coarse}} \equiv S_{\text{BH}} = \frac{A}{G_N}. \tag{7b}$$

Here "coarse" represents the "coarse-grained entropy"(thermal entropy). For a black hole, its coarse-grained entropy is the Bekenstein-Hawking entropy, which is proportional to the area and and decreases with the evaporation. (7a) derives from the fact that Hawking radiation is approximated as the two-dimensional photon gas whose thermal entropy is proportional to its temperature.

- At Page time ($t = t_{\text{Page}}$): On the one hand, $S_R$ increase linearly with time; On the other hand, $S_{\text{BH}}$ decreases with time. They both reach the same value at some point. This time is called the Page time.

- After Page time $\left(t > t_{\text{Page}}\right)$: Hawking radiation $R$ dominates, while the black hole system $B$ is small. Their identities are interchanged. We apply the Page theorem once again and obtain the contrary conclusion:

$$S_R \equiv S_B \approx S_B^{\text{coarse}} \equiv S_{\text{BH}} = \frac{A}{G_N}. \tag{8}$$

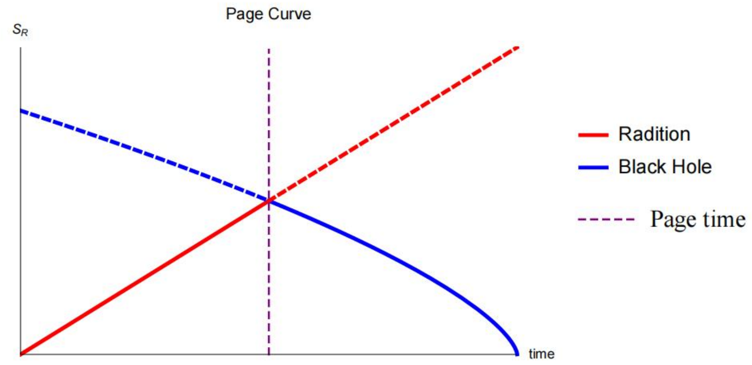
*In the first identity, we use the complementarity of von Neumann (entanglement) entropy: For a bipartite system, if the total system is a pure state, then the entanglement entropy of the two subsystems is always the same: $S_A \equiv S_B$.

# Page Curve

In conclusion, the entanglement entropy of radiation is given by substituting (5) and (6):

$$S_R = \begin{cases} T_H t, & t < t_{\text{Page}} \\ \left(1 - \frac{t}{t_{\text{life}}}\right)^{\frac{1}{3}}, & t > t_{\text{Page}} \end{cases} \tag{9}$$

which is plotted a Page curve.




Comment: the Page curve constituted a milestone. It is worthy of emphasis that it is derived from the spacetime supposition that the black hole evaporation satisfies the unitary.

# Relationship between Two Paradoxes

If universe as a holographic screen, in order to perceive the structure of the universe, it is necessary to be aware that the distinct pixels on the holographic screen at a given time constitute a distinct and high-quality image of the universe. The clearer the image, the less conspicuous the motion or change.

If there is merely one finite-sized black hole in the initial time of the universe. Spacetime outside the black hole is the vacuum. This is the precise vacuum solution to Einstein's equation. For the information paradox, Due to the presence of Hawking radiation, it exhibits a blackbody spectrum. We have no knowledge of the initial information that gives rise to a black hole, nor do we know the state of matter formed by Hawking radiation, resulting in an increasingly blurred picture of the universe. It is contrary to Zeno's statement that “objects in motion do not move.”.

Therefore, to solve the issue of information loss, we also need to take the “limit”, which corresponds mathematically to the “replica trick”.

# Island Rule

Recently, there is a breakthrough in Page curve [5-8] : Employing the QES (quantum extremal surfaces) prescription [9], *Almheiri* and *Penington* et al. obtained the unitary Page curve in the semi-classical gravity and summarized the island rule/formula for computing the entropy of Hawking raditiaon:

$$S_{\text{Rad}} = \min\left[\text{ext}\left(S_{\text{gen}}\right)\right] = \min\left[\text{ext}\left(\frac{\text{Area}(\partial I)}{4G_N}\right) + S_{\text{bulk}}(R \cup I)\right], \qquad (10)$$

where "min" and "ext" stands for "minimization" and "extemization" respectively; $S_{\text{gen}}$ is denoted the generalized entropy; $R$ and $I$ is the region of radiation and island, $\partial I$ is denoted as the boundary of island.

