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WCCC is Safe in Gedankenexperiments

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Outline

- **A review of WCCC in Gedankenexperiments**
  - Wald's original proposal for WCCC
  - Counter proposals against WCCC
  - Resolutions of contradictions
- **New approach**
  - Different proposals in new language
  - General formulae
- **WCCC will not be violated by Gedankenexperiments.**

Cosmic Censorship Conjecture

The Cosmic Censorship Conjecture (CCC) was proposed by Roger Penrose in 1969.

- *Proc. Roy. Soc. Lond. A 314, 529-548 (1970)*

Also, see the following paper by Penrose:

- *General Relativity and Gravitation, Vol. 34, No. 7 (2002)*

If a spacetime contains a singularity that is not hidden behind an event horizon, i.e., far-away observers can receive signals from it, the singularity is said to be "naked."

Since initial conditions cannot be specified at a singularity–meaning anything can come out of the singularity–a naked singularity would prevent predictability in spacetime.

Weak Cosmic Censorship Conjecture

Two versions of CCC:

- **Weak CCC:** In a geodesically complete and asymptotically flat spacetime, the evolution of matter fields satisfying the null energy condition cannot lead to a naked singularity.
- **Strong CCC:** The inner Cauchy horizon is unstable for an infalling test particle or field.

–The Cosmic Censorship Conjecture plays a crucial role in maintaining the deterministic nature of general relativity. We focus on the Weak CCC (WCCC) in this talk.

Wald's Gedankenexperiment on WCCC

Proving the WCCC in general is challenging. One approach involves perturbing a black hole (M, Q_i) with a test particle ($E > \delta M$, $q_i = \delta Q_i$) to see if it leads to a visible singularity.

Wald focused on the Kerr-Newmann (KN) black hole of mass M, charge Q and angular momentum $J = Ma$, with the remaining thermodynamic quantities

$$\Omega = \frac{a}{r_+^2 + a^2}, \quad \Phi = \frac{r_+ Q}{r_+^2 + a^2}, \quad T = \frac{r_+ - M}{2\pi(r_+^2 + a^2)}, \quad S = \pi(r_+^2 + a^2).$$

[Wald, 1974]

Future pointed geodesic motion

For an infalling particle with conserved energy E, angular momentum L, and charge q, the future-pointed velocity $\dot{t} \geq 0$ of the geodesic motion implies

$$E \geq \frac{aL + qQr_+}{a^2 + r_+^2}.$$

If the particle crosses through the outer horizon, then the mass, charge, and angular momentum of the BH change, namely

$$\delta M = E, \quad \delta J = L, \quad \delta Q = q\,,$$

which leads to

$$\delta M \geq \frac{a\delta J + Qr_+\delta Q}{a^2 + r_+^2} = \Omega\delta J + \Phi\delta Q.$$

This inequality, derived from the geodesic motion, resembles part of the first law.

Second law and NEC

The inequality can also be derived from two seemingly different principles:

- The first and second laws: $\delta S \geq 0 \to \delta M - \Phi_i \delta Q_i = T\delta S \geq 0$:

$$\delta M \geq \Omega\delta J + \Phi\delta Q = \frac{a\delta J + Qr_+\delta Q}{a^2 + r_+^2}.$$

- Null energy condition (NEC). Specifically, a test body with energy momentum tensor $T_{\mu\nu}$ crossing the horizon satisfies the condition [E. Poisson, lecture notes, (2002); Chirco,Liberati, Sotiriou,1006.3655]

$$\delta M - \Omega\delta J = \int T_{ab}\chi^a d\Sigma^b \geq 0\,,$$

where $\chi = \partial_t + \Omega\partial_\phi$ is the degenerate Killing vector on the horizon.

  - The NEC needs only to be imposed on the horizon here.
  - The NEC is required for the test body, not black hole itself.

Horizon condition

When a charged particle of $(\delta M, \delta Q, \delta J)$ enters the horizon of KN, would it destroy its horizon and reveals its singularity?

