Energy Flux of Gravitaional Wave for Kerr black hole

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Outline of the talk

1. Motivation
2. Research contents
   - 2.1 Green Function Method
   - 2.2 Exact Solution of Teukolsky Equation
   - 2.3 Radiative energy fluxes
3. Comparison with other methods
4. Conclusion and Outlook

Motivation


2017 诺贝尔奖
For decisive contributions to the LIGO detector and the observation of gravitational waves
Barry C. Barish
Kip S. Thorne
Rainer Weiss
Ronald Drever

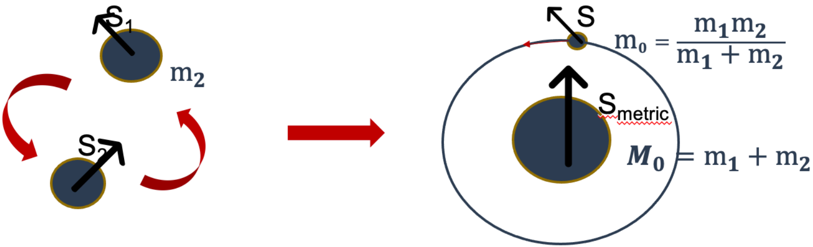
引力波探测意义

- 不仅可以看宇宙，而且还可以听宇宙

- 开启了多波段、多信使引力波天文学

EOB model

Map two–body problem onto an EOB problem
A test particle moves in an effective spacetime



Real spin two-body system   EOB system

The key point to construct EOB model is to study dynamical evolution of a coalescing binary system

The Hamilton equations can be expressed as

$$\frac{d\boldsymbol{r}}{d\hat{t}} = \{\boldsymbol{r}, \hat{H}[g_{\mu\nu}^{\text{eff}}]\} = \frac{\partial \hat{H}[g_{\mu\nu}^{\text{eff}}]}{\partial \boldsymbol{p}},$$

$$\frac{d\boldsymbol{p}}{d\hat{t}} = \{\boldsymbol{p}, \hat{H}[g_{\mu\nu}^{\text{eff}}]\} + \hat{\boldsymbol{\mathcal{F}}}[g_{\mu\nu}^{\text{eff}}] = -\frac{\partial \hat{H}[g_{\mu\nu}^{\text{eff}}]}{\partial \boldsymbol{r}} + \hat{\boldsymbol{\mathcal{F}}}[g_{\mu\nu}^{\text{eff}}],$$

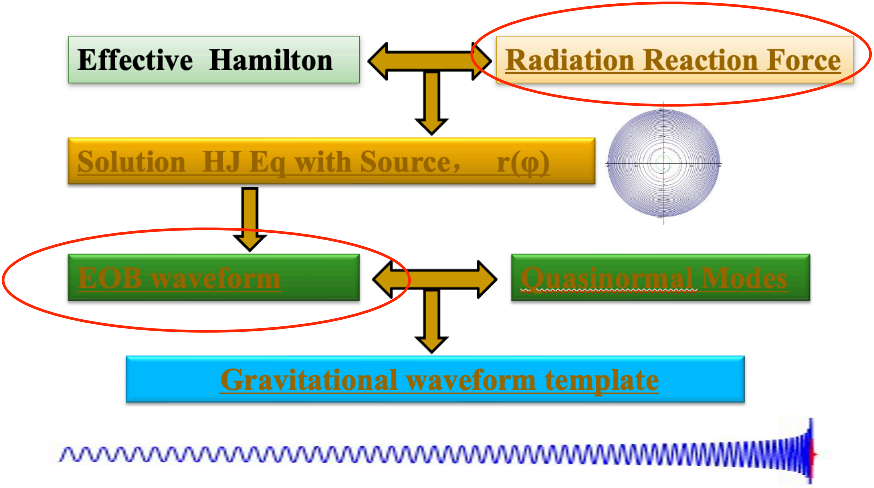
$$\frac{d\boldsymbol{S}_{1,2}}{d\hat{t}} = \{\boldsymbol{S}_{1,2}, \nu\hat{H}[g_{\mu\nu}^{\text{eff}}]\} = \nu\frac{\partial \hat{H}[g_{\mu\nu}^{\text{eff}}]}{\partial \boldsymbol{S}_{1,2}} \times \boldsymbol{S}_{1,2} + \hat{\boldsymbol{\mathcal{F}}}^{s}_{1,2},$$

Spinless (2 parameters) **Spinning** (8 parameters)

(m_1, m_2) $(m_1, \boldsymbol{S}_1, m_2, \boldsymbol{S}_2)$

For a SCEOB model, Hamiltonian, radiation reaction force and waveform should be based on same physical model

EOB theory Based on PM approximation



Brief history for theoretical studies of GWs

1. 1915-1916.9 Einstein, gravitational waves does not exist
2. 1916 Einstein introduces first theory of gravitational waves (monopole)
3. 1918 Einstein first calculates the quadrupole formula

perturbation of Minkowski spacetime $g_{ab} = \eta_{ab} + \epsilon h_{ab}$

Trace-reversed wave perturbation $\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2} h \eta_{ab}$

Harmonic gauge condition $\partial^b \bar{h}_{ab} = 0 \Leftrightarrow \nabla^a \nabla_a x^i = 0$

Linearized Einstein field equations $\partial^c \partial_c \bar{h}_{ab} = -16\pi T_{ab}$

at wave zone with large distant form the source

$$\bar{h}_{ij}(t, x^k) = \frac{2}{r} \frac{\partial^2}{\partial t^2} \int \rho(t - r, x'^k)\, x'^i x'^j \, d^3x' = \frac{2}{r} \ddot{Q}_{ij}$$

Quadrupole formula

1. 1922 Eddington corrects a factor of 2 error in Einstein's formula
2. 1923 Eddington shows that the quadrupole formula applies to a spinning rod with a finite speed of propagation of gravity

1. 1936 年, Einstein 在给 Born 的信中说: “I arrived at the interesting result that gravitational waves do not exist”

1936: Einstein, Do gravitational wave exists?
(submit to PR but be rejected)
爱因斯坦与罗森合作：不存在引力波
罗伯逊在审稿中指出：此论文有错
改正后为有引力波

ON GRAVITATIONAL WAVES.