The key of this formula is that the entanglement entropy of the quantum field $S_{\text{bulk}}$, we take into account not only the contribution of the radiation region $R$ but also that of the island region $I$ inside the black hole.

In addition, there are two "radiations" in the island formula. On the left is the complete entropy of Hawking radiation , i.e. completely accurate entropy of radiation for the quantum state; but the one on the right is just the semi-classical description of the state of radiation.


5. A. Almheiri et al, "The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole," JHEP **12**, 063 (2019).
6. A. Almheiri et al, "The Page curve of Hawking ridiation from semiclassical geometry," JHEP **03**, 149 (2020).
7. G. Penington, "Entanglement Wedge Reconstruction and the Information Paradox," JHEP **09** 002 (2020).
8. A. Almheiri et al, "The entropy of Hawking radi_x0002_ation," Rev. Mod. Phys. **93**, 35002 (2021).
9. N. Engelhardt and A. Wall, "Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime," JHEP **01**, 073 (2015).

# Island Rule for the Whole Process of Evaporation

- Before Page time $(t < t_{\text{Page}})$ : The extremizaiton (ext) of generalized entropy indicates that QES is a trivial surface — "the vanishing surface", $I = \emptyset$, namely, the island is absent at early times.

$$S_{\text{Rad}} = S_{\text{bulk}}(R) \sim T_H t, \tag{11}$$

which lead to the information paradox at late times: $S_{\text{Rad}} > S_{\text{BH}}$ (Benekstein bound is violated [10])

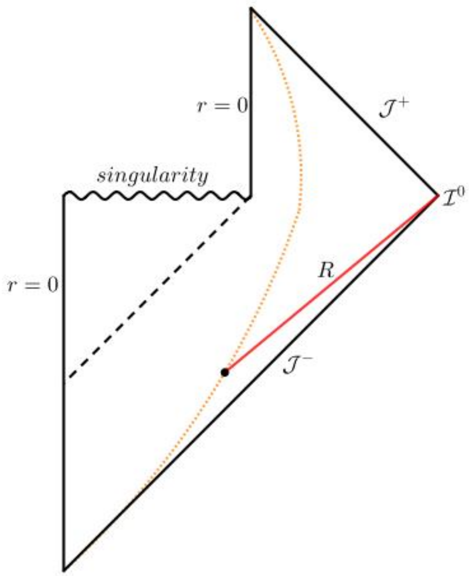



FIG1: The Penrose diagram for an evaporating black hole at early times. $R$ is the region of raidaiton. The orange curve represents the cut-off surface, where the radiation is emitted (not strictly from the event horizon when considering the back-reaction). The QES is contracted to a radius of $r = 0$. It is a vanishing surface, which leads to the entropy of radiation keeps growing.


[10] J. D. Bekenstein, "A Universal Upper Bound on the Entropy to Energy Ratio for Bounded Systems," Phys. Rev. D **23**, 287 (1981).

- After Page time $(t > t_{\text{Page}})$ : There ia another candidate. The extremizaiton (ext) of generalized entropy indicates that QES is an non-trivial surface — "non-vanishing surface", $I \approx$ horizon, then

$$S_{\text{Rad}} = \frac{\text{Area}(\partial I)}{4G_N} + S_{\text{bulk}}(R \cup I) \approx \frac{\text{Area(horizon)}}{4G_N} = S_{\text{BH}}, \quad (12)$$

which lead to a desceasing curve.

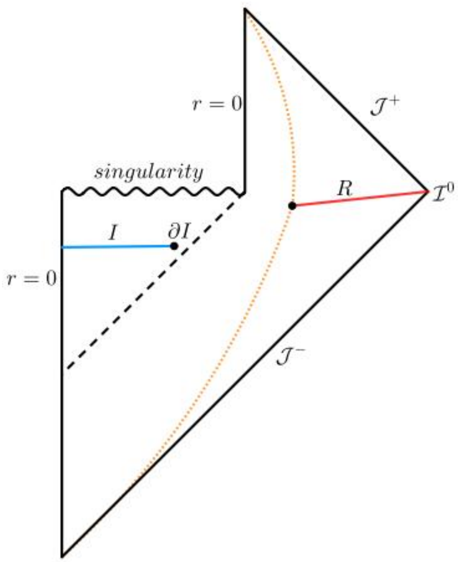



FIG2: The Penrose diagram for an evaporating black hole at late times. $I$ is the region of island. Its boundary is denoted as $\partial I$, which is the non-vanishing QES, also called "quantum extremal island" (QEI). At this time, the Cauchy is devided to two disconnected intervals. The interior of black holes (the region $I$) is entangled with the exterior radiation $R$.