Intuitively, the most vulnerable situation is the extremal limit, with

$$M_{\text{ext}}(Q,J) = r_{\text{ext}}(Q,J) = \sqrt{\frac{1}{2}(Q^2 + \sqrt{4J^2 + Q^4})}.$$

For the KN solution of (M, Q, J), we have two situations

- It contains a naked singularity if $M < M_{\text{ext}}(Q,J)$.
- It describes a black hole if $M \geq M_{\text{ext}}(Q,J)$. (For the KN, it can be equivalently expressed as $M^2 \geq Q^2 + \frac{J^2}{M^2}$, typically adopted in literature, e.g. [Wald; ...])

When the test body enters the horizon of the extremal KN, we have the $(M_{\text{ext}} + \delta M, Q + \delta Q, J + \delta J)$. We can define a quantity

$$X = M_{\text{ext}}(Q,J) + \delta M - M_{\text{ext}}(Q + \delta Q, J + \delta J) = \delta M - \delta M_{\text{ext}}\,.$$

- $X \geq 0$: WCCC is protected;
- $X < 0$: WCCC is violated.

WCCC is protected in the Wald's Gedankenexperiment

From various arguments, including the NEC condition, we saw that

$$\delta M \geq \Omega\delta J + \Phi\delta Q = \delta M_{\text{ext}}\,.$$

It follows that

$$X = \delta M - \delta M_{\text{ext}} \geq 0\,.$$

In other words, WCCC of the extremal KN is protected.

In this Gedankenexperiment, two independent inequalities are used

- **Energy Condition**, (or equivalently the first and second laws) are given respectively by

  $$\delta M \geq \phi_i \delta Q_i \quad \text{(first law, imposing } \delta S > 0\text{)}$$

  This also coincides with the $\dot{t} \geq 0$ condition of ingoing particles.

- **Horizon Condition**: Ensures $M \geq M_{\text{ext}}$ before and after the perturbation to protect the WCCC.

Alternative proposals that violate WCCC

After Wald's demonstration of WCCC for KN in 74, counter arguments emerged

- **Hubeny** [9808043] suggested that the WCCC could be violated for the near-extremal RN black hole; also for KN [Saa,Santarelli,1105.3950] (Missing type)

- **Gao and Zhang** [1211.2631] proposed WCCC violation for an extremal KN black hole by expanding the horizon condition to the second order in δM and δQ_i, while the energy conditions remained first order (Mixed type).

Resolving contradictions

- To resolve the contradictions in Hubeny's missing type, Sorce and Wald (SW) identified that their analysis was insufficient at the linear order of δQ_i and δM. The near-extremal case requires a second-order expansion. [Sorce,Wald,1707.05862]

– SW examined WCCC for near-extremal KN black holes up to the **second-order variation** and showed there was no violation of WCCC for KN.

– Regarding the mixed type, we shall point out that mixing different orders in the analysis led to confusion.

SW Analysis

SW introduced an order parameter λ to express the horizon condition for KN as (equivalent to our earlier $M - M_{\text{ext}}(Q,J) > 0$)

$$f(\lambda) = M(\lambda)^2 - Q(\lambda)^2 - \frac{J(\lambda)^2}{M(\lambda)^2} \geq 0\,,$$

which, using Taylor expansion, contains all orders of perturbation. SW proposed, based on argument of NEC of perturbating matter, a new inequality at the second order

$$\delta^2 M - \Omega_H \delta^2 J - \Phi_H \delta^2 Q \geq -\frac{\kappa}{8\pi}\delta^2 A^{KN}\,.$$

With this input, SW demonstrated that the above horizon condition is satisfied at the second order

$$f(\lambda) \geq \left(-\frac{(M^4 - J^2)Q\delta Q + 2JM^2\delta J}{M(M^4+J^2)}\lambda + M\epsilon\right)^2 + O(\lambda^3, \lambda^2\epsilon, \ldots),$$

where $\epsilon = \frac{r_+}{M} - 1 \geq 0$ is a dimensionless parameter, and M is the extremal value.