BY

A. EINSTEIN and N. ROSEN.

ABSTRACT.

The rigorous solution for cylindrical gravitational waves is given. For the convenience of the reader the theory of gravitational waves and their production, already known in principle, is given in the first part of this paper. After encountering relationships which cast doubt on the existence of *rigorous* solutions for undulatory gravitational fields, we investigate rigorously the case of cylindrical gravitational waves. It turns out that rigorous solutions exist and that the problem reduces to the usual cylindrical waves in euclidean space.

A. Einstein, N. Rosen. On gravitational waves. Journal of the Franklin Institute, 1937, 223: 43-54.

1. 1938 Einstein Infeld and Hoffman (EIH) do first post-Newtonian order problem of motion
2. 1941 Landau and Lifshitz claim the quadrupole formula applies to waves from binary stars
3. 1947 Ning Hu applies EIH to case of emission of waves from a binary star but finds energy gain
4. 1953 Infeld and Scheidegger claim that binary stars do not emit gravitational waves
5. 1955 Rosen argues that gravitational waves cannot carry energy
6. 1955 Fock recovers the quadrupole formula using his own slow-motion calculation
7. 1957 Bondi and others give thought experiment to show that gravity waves do carry energy
8. 1958 Trautman shows that energy loss by binary stars cannot be transformed away
9. 1959 Peres finds energy gain in binaries
10. 1959, 60 Peres discovers that his previous result was due to incorrect bondary conditions. New result agrees with quadrupole formula
11. 1960 Infeld and Plebansky textbook, in which Infeld still argues that binary stars do undergo radiation damping
12. 1962 Bondi et al show that systems which emit gravitational waves lost mass

1962: Bondi et al, GWs in axisymmetric spacetime

Hermann Bondi (1919–2005)

Mathematician, cosmologist

Bondi , van der Burg and Metzner, 1962

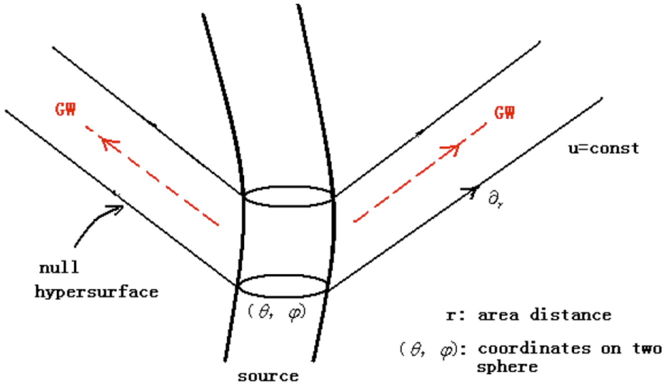


Fig. Bondi-Sachs coordinate system

Summary of Bondi's work

1. Bondi-Sachs framework

2. Asymptotic symmetries (BMS group)

3. News function , Peeling off property of Weyl tensor

4. Bondi mass and mass loss formula

$$\frac{dm}{du} = -\frac{1}{2}\int_0^\pi \left(\frac{\partial c}{\partial u}\right)^2 \sin\theta d\theta$$

Note: Sachs generalized Bondi's work to general spacetime in 1962.

Relationship between BS quantities and quadruople moments in asymptotically flat spacetime

Coleman, 1974

$$\Psi_4 = -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d = -\frac{\partial^2 c}{\partial u^2}\frac{1}{r} + O(r^{-2})$$

$$\dot{c} \sim \frac{1}{2}\eth n^i \eth n^j \frac{\partial^3}{\partial \bar{x}^{03}} \int (T^{00}\bar{x}^i \bar{x}^j) d^3\bar{x}$$

Idea of proof.

step1. Bondi news $\leftrightarrow \{u, r, \theta, \phi\} \leftrightarrow \{\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^2\} \leftrightarrow$ Quadrupole

step2.

$$\bar{g}^{\alpha\beta} = \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\frac{\partial \bar{x}^\beta}{\partial x^\nu} g^{\mu\nu}$$

step3. Weyl tensor in Harmonic coordinate

Energy Flux of GWs for Kerr Black hole

- Regge(1957)和 Zerilli(1970)将 Schwarzschild 时空中引力微扰的单个主方程分解为奇偶两部分。但不能推广到 Kerr 黑洞。
- 通过使用纽曼—彭罗斯的零标架理论,Bardeen 和 Press (1973) 推导出了一个无源的 Schwarzschild 黑洞的曲率微扰的主方程。
- Teukolsky 将这套理论扩展到含源的 Kerr 黑洞中 (1973), 取 $\Psi_4^B = (\bar{\rho}^*)^{-4}\phi_4^B$, 从而推导出 $s=-2$ 曲率微扰方程