Entanglement wedge reconstruction suggestion [11]:
The interior of black holes belongs to the entanglement wedge (causial region) of exterior radiation. It seems to inherit the spirit of "ER=EPR" [12].

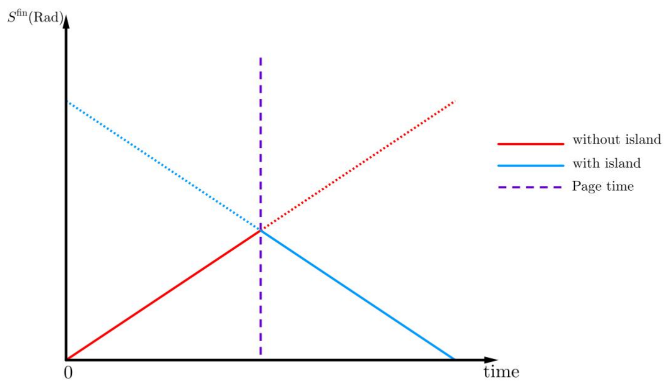

[10] J. D. Bekenstein, "A Universal Upper Bound on the Entropy to Energy Ratio for Bounded Systems," Phys. Rev. D **23**, 287 (1981).
[11] G. Penington, "Entanglement Wedge Reconstruction and the Information Paradox," JHEP **09** 002 (2020).
[12] J. Maldacena and L. Susskind, "Cool horizons for entangled black holes," Fortsch. Phys. **61**, 781-811.

# Page curves from the island rule

In summary, the behavior of entropy of radiation satisfies:

$$S_{\mathrm{Rad}} = \min\left(S_{\mathrm{gen}}^{\text{without island}}, S_{\mathrm{gen}}^{\text{with island}}\right) \approx \min(T_H t, S_{\mathrm{BH}}). \tag{13}$$

which describes an unitary Page curve.




Emphasis: We may be naive to assume that the whole process merely includes the interior of the black hole in the calculation, which is just a “trick”. However, it is not that we “actively” include the interior of a black hole; rather, it is that “gravity itself guides us to include the interior” in our calculations. Therefore, gravity informs us that the evaporation of black holes is unitary, yet it fails to provide us with the details of the quantum state of the Hawking radiation. In fact, the island formula (10) can be strictly derived from the gravitational path integral, as we will see next.

# The End of the World (EOW) model

The EOW model is consists of Jackiw-Teitelboim (JT) gravity, the EOW brane at the AdS boundary, and a coupled auxiliary system (the flat bath without the gravitational effect) [13,14].

The total action is [13]:

$$I = I_{\text{JT}} + \mu \int_{\text{brane}} \mathrm{d}s,$$
$$I_{\text{JT}} = -\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}} \sqrt{g}R + \int_{\partial\mathcal{M}} \sqrt{h}K\right] - \left[\frac{1}{2}\int_{\mathcal{M}} \sqrt{g}\phi(R+2) + \int_{\partial\mathcal{M}} \sqrt{h}\phi K\right]. \tag{14}$$

Here $\mu$ is the mass of brane and the integral is along the worldline of the brane. The $k$ internal states living in the EOW brane can be used to describe the entanglement partner inside the black hole of early Hawking radiation.

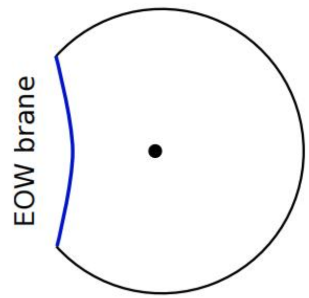



FIG3: The Euclidean geometry for a black hole with an EOW brane behind the horizon.

[13] G. Penington, S. Shenker, D. Stanford and Z. Yang, "Replica wormholes and the black hole interior," arXiv:1911.11977
[14] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, "Replica Wormholes and the Entropy of Hawking Radiation," JHEP **05**, 013 (2020).