[Sorce,Wald,1707.05862]

Our approach

We shall follow largely the SW proposal, but apply for a more general class of black holes in diverse dimensions

- that are specified by conserved quantities, such as mass M and charges Q_i, including the electric and magnetic charges of a single or multiple Maxwell fields, and/or angular momenta,
- that have extremal limit ($T=0$), with $M(T=0)=M_{\text{ext}}(Q_i)$.

We consider general perturbations

$$\begin{aligned} M &\to M+\Delta M = M+\lambda\delta M+\frac{1}{2}\lambda^2\delta^2 M+\cdots\,, \\ Q_i &\to Q_i+\Delta Q_i = Q_i+\lambda\delta Q_i+\frac{1}{2}\lambda^2\delta^2 Q_i+\cdots\,, \end{aligned}$$

and examine the horizon condition $X_\epsilon \ge 0$, where

$$\begin{aligned} X_\epsilon &\equiv M(T_\epsilon,Q_i)+\Delta M - M_{\text{ext}}(Q_i+\Delta Q_i) \\ &= \Big(M(T_\epsilon,Q_i)-M_{\text{ext}}(Q_i)\Big)+\Delta M-\Delta M_{\text{ext}}. \end{aligned}$$

In the extremal case, the first bracket vanishes.

SW underlying condition

The underlying condition of SW for the KN, based on NEC, was

$$\delta^2 M - \Omega_H \delta^2 J - \Phi_H \delta^2 Q \geq -\frac{\kappa}{8\pi}\delta^2 A^{KN}\,.$$

However, it is rather difficult to follow the logic behind this second-order inequality, at least to us, since, according to Eqn (113) of SW paper,

- $\delta^2 A$ here is the Hessian of $A(M,Q,J)$, which we denote as DA.

Thus δ^2 means different things in different terms in the SW inequality.

Note that $DA = 4DS = -4\frac{DM}{T}$, thus SW inequality also means

$$\delta^2 M - \Omega_H \delta^2 J - \Phi_H \delta^2 Q \geq DM\,.$$

We shall come to this inequality later through a different underlying principle.

Our underlying condition

We adopt a simpler-looking argument, based on the second law

$$\Delta S = \lambda \delta S + \frac{1}{2}\lambda^2 \delta^2 S + \cdots \geq 0 \,.$$

- $\delta S > 0$, then we can absorb the higher-order ones to the leading one, in which case, the WCCC has already been proven.
- $\delta S = 0$, then $\delta^2 S \geq 0$.

This second-order condition was independently proposed earlier in [2405. 07728, Lin, Ning] for studying the WCCC of charged KN-dS black holes.

It can be proven for KN, the above two seemingly different proposal leads to the same outcome. In fact, we shall see presently that

$$(\delta S = 0 \,, \delta^2 S \geq 0) \quad \rightarrow \quad \delta^2 M - \Omega_H \delta^2 J - \Phi_H \delta^2 Q \geq -\frac{k}{8\pi}\delta^2 A^{KN} \,.$$

WCCC types in the new language

With this setup, we can reexamine the previous WCCC types for general class of black holes, instead of specific examples typically done in literature.

We introduce the quantity W of extremal black holes, defined by

$$W = \lim_{T \to 0} \left(\frac{\partial S}{\partial T} \right)_{Q_i} \equiv \left(\frac{\partial S}{\partial T} \right)_{Q_i; T=0} = \lim_{T \to 0} \frac{C_Q}{T} \,,$$

Its sign, namely

$$W > 0 \,,$$

plays a determining role in our discussion.

The missing type

Hubeny's test of WCCC involves replacing Δ in X_ϵ with δ:

$$X_{\mathsf{H}} = \Big(M(T_\epsilon, Q_i) - M_{\mathsf{ext}}(Q_i)\Big) + \delta M - \delta M_{\mathsf{ext}}.$$

Near the extremal case ($T = T_\epsilon \propto \epsilon \to 0$,) expanding up to the first order in λ gives:

$$\begin{aligned} X_{\mathsf{H}} &= T_\epsilon \delta S + \frac{1}{2} W T_\epsilon^2 - \lambda \frac{\partial S_{\mathsf{ext}}}{\partial Q_i} \delta Q_i T_\epsilon \\ &\geq \frac{1}{2} W T_\epsilon^2 - \lambda \left(\delta S_{\mathsf{ext}}\right) T_\epsilon \,. \end{aligned}$$

The inequality arises from $\delta S \geq 0$. The first term is of ϵ^2 order, while the second term is of $\lambda\epsilon$ order. In the extremal limit ($T_\epsilon = 0$), $X_{\mathsf{H}} = 0$.