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\phi_4^B}{\partial t^2} + \left[\frac{2(r^2+a^2)\Delta'}{\Delta} - 8r - 4ia\cos\theta\right]\frac{\partial\phi_4^B}{\partial t}$$
$$-\Delta^2\frac{\partial}{\partial r}\left(\frac{1}{\Delta}\frac{\partial\phi_4^B}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi_4^B}{\partial\theta}\right) + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2\phi_4^B}{\partial\varphi^2}$$
$$+\left[\frac{2a(r^2+a^2)}{\Delta} - 2a\right]\frac{\partial^2\phi_4^B}{\partial t\partial\varphi} + \left(\frac{2a\Delta'}{\Delta} + \frac{4i\cos\theta}{\sin^2\theta}\right)\frac{\partial\phi_4^B}{\partial\varphi} + \left[4\cot^2\theta + 2\right]\phi_4^B \equiv \mathcal{T}^{(-2)}, \tag{1}$$

Teukolsky 方程

把 ϕ_4^B 和 $T^{(-2)}$ 用 ${}_sS_{lm}^{a\omega}(\theta)$ 展开

$$\phi_4^B = \int d\omega \sum_{l,m} R_{lm\omega}^{(-2)}(r)\, {}_sS_{lm}^{a\omega}(\theta) e^{-i\omega t} e^{im\varphi}, \qquad \mathcal{T}^{(-2)} = \int d\omega \sum_{l,m} \mathcal{T}_{lm\omega}^{(-2)}(r)\, {}_sS_{lm}^{a\omega}(\theta) e^{-i\omega t} e^{im\varphi}. \tag{2}$$

Teukolsky 方程:

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right) R = {}_sT_{\ell m\omega} \tag{3}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\,{}_sS_{lm}^{a\omega}}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2 + 2sm\cos\theta}{\sin^2\theta} 2sa\omega\cos\theta - s^2\cot^2\theta + s + A\right){}_sS_{lm}^{a\omega} = 0, \tag{4}$$

应用于任意的自旋权重场:

- 无源:似正模、霍金辐射、超辐射、伽玛射线暴的中心引擎和宇宙喷流
- 含源: 自力、微扰场的辐射、引力波波形建模、引力波回声和潮汐耦合

点源 Teukolsky 方程的求解

- Sasaki 的低阶模的后牛顿的部分格林函数:

  (1)计算 RW 方程,得到 Schwarzschild 黑洞的 5.5PN 的入射波解(PLA 87, 85 (1981), PTP 1994, 92(4):745–771)。

  (2) 用 Sasaki-Nakamura 变换,得到 Kerr 黑洞的 4PN 的入射波解 (PRD, 1996, 54:1439–1459)。

  (3) 用球 Hankel 函数,仅能得到 Schwarzschild 黑洞的 1PN 的出射波解 (PRD, 1995, 51(10):5753–5767),

  但不适用 Kerr 黑洞。

点源 Teukolsky 方程的求解 - 高精度求解

- Mano-Suzuki-Takasugi (MST) 方法的完整格林函数
  Mano 等人改进了 Leaver 解，提出了二阶的后闵可夫斯基展开的完整解
  (PTP, 1996,96(3):549; PTP, 1996, 95(6):1079.)。
- Fujita 改进了 MST 方法的重整化角动量：
  (1)任意的浮点精度的数值解(PTP, 2004, 112(3):415; 2005, 113(6):1165)。
  (2)Schwarzschild 黑洞的 23PN 和 Kerr 黑洞的 16PN 的引力辐射(PTP, 2012, 128(5):971–992; PTEP, 2015, 2015(3):33E01)：
- 蔡荣根、曹周键、韩文标(Chin Sci Bull, 2016, 61: 1525–1535)：
  Teukolsky 方程的精确的解析解至今还未得到。
- **我们得到了Teukolsky 方程的精确的解析解，进而得到精确引力波能流**

Energy Flux 0

Research contents

Formal Solution for Energy Flux of GWs

径向 Teukolsky 方程的通用形式:

$$\left[\Delta_n^{-s+1}\frac{d}{dr}\Delta_n^{s+1}\frac{d}{dr}+V(r)\right]R_{\ell m\omega}=\Delta_n\ {}_sT_{\ell m\omega}, \tag{5}$$

其中,

$$\Delta_n=\sum_{i=0}^{n}b_ir^i=\prod_{i=1}^{n}(r-r_i), \tag{6}$$

本文关注 Δ_2 型的 Teukolsky 方程,即 $\Delta_2=(r-r_-)(r-r_+)\equiv\Delta$, r_- 是内视界,r_+ 是外(事件)视界。

格林函数法

齐次 Teukolsky 方程解 $R^{\text{in,up}}_{\ell m\omega}(r)$ 满足边界条件:

$$R^{\text{in}}_{\ell m\omega} \to \begin{cases} B^{\text{trans}}_{\ell m\omega}\Delta^{-s}e^{-iPr^*}, & r \to r_+ \\ B^{\text{ref}}_{\ell m\omega} r^{1-s}e^{i\omega r^*} + B^{\text{inc}}_{\ell m\omega} r^{-1}e^{-i\omega r^*}, & r \to +\infty, \end{cases} \tag{7}$$

$$R^{\text{up}}_{\ell m\omega} \to \begin{cases} C^{\text{up}}_{\ell m\omega}e^{iPr^*} + C^{\text{ref}}_{\ell m\omega}\Delta^{-s}e^{-iPr^*}, & r \to r_+, \\ C^{\text{trans}}_{\ell m\omega} r^{1-s}e^{i\omega r^*}, & r \to +\infty, \end{cases} \tag{8}$$