# Gravitational Path Integral

These $k$ states are entangled with the auxiliary system $R$. We can therefore construct a model of the evaporating black hole $B$. The wave function of the whole system is:

$$|\Psi\rangle = \frac{1}{k}\sum_{i=1}^{k} |\psi_i\rangle_B\, |i\rangle_R, \tag{15}$$

where $|\psi_i\rangle_B$ is the state for black holes, the subscript $i$ represnets the EOW brane at the $i$-th state. $|i\rangle_R$ is the state for the radiation. Then the (reduced) density matrix can be expressed as:

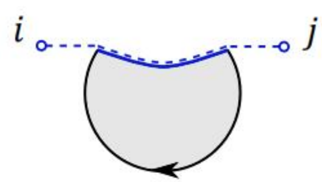
$$\rho_R = \frac{1}{k}\sum_{i=1}^{k} |j\rangle\langle i|_R \langle\psi_i\left|\psi_j\right\rangle_B, \tag{16}$$

where each  matrix element is the gravity amplitude:

$$\langle\psi_i|\psi_j\rangle = \tag{17}$$


The arrows here represent the direction of time evolution. At the intersection of the dash line and the solid line, we give an EOW brane with an asymptotic boundary. Considering the leading order of the gravitational configuration, the following classical order satisfies the boundary conditions:

$$\langle\psi_i|\psi_j\rangle \approx \tag{18}$$



# Why Replica Trick?

If we directly the von Neumann entropy for the radiation, the process is very difficult since the term $\log \rho_R$ is involved. A very mathematical trick is first to evaluate the corresponding n-th Renyi entropy and take the limit of $n \to 1$ to obtian the von Neumann entropy:

$$S_n = \frac{1}{1-n} \log\left[\mathrm{Tr}(\rho_R^n)\right],$$
$$S_{\mathrm{vN}} = \lim_{n\to 1} \frac{1}{1-n} \log\left[\mathrm{Tr}(\rho_R^n)\right]. \tag{19}$$

We consider the $n = 2$ case as the example, at this time, our calculation involve the purity:

$$\mathrm{Tr}(\rho_R^2) = \frac{1}{k^2} \sum_{i,j=1}^{k} \left|\langle\psi_i|\psi_j\rangle\right|^2. \tag{20}$$

Different from the von Neumann entropy ($n = 1$), we can sum $i$ and $j$ in two ways by connecting dash lines:

- **Disconnected Geometry**

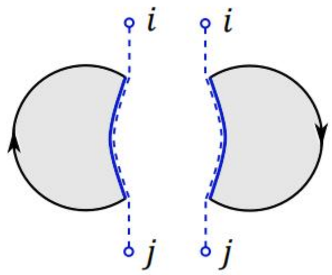



FIG4: Diagram for the disconnected geometry. It is aslo called the "Hawking saddle" , which is dominates the evaporation at early times.

# Replica Wormholes Saddle

- **Connected Geometry**

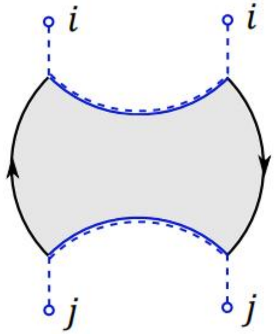



FIG5: Diagram for the connected geometry. It is aslo called the "replica wormholes saddle" , which is dominates the evaporation at late times.

In order to describe the contribution of these geometries, we denote $Z_n = Z_n(\beta)$ as the gravitational path integral on the disk, the boundary of which consists of $n$ physical boundaries (each with the renormalization length $\beta$) and $n$ EOW branes. Then the purity has the following form:

$$\mathrm{Tr}(\rho_R^2) = \frac{kZ_1^2+k^2Z_2}{(kZ_1)^2} = \frac{1}{k} + \frac{Z_2}{Z_1^2} \approx k^{-1} + e^{-S_0}. \quad (21)$$

in which the numerator come from the sum of the contributions of connected ($k^{-1}$) and disconnected ($e^{S_0}$) geometries. In the final simplification, we ues the planar approximation $Z_n \sim e^{S_0}$, because $Z_n$ has the disk-like topology. Accordingly, when $k$ is small enough (long before the Page time), the disconnected geometry dominates, which reproduce the Hawking's curve; When $k$ becomes very large, the connected geomety dominates, then the entropy is independent of $k$, and stops increasing. It is the competition between these two saddles during the evaporation leads to an unitary Page curve:

$$S_R \approx \begin{cases} \log k, & t < t_{\mathrm{Page}} \\ S_{\mathrm{BH},} & t > t_{\mathrm{Page}} \end{cases} \quad (22)$$