For example, for the Reissner-Nordström (RN) black hole, $X_{\mathsf{H}} \geq \frac{M\epsilon^2}{2} - \lambda\epsilon\delta Q$.

Conclusion: Hubeny's method is incomplete, as it *lacks* the appropriate λ^2 term in the expansion, which becomes important when X_{H} changes the sign.

The mixed type

The mixed type approach, initiated by Gao and Zhang for KN, investigates WCCC by mixing orders in the extremal case ($\epsilon = 0$). We find that in general it leads to

$$X_{\text{ext,mix}} = -\frac{1}{2}\lambda^2 \frac{\partial^2 M_{\text{ext}}}{\partial Q_i \partial Q_j} \delta Q_i \delta Q_j + \cdots .$$

This result represents the Hessian metric of the extremal mass $M_{\text{ext}}(Q_i)$ in terms of charges, providing a general formula for various examples.

Examples that are counterintuitive:

$$\begin{cases} M_{\text{ext}} = \sqrt{P^2 + Q^2}, & X_{\text{ext}} < 0; \\ M_{\text{ext}} = P + Q, & X_{\text{ext}} = 0; \\ M_{\text{ext}} = (P^{\frac{2}{3}} + Q^{\frac{2}{3}})^{\frac{3}{2}}, & X_{\text{ext}} > 0. \end{cases}$$

Expansions up to the second order in λ have been explored, but with $\delta^2 M = \delta^2 Q = 0$.

- **Conclusion:** The mixed type expansion uses different orders for ΔM and ΔM_{ext}, making the sign of the above equation insufficient to conclusively determine WCCC violation.

SW type, but now with ($\delta S = 0, \delta^2 S \geq 0$)

By expanding:

$$S[\lambda] - S[M(\lambda), Q_i(\lambda)] = 0, \quad M[\lambda] - M[S(\lambda), Q_i(\lambda)] = 0,$$

we obtain, first-order expressions (equivalent to the first law):

$$\delta S = -\frac{\phi_i}{T}\delta Q_i + \frac{1}{T}\delta M, \quad \delta M = T\delta S + \phi_i \delta Q_i.$$

and the second-order relations (after using $\delta S = 0$)

$$\delta^2 S = \frac{1}{T}\delta^2 M - \frac{\phi_i}{T}\delta^2 Q_i + DS, \quad \delta^2 M = T\delta^2 S + \phi_i \delta^2 Q_i + DM,$$

where DS and DM are the Hessian metrics, satisfying $DM = \delta T \delta S + \delta\phi_i \delta Q_i = -TDS$. Using the condition $\delta S = 0$ and $\delta^2 S \geq 0$ **leads to the SW inequality.** It also implies

$$\delta^2 M \geq \phi_i \delta^2 Q_i + \delta\phi_i \delta Q_i.$$

The inequality is saturated for the extremal limit $T = 0$.

Note that requiring $\delta S = 0$ allows us to avoid dealing with δT, which we do not know in principle since the end product may not be a black hole.

SW type (cont.)

For the near-extremal case, we expand X_ϵ

$$\begin{aligned} X_\epsilon &= \Big(M(T_\epsilon, Q_i) - M_{\text{ext}}(Q_i)\Big) + \Delta M - \Delta M_{\text{ext}} \\ &\geq \frac{1}{2} W T_\epsilon^2 - \lambda T_\epsilon \delta S_{\text{ext}} + \frac{1}{2}\lambda^2 (\delta\phi_i - \delta\phi_{\text{ext},i})\delta Q_i. \end{aligned}$$

Further evaluation shows:

$$(\delta\phi_i - \delta\phi_{\text{ext},i})\delta Q_i = W^{-1}(\delta S_{\text{ext}})^2,$$

resulting an SW-type inequality:

$$X_\epsilon \geq \frac{1}{2} W \left(T_\epsilon - \lambda \frac{\delta S_{\text{ext}}}{W}\right)^2.$$

Applying to KN, this general formula reproduces the SW result.