非齐次 Teukolsky 方程的解

$$R^{s}_{\ell m\omega} = \frac{1}{2i\omega C^{\text{trans}}_{\ell m\omega}B^{\text{inc}}_{\ell m\omega}}\left\{R^{\text{up(s)}}_{\ell m\omega}\int_{r_+}^{r} d\tilde{r}\frac{R^{\text{in(s)}}_{\ell m\omega}(\tilde{r})\,T^{(s)}_{\ell m\omega}(\tilde{r})}{\Delta^2(\tilde{r})} + R^{\text{in(s)}}_{\ell m\omega}\int_{r}^{\infty} d\tilde{r}\frac{R^{\text{up(s)}}_{\ell m\omega}(\tilde{r})\,T^{(s)}_{\ell m\omega}(\tilde{r})}{\Delta^2(\tilde{r})}\right\}, \tag{9}$$

可写为:

$$R^{s}_{\ell m\omega}(r) = {}_s\tilde{Z}^{\infty}_{\ell m\omega}R^{\text{up(s)}}_{\ell m\omega}(r) + {}_s\tilde{Z}^{\text{H}}_{\ell m\omega}R^{\text{in(s)}}_{\ell m\omega}(r), \tag{10}$$

其中渐近振幅 $Z^{\infty,\text{H},s}_{\ell m\omega}$:

$${}_s\tilde{Z}^{\infty,\text{H}}_{\ell m\omega} = \frac{1}{W_\text{C}}\int_{r_+}^{\infty} dr'\,\frac{{}_sT_{\ell m\omega}(r')\,R^{\text{in,up(s)}}_{\ell m\omega}(r')}{\Delta^2(r')}, \tag{11}$$

准圆轨道的辐射能流

对于在准圆轨道上的非自旋黑洞双星,渐近振幅 ${}_s\tilde{Z}^{\mathrm{H},\infty}_{\ell m\omega}$ 的形式为

$$ {}_s\tilde{Z}^{\infty,\mathrm{H}}_{\ell m\omega} = \frac{1}{2i\omega B^{\mathrm{inc}}_{\ell m\omega}} \int_{r_+}^{\infty} dr' \frac{{}_sT_{\ell m\omega}(r') R^{\mathrm{in,up}}_{\ell m\omega}(r')}{\Delta^2(r')}, \tag{12} $$

$$ {}_sZ^{\mathrm{H},\infty}_{\ell m\omega} = {}_s\tilde{Z}^{\mathrm{H},\infty}_{\ell m\omega}\,\delta(\omega - m\Omega). \tag{13} $$

在无穷远处的辐射能流:

$$ {}_s\left\langle \frac{dE}{dt} \right\rangle_{\infty} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} {}_s\beta_{\ell m\omega} \frac{|{}_s\tilde{Z}^{\infty}_{\ell m\omega}|^2}{4\pi\omega^{2(|s|-1)}}, \tag{14} $$

在事件视界处的辐射能流:

$$ {}_s\left\langle \frac{dE}{dt} \right\rangle_{\mathrm{H}} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} {}_s\alpha_{\ell m\omega} \frac{|{}_sZ^{\mathrm{H}}_{\ell m\omega}|^2}{4\pi\omega^2}, \tag{15} $$

无穷远处和事件视界处的角动量流由以下给出:

$$ {}_s\left\langle \frac{dJ}{dt} \right\rangle_{\mathrm{H},\infty} = \frac{1}{\Omega}\, {}_s\left\langle \frac{dE}{dt} \right\rangle_{\mathrm{H},\infty} \tag{16} $$

Energy Flux 0

一般轨道的引力波能流

一般轨道的引力波振幅 $Z^{\infty,\mathrm{H}}_{lm\omega}$ 改写:

$$Z^{\infty,\mathrm{H}}_{lm\omega} = \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} Z^{\infty,\mathrm{H}}_{\ell mkn}\delta(\omega-\omega_{mkn}) \;, \tag{17}$$

$$Z^{\infty,\mathrm{H}}_{\ell mkn} = e^{-im\phi_0} J^{\infty,\mathrm{H}}_{\ell mkn}/\Upsilon_t \;. \tag{18}$$

$$J^{\infty,\mathrm{H}}_{\ell mkn} = \frac{1}{\Lambda_r\Lambda_\theta}\int_0^{\Lambda_r} d\hat{\lambda}_r\, e^{in\Upsilon_r\hat{\lambda}_r}\int_0^{\Lambda_\theta} d\hat{\lambda}_\theta\, e^{ik\Upsilon_\theta\hat{\lambda}_\theta}\, J^{\infty,\mathrm{H}}_{lm\omega}[r(\hat{\lambda}_r),\theta(\hat{\lambda}_\theta)] \;. \tag{19}$$

其中,$\omega_{mkn} = m\Omega_\phi + k\Omega_\theta + n\Omega_r$。一般轨道的引力波能流、角动量流和 Carter 常数:

$$\left\langle \frac{dE}{dt}\right\rangle = \sum_{\ell mkn}\frac{1}{4\pi\omega_{mkn}^2}\left(|Z^{\infty}_{\ell mkn}|^2 + \alpha_{\ell m\omega}|Z^{\mathrm{H}}_{\ell mkn}|^2\right) \;, \tag{20}$$

$$\left\langle \frac{dL_z}{dt}\right\rangle = \sum_{\ell mkn}\frac{m}{4\pi\omega_{mkn}^3}\left(|Z^{\infty}_{\ell mkn}|^2 + \alpha_{\ell m\omega}|Z^{\mathrm{H}}_{\ell mkn}|^2\right) \;, \tag{21}$$

$$\left\langle \frac{dQ}{dt}\right\rangle = \sum_{\ell mkn}\frac{(\mathcal{L}_{mkn}+k\Upsilon_\theta)}{2\pi\omega_{mkn}^3}\left(\left|Z^{\infty}_{\ell mkn}\right|^2 + \alpha_{\ell m\omega}\left|Z^{\mathrm{H}}_{\ell mkn}\right|^2\right). \tag{22}$$