# Replica Metric

The next step is to obtain the replica geometry. We can glue together $n$ copies of spacetime and along a set of branch cut. For JT gravity, we consider this in the manifold $\widetilde{M}_n$, which can be regarded as an $n$-fold manifold with the $\mathbb{Z}_n$ symmetry. Then we use the uniformization map. At last, we obtain the metric with $n$-dependence:

$$ds_n^2 = -\frac{4|d\widetilde{w}|^2}{(1-|\widetilde{w}|^2)^2}, \qquad \text{dilaton: } \phi_n = \phi_0 + \frac{2\pi\phi_r}{\beta}\frac{1+|\widetilde{w}|^2}{1-|\widetilde{w}|^2}, \tag{32}$$

with the uniformization coordinate:

$$\widetilde{w}^n = w. \tag{33}$$

Note that this process will introduce the conical singularity on the $\widetilde{M}_n$. To determine the dilaton with $n$-dependence in the presence of concial singularities, there exists two equivalent geometric description as follows:

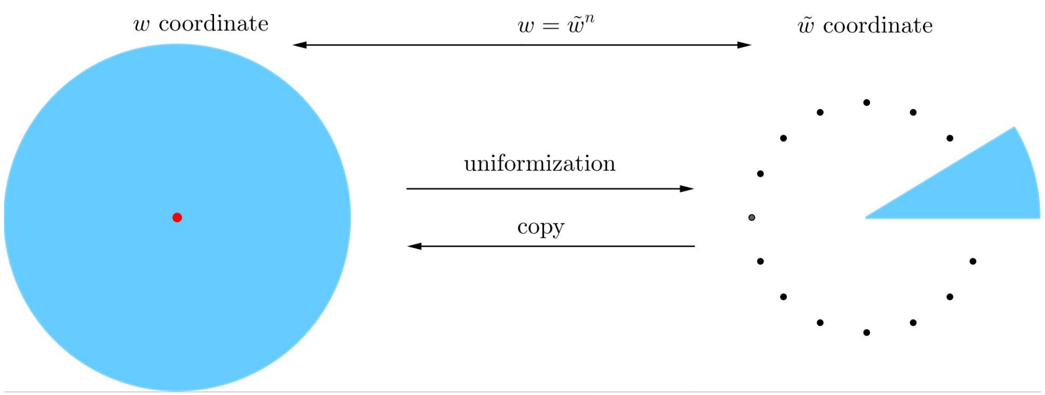



FIG7: The two equivalent ways to describing the conical singularity for the Euclidean signature. On the left, we parameterize the manifold with the coordinate $w = e^{\frac{2\pi}{\beta}(\sigma+i\theta)}$, where the Euclidean time are periodic: $\theta \sim \theta + \beta$ by the Wick rotation $t \to i\theta$, and $-\infty < \sigma <- \epsilon$; On the right, the geometry is uniformized by coordinate $\widetilde{w} = e^{\frac{2\pi}{\beta}(\widetilde{\sigma} + i\widetilde{\theta})}$. **The metric identify with a $\mathbf{AdS_2}$ disk with the temperature $\frac{1}{n}\beta$.**

# Modular Entropy

- Without Island

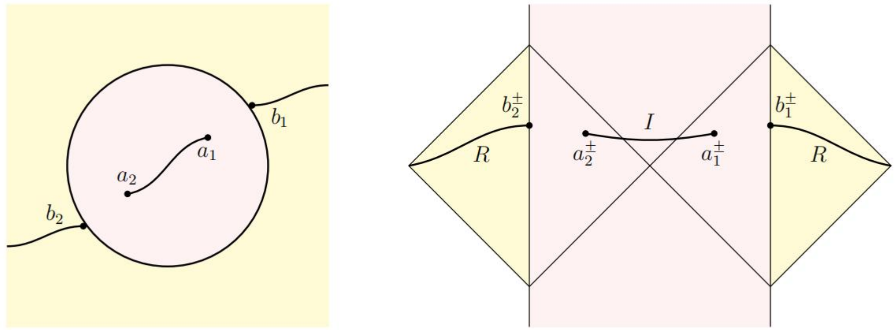



FIG8: : The set up in (left) Euclidean (right) Lorenztian signature with two QES in the context of the eternal black hole and a radiation region $R$ that covers the left and right bands, as shown. There is a point in the right Minkowski bath and one QES as shown ultimately outside the horizon of the right black hole in the Lorentzian picture and the mirror image on the left.