Compared to Hubeny's type, the positive λ^2 term ensures a more complete test for WCCC.

In the extremal limit where $\epsilon = 0$ (or equivalently $T_\epsilon = 0$), we have the equal sign.

On the possibility of WCCC violation

Based on $\delta S = 0$ and $\delta^2 S \geq 0$, we find

$$X_\epsilon \geq \frac{1}{2} W \left(T_\epsilon - \lambda \frac{\delta S_{\text{ext}}}{W} \right)^2 ,$$

where

$$W = \lim_{T \to 0} \left(\frac{\partial S}{\partial T} \right)_{Q_i} \equiv \left(\frac{\partial S}{\partial T} \right)_{Q_i ; T=0} ,$$

The sign of W becomes crucial for the WCCC in these Gedanken experiments

WCCC is protected if $W \geq 0$.

Could W be negative for some specific black holes?

$W > 0$ for spherically-symmetric and static black holes

Consider $ds^2 = -e^{2\chi} f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2$, where $f = f(r, M, Q_i)$ and $\chi = \chi(r, M, Q_i)$. The horizon radius r_+ satisfies $f(r_+, M, Q_i) = 0$, which implies that $M = M(r_+, Q_i)$. The temperature is then given by

$$T = \frac{e^{\chi} f'(r, M(r_+), Q_i)}{4\pi}\Big|_{r=r_+}.$$

A prime denotes a derivative with respect to r. Thus we have

$$\left(\frac{\partial S}{\partial T}\right)_{Q_i} = \frac{8\pi^2 r_+ \, e^{-\chi}}{f'' + 2\pi T(2e^{-\chi}\chi' + r_+ \partial_M f') - 8\pi^2 r_+ T^2 \partial_M e^{-\chi}}\Big|_{r_+,\, M(r_+)}.$$

In the extremal limit, we have

$$T \sim f'(r_+) = 0\,, \qquad f''(r_+) > 0\,.$$

Thus we have

$$W = \left(\frac{\partial S}{\partial T}\right)_{Q_i; T=0} = \frac{8\pi^2 r_+ e^{-\chi(r_+)}}{f''(r_+)} > 0\,.$$

$W > 0$ is related to the no-hair theorem

It is easy to see that

$$M(T_\epsilon, Q_i) - M_{\text{ext}}(Q_i) = \frac{1}{2} W\, T_\epsilon^2 > 0\,, \quad \leftrightarrow \quad W > 0\,.$$

Therefore, $W > 0$ is closely related to the black hole no-hair theorem

We should have $M > M_{\text{ext}}$; otherwise, two black holes with the same mass and charges could have different temperatures. This is because, for fixed charges, the mass is a function of temperature with $M(T = 0) = M_{\text{ext}}$.

If $M(T) < M_{\text{ext}}$ as $T \to 0$, there should be a minimum mass $M_{\text{min}} < M_{\text{ext}}$ at certain non-zero temperature, since it is reasonable to assume that black hole mass is unbound above.

In this scenario, there would then exist at least two black holes of the same mass slightly above the minimum mass M_{min}, but with different temperatures, violating the uniqueness property of black holes.

However, our spherically-symmetric and static proof does not rely on the no-hair theorem. This leads us to expect that $W > 0$ for general black holes, independent of the no-hair theorem, but we cannot prove it.

What if $W < 0$

No-hair theorem is weak and we cannot prove $W > 0$ in general, we therefore have to consider the possibility of $W < 0$, although there is hitherto no such a known example.

$W < 0$ implies that the mass of the extremal black hole for fixed charges is a local maximum. In this case, *the WCCC is protected provided that $X_\epsilon < 0$.* In other words, the horizon condition switches the sign.