问题的关键点

问题的关键是求解渐近振幅 $\tilde{Z}^{\infty,\mathrm{H},s}_{\ell m\omega}$:

$$
{}_s\tilde{Z}^{\infty,\mathrm{H}}_{\ell m\omega} = \frac{1}{W_{\mathrm{C}}} \int_{r_+}^{\infty} dr' \frac{{}_sT_{\ell m\omega}(r') R^{\mathrm{in,up(s)}}_{\ell m\omega}(r')}{\Delta^2(r')}, \tag{23}
$$

无源 Teukolsky 方程的解析解

无源的 Teukolsky 方程：

$$\left[\Delta^{-s+1}\frac{d}{dr}\Delta^{s+1}\frac{d}{dr}+V(r)\right]R_{\ell m\omega}=0, \tag{24}$$

其通解为：

$$R_{\ell m\omega}(r)=C_1R_0^{\beta}(r)+C_2R_0^{-\beta}(r), \tag{25}$$

和

$$R_0^{\pm\beta}(r)=S_0^{\pm\beta}(x)\mathbb{H}_0^{\pm\beta}(x), \tag{26}$$

$$S_0^{\pm\beta}(x)=(-x)^{\frac{1}{2}(\pm\beta-s)}(1-x)^{\frac{1}{2}(\gamma-s)}\mathrm{e}^{\frac{1}{2}\alpha x}, \tag{27}$$

$$x=-\frac{r-r_+}{r_+-r_-}. \tag{28}$$

如何确定 $\mathbb{H}_0^{\pm\beta}(x)$?

无源 Teukolsky 方程的解析解

将方程(26)、(28) 和(27) 代入方程(24)：

$$\mathbb{H}''(x) - \frac{\left(-x^2\alpha + (-2-\beta-\gamma+\alpha)\,x + 1 + \beta\right)}{x\,(x-1)}\mathbb{H}'(x) - \left(\begin{array}{l} ((-2-\beta-\gamma)\,\alpha - 2\delta)\,x \\ +\,(\beta+1)\,\alpha + (-\gamma-1)\,\beta - \gamma - 2\eta \end{array}\right)\frac{\mathbb{H}(x)}{2x\,(x-1)} = 0. \tag{29}$$

方程(29)是标准的合流 Heun 函数对应的微分方程。其两个线性独立的特解 $\mathbb{H}_0^{\pm\beta}(x)$：

$$\mathbb{H}_0^{\beta}(x) = \mathrm{HeunC}(\alpha,\beta,\gamma,\delta,\eta;x), \tag{30a}$$

$$\mathbb{H}_0^{-\beta}(x) = (-x)^{-\beta}\mathrm{HeunC}(\alpha,-\beta,\gamma,\delta,\eta;x), \tag{30b}$$

其中，α、β、γ、δ 和 η 都是需要根据给定黑洞时空才可确定的参数。

无源 Teukolsky 方程的解析解

无源的 Teukolsky 方程的通解为：

$$R_{\ell m\omega} = C_1 S_0^{\beta}(x)\mathrm{HeunC}(\alpha,\beta,\gamma,\delta,\eta;x) + C_2 S_0^{-\beta}(x)\mathrm{HeunC}(\alpha,-\beta,\gamma,\delta,\eta;x). \tag{31}$$

合流 Heun 函数在外视界的渐近行为：

$$\lim_{x\to 0}\mathrm{HeunC}(x) = 1, \quad r\to r_+. \tag{32}$$

合流 Heun 函数在无穷远处的渐近行为：

$$\lim_{|x|\to\infty}\mathrm{HeunC}(x) \to D_{\odot}^{\beta}\, x^{-\frac{\beta+\gamma+2}{2}-\frac{\delta}{\alpha}} + D_{\otimes}^{\beta}\mathrm{e}^{-\alpha x}x^{-\frac{\beta+\gamma+2}{2}+\frac{\delta}{\alpha}}, \quad r\to\infty, \tag{33}$$

只有当常数 $D_{\otimes}^{\beta}$ 和 $D_{\odot}^{\beta}$ 已知时，才能根据边界条件确定 C_1 和 C_2。

- 菲齐耶夫认为(CQG, 2006, 23(7):2447; arXiv: 0902.2411v2)：
  求解 C_1 和 C_2 是一个尚未解决的数学难题

合流 Heun 函数在无穷远处的渐近解析式

渐近行为 (33) 中的常数 $D_{\odot}^{\beta}$ 和 $D_{\otimes}^{\beta}$ 可以表示为

$$D_{\odot}^{\beta}=\Xi_{n,\nu}^{\beta}D_{\odot,n,\nu}^{\beta}+\mathrm{e}^{-i\pi\left(\nu+\frac{1}{2}\right)}\frac{\sin\pi\left(\nu+\frac{\delta}{\alpha}\right)}{\sin\pi\left(\nu-\frac{\delta}{\alpha}\right)}\Xi_{-n,-\nu-1}^{\beta}D_{\odot,-n,-\nu-1}^{\beta}, \tag{34}$$

$$D_{\otimes}^{\beta}=\Xi_{n,\nu}^{\beta}D_{\otimes,n,\nu}^{\beta}+\mathrm{e}^{i\pi\left(\nu+\frac{1}{2}\right)}\Xi_{-n,-\nu-1}^{\beta}D_{\otimes,-n,-\nu-1}^{\beta}, \tag{35}$$