Set the coordinate for $a$ (the boundary of islands) and $b$ (the boundary of radiation) are ($\bar{a}$ and $\bar{b}$ are symmetry points on the left wedge):

$$b_1 =- b_1^- =- \bar{b}_2 =- b_2^+ = (-t_b, b), \qquad \bar{b}_1 = b_1^+ =- b_2 = b_2^- = (+t_b, b), \tag{39a}$$

$$a_1 =- a_1^- =- \bar{a}_2 =- a_2^+ = (-t_a, a), \qquad \bar{a}_1 = a_1^+ =- a_2 = a_2^- = (+t_a, a), \tag{39b}$$

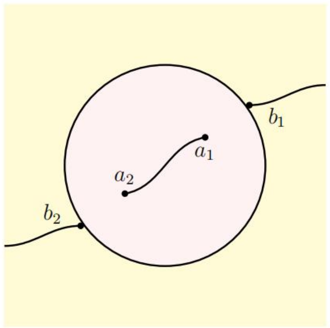
Then the modular entropy without island is [19]:

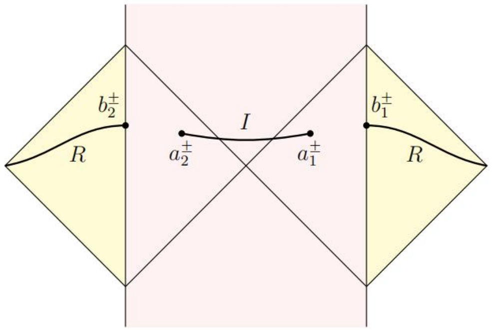
$$\tilde{S}_n(\text{without islnad}) = \tilde{S}_n(R) = \frac{c}{6n}\log\frac{|F(b_1)-F(b_2)|^2}{\varepsilon_{UV}^2 F^{'}(b_1)F^{'}(b_2)\Omega_{\text{bath}}(b_1)\Omega_{\text{bath}}(b_2)} \approx \frac{2\pi c}{3n\beta}t_b. \tag{40}$$


[19] H. Casini, C.D. Fosco and M. Huerta, “Entanglement and alpha entropies for a massive Dirac field in two dimensions”, J. Stat. Mech. **0507** (2005) P07007.

# Modular Entropy

- With Island




Then the generalized modular entropy with island is:

$$\tilde{S}_{\text{gen}}(\text{with islnad}) = 2\phi_0 + \frac{4\pi}{n\beta}\frac{\phi_r}{\text{Tanh}\left(\frac{2\pi a}{n\beta}\right)} + \tilde{S}_n(R \cup I) = \qquad = 2\phi_0 +$$

$$\frac{4\pi}{n\beta}\frac{\phi_r}{\text{Tanh}\left(\frac{2\pi a}{n\beta}\right)} + \frac{c}{3n}\log\frac{|F(b_1)-G(a_1)|^2}{\varepsilon_{UV}^2 G'(a_1)F'(b_1)\Omega_{\text{bath}}(b_1)\Omega_{JT}(a_1)}$$

$$= 2\phi_0 + \frac{4\pi}{n\beta}\frac{\phi_r}{\text{Tanh}\left(\frac{2\pi a}{n\beta}\right)} + \frac{c}{3n}\log\frac{\beta}{\pi\varepsilon_{UV}^2}\frac{\text{Cosh}\left(\frac{2\pi}{\beta}(a+b)\right)-\text{Cosh}\left(\frac{2\pi}{\beta}(t_a-t_b)\right)}{\text{Sinh}\left(\frac{2\pi}{a}\right)}\,. \qquad (41)$$

Extremizing the above equation with respect to $t_a$ first:

$$\frac{\partial\,\tilde{S}_{\text{gen}}(\text{with islnad})}{\partial t_a} \sim \text{Sinh}\left(\frac{2\pi}{\beta}(t_a - t_b)\right) = 0 \;\rightarrow\; t_a = t_b. \qquad (42)$$