From $\delta S = 0$ and $\delta^2 S \geq 0$, we had

$$T_\epsilon = 0: \qquad X_{\text{ext}} = \frac{1}{2}\lambda^2 W^{-1}(\delta S_{\text{ext}})^2,$$
$$T_\epsilon > 0: \qquad X_\epsilon \geq \frac{1}{2}W\left(T_\epsilon - \lambda\frac{\delta S_{\text{ext}}}{W}\right)^2.$$

Thus for $W < 0$, WCCC is still protected for extremal black holes, but could be violated for near extremal ones.

In other words, the condition $\delta S = 0$ and $\delta^2 S \geq 0$ cannot protect the WCCC for near-extremal black holes with $W < 0$.

Asymptotic (A)dS black holes

The conclusion is largely the same for (A)dS black holes, since our analysis does not dependent the details of the black hole geometry.

but for dS black holes, there can be a subtlety, since it has also the cosmic horizon. For KN-dS black holes, there can be inner, outer and cosmic horizons (r_-, r_+, r_c). We shall not consider the case with $r_+ \sim r_c$, where a perturbation can destroy the "world". Instead, consider $r_- \sim r_+ \ll r_c$.

The new subtlety is that we now have an option to consider "total entropy" [2405.07728, Lin, Ning]

$$\widetilde{S} = S_+ + S_c\,,$$

and impose the condition

$$\delta\widetilde{S} = 0\,, \qquad \delta^2\widetilde{S} \geq 0\,.$$

dS black holes are safe

To make sense of $\tilde{S}$ as being the entropy, the thermodynamic quantities should be modified, given by

$$\tilde{T} = \frac{T_+ T_c}{T_+ + T_c}, \quad \widetilde{\Phi}_i = \frac{T_c}{T_+ + T_c}\Phi_{+,i} + \frac{T_+}{T_+ + T_c}\Phi_{c,i}.$$

The first law is satisfied: $dM = \tilde{T} d\tilde{S} + \widetilde{\Phi}_i dQ_i$. (The conserve quantities remain the same.)

Thus the drivation is the same for X_ϵ, but with tilded variables.

Conclusion: WCCC is not violated if $\widetilde{W} > 0$. We verified that in the extremal limit that

- $\tilde{T} = T_+ = 0$.
- $\widetilde{W} = W_+ > 0$. (We verified this for the RN-dS and KN-dS cases)

Thus our main conclusion does not change, regardless of the entropy definition used.

Summary and Results

- Innovated a new approach to examine the WCCC using Gedanken-experiments, providing general formulae for near-extremal and extremal black holes.
- This approach allowed us to systematically address contradictions found in numerous studies and categorize well-known types of WCCC.
- **Key Findings**: General testing of WCCC depends on the sign of W
  - $W > 0$: WCCC is protected for both near-extremal and extremal cases.
  - $W < 0$: Indicates potential violation in near-extremal cases but does not affect extremal black holes.
- The condition $W > 0$ is an intrinsic property of extremal black holes and is linked to $M(T_\epsilon, Q_i) > M_{\text{ext}}(Q_i)$. This is closely related to the black hole no-hair theorem.
- We provided an independent proof that $W > 0$ for spherically symmetric and static extremal black holes. This leads to our belief that $W > 0$ irregardless of the no-hair theorem.
- We thus state that WCCC cannot be violated by the Gedanken experiments at least at the second order. (Higher-order?)

A further comment

The Gedankenexperiments are perturbative analysis of extremal and near extremal black holes; the conclusion may be overthrown by the nonlinear numerical analysis.

In [PKLT, 1702.01755], authors used the numerical approach to show that some unstable points of the single-rotating Myers-Perry black hole in six dimensions led to the violation of WCCC.

However, since single-rotating black hole in $D = 6$ do not have the extremal limit, this does not provide a direct contradiction to the Gedankenexperiment analysis.

Nevertheless, the possibility should be kept in mind, since Einstein gravity is highly nonlinear.

Thank you for your attention!