其中，

$$D_{\odot,n,\nu}^{\beta}=(-1)^{\frac{\gamma+\beta+2}{2}+\frac{\delta}{\alpha}}\left(\frac{\alpha}{2}\right)^{\tau}\left(-\frac{i\alpha}{2}\right)^{-\frac{\gamma+\beta+2}{2}-\frac{\delta}{\alpha}}\mathrm{e}^{-\frac{i\pi\tau+\alpha}{2}}\times 2^{-1-\frac{\delta}{\alpha}}\mathrm{e}^{\frac{i\pi}{2}\left(\nu+1+\frac{\delta}{\alpha}\right)}\Xi_{n,\nu}^{\beta}\frac{\Gamma\left(\nu+1+\frac{\delta}{\alpha}\right)}{\Gamma\left(\nu+1-\frac{\delta}{\alpha}\right)},$$

$$D_{\otimes,n,\nu}^{\beta}=(-1)^{\frac{\gamma+\beta+2}{2}-\frac{\delta}{\alpha}}\left(\frac{\alpha}{2}\right)^{\tau}\left(-\frac{i\alpha}{2}\right)^{-\frac{\gamma+\beta+2}{2}+\frac{\delta}{\alpha}}$$
$$\times\,\mathrm{e}^{-\frac{i\pi\tau-\alpha}{2}}\Xi_{n,\nu}^{\beta}\frac{2^{-1+\frac{\delta}{\alpha}}\mathrm{e}^{-\frac{i\pi}{2}\left(\nu+1-\frac{\delta}{\alpha}\right)}}{\sum_{n=-\infty}^{+\infty}f_n^{\nu}}\times\sum_{n=-\infty}^{+\infty}(-1)^n\frac{\left(\nu+1-\frac{\delta}{\alpha}\right)_n}{\left(\nu+1+\frac{\delta}{\alpha}\right)_n}f_n^{\nu}.$$

归一化的入射波解和出射波解的振幅

归一化的入射波解 $R^{\text{in}}_{\ell m\omega}$ 和出射波解 $R^{\text{up}}_{\ell m\omega}$：

$$R^{\text{in}}_{\ell m\omega} = S_0^{\beta}(x)\text{HeunC}(\alpha,\beta,\gamma,\delta,\eta;x). \tag{36}$$

$$R^{\text{up}}_{\ell m\omega} = (-1)^{\beta+1}\frac{D_{\odot}^{-\beta}}{D_{\odot}^{\beta}}S_0^{\beta}(x)\text{HeunC}(\alpha,\beta,\gamma,\delta,\eta;x)+S_0^{-\beta}(x)\text{HeunC}(\alpha,-\beta,\gamma,\delta,\eta;x), \tag{37}$$

和归一化的渐近振幅($B^{\text{trans}}_{\ell m\omega} = C^{\text{trans}}_{\ell m\omega} = 1$)：

$$B^{\text{inc}}_{\ell m\omega} = (-r_{\text{x}})^{1-2s+i\frac{ma}{r_+}}(2M)^{-i\frac{ma}{r_+}}e^{i\frac{ma}{2M}}(-1)^{-\frac{\beta+\gamma+2}{2}-\frac{\delta}{\alpha}}D_{\odot}^{\beta}, \tag{38a}$$

$$B^{\text{ref}}_{\ell m\omega} = (-r_{\text{x}})\left(-\frac{2M}{r_{\text{x}}}\right)^{2iM(P+\omega)}e^{-i(P+\omega)r_+}(-1)^{-\frac{\beta+\gamma+2}{2}+\frac{\delta}{\alpha}}D_{\otimes}^{\beta}, \tag{38b}$$

$$C^{\text{ref}}_{\ell m\omega} = (-r_{\text{x}})^{-1+2iM(P+\omega)}(2M)^{-2iM(P+\omega)}e^{i(P+\omega)r_+}(-1)^{2+\frac{\beta+\gamma}{2}-\frac{\delta}{\alpha}}\frac{D_{\odot}^{-\beta}}{D_{\odot}^{\beta}}\tilde{D}, \tag{38c}$$

$$C^{\text{up}}_{\ell m\omega} = (-r_{\text{x}})^{-2s-1+i\frac{ma}{r_+}}(2M)^{-i\frac{ma}{r_+}}e^{i\frac{ma}{2M}}(-1)^{\frac{-\beta+\gamma+2}{2}-\frac{\delta}{\alpha}}\tilde{D}. \tag{38d}$$

其中，$\tilde{D} = \left(D_{\otimes}^{-\beta} - \frac{D_{\odot}^{-\beta}}{D_{\odot}^{\beta}}D_{\otimes}^{\beta}\right)^{-1}$.

关键问题得到解决

$$
{}_s\tilde{Z}^{\infty,\mathrm{H}}_{\ell m\omega} = \frac{1}{W_{\mathrm{C}}} \int_{r_+}^{\infty} dr' \, \frac{{}_sT_{\ell m\omega}(r')\, R^{\mathrm{in,up(s)}}_{\ell m\omega}(r')}{\Delta^2(r')}, \tag{39}
$$

入射波解和出射波解: 合流 Heun 函数的渐近行为

- JCAP 的评审意见:

  文献 [Commun. Math. Phys, 2023,397(2):635–727] 也推导出了类似公式(33)的结果, 但其需要低频或者高频展开。而该文的渐近解析式(33)则不受任何物理约束,其他人的结果是该文结果的特殊情况。

Comparison with other methods

合流 Heun 函数在无穷远处的渐近解析式的验证

例子：当 $\omega = 0.1$(低频)和 $s = -2$(引力场) 的 (2,2) 模式时，三大数学软件和我们的解析式关于合流 Heun 函数在大 r_0 处的渐近行为