# Modular Entropy

- With Island

Substituting the relation $t_a = t_b$ to the generalized modular entropy and extremize it with respect to $a$:

$$\frac{\partial\, \tilde{S}_{\text{gen}}(\text{with islnad})}{\partial a} = -\frac{\pi\left[cn\beta\text{Coth}\left(\frac{2\pi}{\beta}a\right) - cn\beta\text{Coth}\left(\frac{2\pi}{\beta}(a+b)\right) + 12\pi\phi_r\text{Csch}^2\left(\frac{2\pi}{n\beta}a\right)\right]}{3n^2\beta^2} = 0. \tag{43}$$

The above equation is equivalent to the following condition:

$$\frac{12\pi\phi_r}{cn\beta} = \frac{1}{2n\kappa} = \frac{\text{Sinh}\left(\frac{2\pi}{\beta}(a-b)\right)\text{Sinh}\left(\frac{2\pi}{\beta}a\right)}{\text{Sinh}\left(\frac{\pi}{\beta}(a+b)\right)}. \tag{44}$$

In the high temperature limit $\kappa \sim 0$, we obtain the location of island for the finite $n$ is:

$$a \to \infty, \tag{45}$$

which indicates that the island is located at the center of the $\text{AdS}_2$ disk. Then the modular entropy with island is:

$$\tilde{S}_{\text{gen}}(\text{with islnad}) = 2\phi_0 + \frac{4\pi}{n\beta}\frac{\phi_r}{\text{Tanh}\left(\frac{2\pi\infty}{n\beta}\right)} + \frac{c}{3n}\log\left(\frac{\beta}{\pi\varepsilon_{UV}^2}e^{-\frac{2\pi b}{\beta}}\right) \approx 2\tilde{S}_{\text{BH}}^n + \mathcal{O}\left(\frac{b}{n\beta}\right). \tag{46}$$

Combing the result without island, we find

$$\tilde{S}_n(\text{Rad}) = \min\left(\frac{2\pi c}{3n\beta}t,\quad 2\tilde{S}_{\text{BH}}^n\right). \tag{47}$$

Then, the Page time is given by:

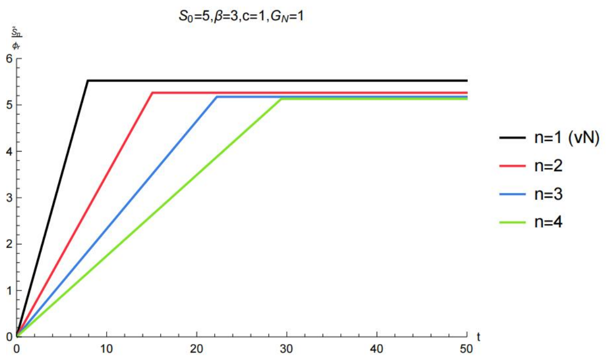
$$t_{\text{Page}} = \frac{3n\beta\tilde{S}_{\text{BH}}^n}{\pi c} = \frac{3n\beta}{\pi c}S_0 + \frac{3\phi_r}{cG_N}. \tag{48}$$

# Modular Page Curve and Entanglement Capacity

Then the entanglement capacity is given by taking the derivative of the modular entropy with respect to $n$:

$$C_n = -n \frac{\partial \tilde{S}_n(\mathrm{Rad})}{\partial n} = \begin{cases} \frac{2\pi c}{3n\beta} t, & t > t_{\mathrm{Page}} \\ C_n^{\mathrm{thermal}} = \frac{\pi \phi_r}{2n\beta G_N}, & t < t_{\mathrm{Page}} \end{cases} \quad (49)$$

Finally, we plot the time-depend of the entropy and the capacity through (47) and (49):



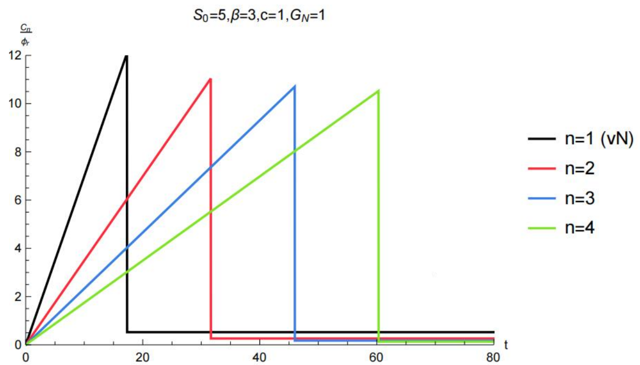




FIG9: Left: The Page curve for the modular entropy; Middle: The capacity as the function of $t$; Right: The zoomed plots for the middle subfigure.