左图：实部 右图：虚部

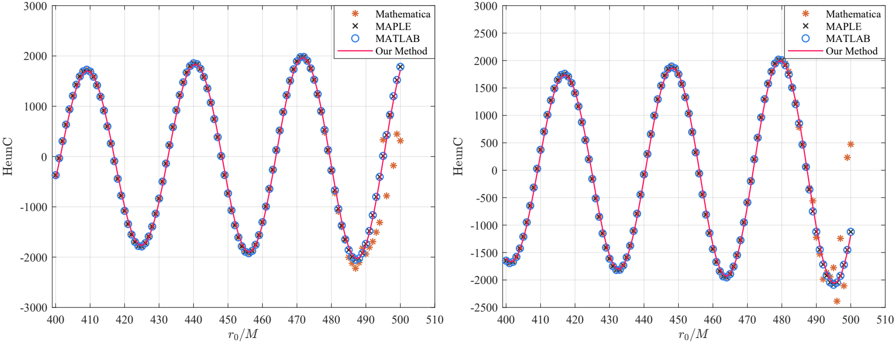


Figure 1: Mathematica 的合流 Heun 函数的发散行为

HeunC 函数在无穷远处的渐近解析式的验证

左图:实部 右图:虚部

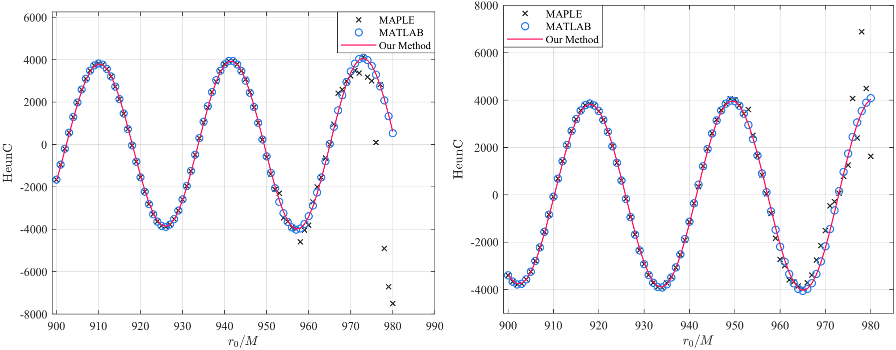


Figure 2: Maple 的合流 Heun 函数的发散行为

合流 Heun 函数在无穷远处的渐近解析式的验证

左图:实部 右图:虚部

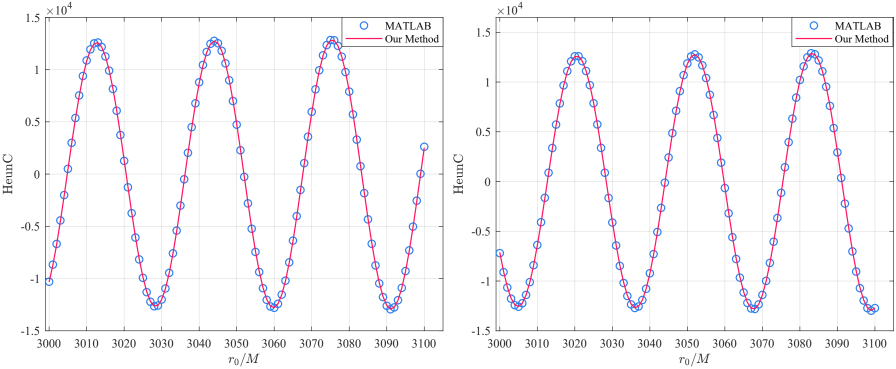


Figure 3: 我们的渐近解析式与 MATLAB 在大 r_0 处是收敛的

合流 Heun 函数在无穷远处的渐近解析式的验证

Table 1: 当 $\omega = 0.1$ 和 $s = -2$ 时, (2,2) 模的四种方法的合流 Heun 函数的计算时间 (秒).

r_0	本文的解析式(33)	MATLAB	MAPLE	Mathematica
$[400M, 500M]$	0.0005209	0.081162	26.281	2.51563
$[900M, 980M]$	0.0005361	0.132998	40.953	发散
$[3000M, 3100M]$	0.0005731	0.405863	发散	发散

Schwarzschild 黑洞中准圆轨道的辐射能流

例子:任意自旋权重场的总能流

Table 2: 当 $\mathbf{N} = 100$ 和 $r_0 = r_{\rm ISCO}$ 时,四种方法的总能流 ${}_s\langle dE/dt\rangle_{\infty,\rm H}$ 与精确解之间的相对误差 (**RE**)。

		${}_s\langle dE/dt\rangle_\infty$ 的相对误差 $(2 \le \ell \le 25)$				${}_s\langle dE/dt\rangle_{\rm H}$ 的相对误差 $(2 \le \ell \le 11)$			
s	Method	HeunC	MST-RTE	MST-RWE	23PN	HeunC	MST-RTE	MST-RWE	23PN
-2	**RE**	6.354E-73	2.769E-52	9.458E-51	1.496E-4	3.135E-76	2.308E-51	5.370E-51	1.750E-3
	CPU(s)	197.9	591.7	588.1	NA	53.0	91.2	91.9	NA
$+2$	**RE**	7.421E-72	7.993E-53	1.303E-50	NA	6.222E-74	9.458E-51	5.370E-51	NA
	CPU(s)	193.6	571.3	576.1	NA	54.7	88.8	93.1	NA
-1	**RE**	6.274E-72	1.136E-52	NA	NA	4.633E-76	5.302E-52	NA	NA
	CPU(s)	192.0	574.8	NA	NA	51.2	94.9	NA	NA
$+1$	**RE**	3.880E-72	2.132E-52	NA	NA	9.675E-75	9.867E-51	NA	NA
	CPU(s)	189.5	599.2	NA	NA	51.4	98.0	NA	NA
0	**RE**	5.539E-59	1.343E-52	NA	NA	4.802E-53	3.451E-51	NA	NA
	CPU(s)	139.2	295.5	NA	NA	29.2	52.5	NA	NA

Kerr 黑洞中一般轨道上运动的引力波能流

准圆轨道 ♣				球轨道 ♢				椭圆轨道 ♡				一般轨道 ♠			
a	Δ_E	HeunC	MST	x_I	Δ_E	HeunC	MST	e	Δ_E	HeunC	MST	e	Δ_E	HeunC	MST
0.1	10^{-20}	34	38	0.1	10^{-20}	34	39	0.1	10^{-20}	32	44	0.1	10^{-20}	32	43
	10^{-30}	44	56		10^{-30}	44	61		10^{-30}	42	64		10^{-30}	42	63
0.3	10^{-20}	36	38	0.3	10^{-20}	33	39	0.3	10^{-20}	35	44	0.3	10^{-20}	31	43
	10^{-30}	45	57		10^{-30}	43	59		10^{-30}	42	64		10^{-30}	41	62
0.5	10^{-20}	38	38	0.5	10^{-20}	33	39	0.5	10^{-20}	36	44	0.5	10^{-20}	32	43
	10^{-30}	48	57		10^{-30}	44	59		10^{-30}	42	65		10^{-30}	42	63
0.7	10^{-20}	43	43	0.7	10^{-20}	34	39	0.7	10^{-20}	35	42	0.7	10^{-20}	37	44
	10^{-30}	53	57		10^{-30}	44	60		10^{-30}	44	62		10^{-30}	46	64
0.9	10^{-20}	53	54	0.9	10^{-20}	34	40	0.9	10^{-20}	44	44	0.9	10^{-20}	39	46
	10^{-30}	64	78		10^{-30}	44	61		10^{-30}	49	63		10^{-30}	47	68

[♣] $p_{\rm ISCO} = 6M$ 处的圆轨道参数为 $\ell_{\max} = 6$. (x_I 是轨道倾角的余弦值)

[♢] $p = 10M$ 处的球轨道参数为 $a = 0.9, e = 0, \ell_{\max} = 6, 1 \leq n \leq 5$.

[♡] $p = 10M$ 处的椭圆轨道的参数为 $a = 0.9, \theta_{\rm inc} = 0, \ell_{\max} = 5, 1 \leq k \leq 3$.

[♠] $p = 10M$ 处的一般轨道参数为 $a = 0.9, x_I = 0.1, \ell_{\max} = 4, 1 \leq n \leq 3, 1 \leq k \leq 3$.

Schwarzschild 黑洞中准圆轨道上的辐射能流

例子: $r_0 = r_{\text{ISCO}}$ 时, 不同浮点数 N 的四种方法与精确解的能流 ${}_s\langle dE_{22}/dt\rangle_{\infty,\text{H}}$ 的相对误差

左图:无穷远处的辐射能流 右图:视界处的辐射能流

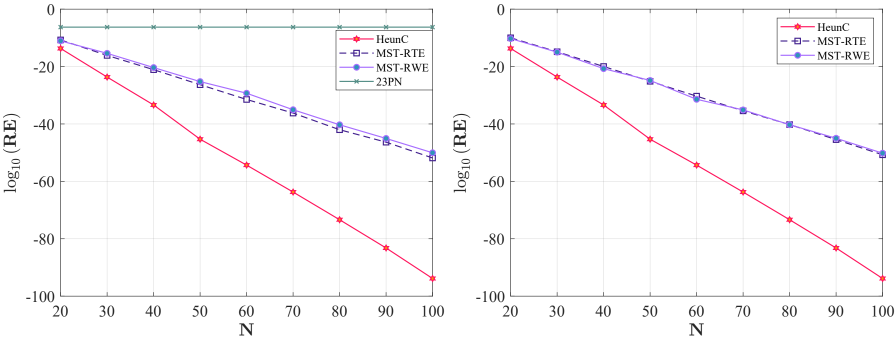


Figure 4: 引力场的能流比较

与近似解的比较

- 与 Sasaki 的后牛顿展开的结果相比,我们的优势:
  1. 一个无需级数展开的完整解析解。
  2. 出射波解 (37) 是满足 Wronskian 行列式的守恒条件。
  3. 不受缓慢运动和弱场近似的限制,(视界处)比后牛顿更准确。
- 与 MST 方法相比,我们的优势:
  1. 一个简单的合流 Heun 函数,无需计算双边无穷级数。
  2. 没有任何的限制,而 MST 方法受低频近似的限制。

Conclusion and Outlook

Conclusions and Outlook

1. 引力波能流的计算，克服计算难点：大离心率、高自旋、高阶模和强场区域。而且，结果优于高阶的后牛顿展开、MST 法和数值积分法。
2. 给出了合流 Heun 函数在无穷远处的渐近解析式；其计算结果优于三大数学软件。
3. 给出了 Kerr 时空的含源 Teukolsky 方程的精确的解析解（无需低频、慢动或弱场的条件约束）。
4. 作为一种新的求解含源 Teukolsky 方程的范式：
   (1) 二阶的黑洞微扰 [Spiers, Pound, Barack (AAK), Wardell, van de Meent]
   (2) 修改引力 [陈雁北的 Teukolsky 方程：PRX, 2023,13(2):021029]
   (3) 复杂时空（Kerr-Newman anti-de Sitter）[PTP. 1998, 100(3):491–505]

Thanks !