1.The modular entropy is a curve and saturated at late times, which implies that modular entropy is conserved during evaporation.

2. The entanglement capacity presents a discontinuity at Page time. This seems to indicate that there is a second-order phase transition at the Page time.

3. Both modular entropy and entanglement capacity are identied to the thermal entropy and heat capacity of black holes at late times, which strong indicates the relation between the modular quantity and statistical physics.

4. In addition, in the limit of $n \to 1$, all our results are consistent with the previous work [19].


[19] A. Almheiri, R. Mahajan and J. Maldacena, "Islands outside the horizon," arXiv:1910.11077.

# Go Beyond the Page Curve

**Motivation:**

We contemplate the Page cuve at a more profound level: Consider the directivity from the Hawking saddle to the replica wormholes saddle (The reversibility problem).

**Gap:**

All previous work has shown that the evolution from the Hawking saddle to the replica wormholes saddle is unitary microscopically, and therefore invertible between the two microstates. However, leads to a new paradox — Why do the Hawking saddle dominates in the early stage of evaporation rather than the replica wormholes saddle ? Vice versa.

Another closely related issue is related to the quantum non-cloning theorem. When we use the "replica trick", we need to make $n$ copies of the state $\rho$. But the quantum non-cloning theorem forbids cloning of an unknown quantum state. We need to prepare $n$ known quantum states in advance. Then taking the limit $n \to 1$ yields a finite von Neumann entropy. Therefore, from the point of view of information, there should exists the directivity. Therefore, information after the evolution of the black hole can only be obtained if there is information of $n$ quantum states first. Then the quantum non-cloning theorem can be satisfied.

# Second Law of Relative Entropy

Based on previous result (FIG9), we find that the behavior of modular entropy is closer to the thermal entropy than Renyi entropy. Here we consider the relative entropy and its second law [20]. For two density matrices $\rho$ and $\sigma$, the relative entropy is defined by:

$$S(\rho||\sigma) = \mathrm{Tr}(\rho\log\rho) - \mathrm{Tr}(\rho\log\sigma). \tag{50}$$

Two properties: Positive definiteness and Monotonicity

Consider $\sigma$ as a reference state and introduce the modular Hamiltonian $H_\sigma = -\log\sigma$ and the replica free energy $F(\rho) = \mathrm{Tr}(\rho H_\sigma) - S(\rho)$, where $S(\rho)$ is the von Neumann entropy for the state $\rho$. Then:

$$S(\rho||\sigma) = [\mathrm{Tr}(\rho\log\rho) - \mathrm{Tr}(\sigma\log\sigma)] + [\mathrm{Tr}(\sigma\log\sigma) - \mathrm{Tr}(\rho\log\sigma)] = F(\rho) - F(\sigma). \tag{51}$$

For the evaporating black hole, we assume that the state of the Hawking saddle is $\rho$, while the state of the replica wormholes saddle is $\rho_n = e^{-nH}$. Accordingly,

$$S(\rho||\rho_n) = \mathrm{Tr}(\rho\log\rho) - \mathrm{Tr}(\rho\log\rho_n) \geq 0. \tag{52}$$

We obtain the second law of relative entropy:

$$n\langle H\rangle_A - S(\rho) \geq 0. \tag{53}$$

Therefore, there is a directivity between the Hawking saddle and the replica wormhole saddle, which is given by the second law of relative entropy. At the same time, the replica parameter $n$ does play the role of inverse temperature.

[20] H. Casini, “Relative entropy and the Bekenstein bound,” Class. Quant. Grav. **25**, 205021 (2008).

# Discussion and conclusion

In conclusion, we can summarize the form of modular entropy by analogy with the second law of thermodynamics in the modular space:

1. There is a directivity between the Hawking saddle and the replica wormhole saddle, which is given by the second law of the relative entropy, and $n$ does play the role of inverse temperature.

2. More specifically, similar to two equivalent statements of the second law of thermodynamics:

The modular entropy cannot flow spontaneously from the replica wormhole saddle to the Hawking saddle without causing other changes. Or equivalently, a quantum state cannot spontaneously transition from one copy to multiple copies without causing other changes.

There might exist more than mere analogies between the physical quantities related to modular and statistical mechanics, and there might a dulaity. This is also the focus of our future work.

# Thank you !