Analyzing gravitational wave effects in general modified gravity

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Based on:
Yu-Qi Dong, Yu-Qiang Liu, and YXL, RRD 109, 044013 (2024).
Yu-Qi Dong, Xiao-Bin Lai, Yu-Qiang Liu, and YXL, arXiv: 2409.11838.

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Content

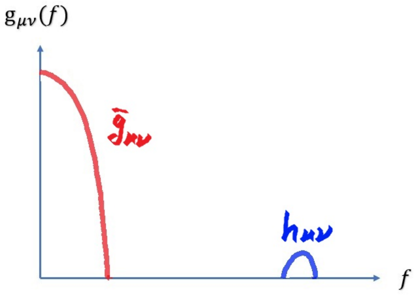
1. Introduction

- There are numerous modified gravity theories, each with distinct predictions about GW effects.
- The detection of gravitational waves (GWs) offers a powerful tool for testing various modified gravity theories.
- The detectable physical quantities of GWs:
  - Polarizations of GWs [Y. Gong et al, PRD 95, 104034 (2017); EPJC 78, 378(2018); Universe 7, 9 (2021).] [P. Wagle et al, PRD 100, 124007 (2019).]
  - Dispersion relationships of various polarization modes
  - Radiated energy/angular momentum
  - Memory effect of GWs (nonlinear effects) [D. Christodoulou, PRL 67, 1486 (1991).] [L. Heisenberg et al, PRD 108, 024010 (2023).]
  - ......
- Can a model-independent general framework be constructed to uniformly analyze the GW effects across various theories?

2. Gravitational wave effects and Isaacson picture

The Isaacson picture [Phys. Rev. 166, 1263 (1968); Phys. Rev. 166, 1272 (1968)]

- Definition of GWs



- Low-frequency background and high-frequency perturbation (GWs):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{1}$$

2. Gravitational wave effects and Isaacson picture

The Isaacson picture [Phys. Rev. 166, 1263 (1968); Phys. Rev. 166, 1272 (1968)]

- Expand Einstein field equation:

$$G_{\mu\nu} = G^{(0)}_{\mu\nu}\left[\bar{g}_{\mu\nu}\right] + G^{(1)}_{\mu\nu}\left[\bar{g}_{\mu\nu}, h_{\mu\nu}\right] + G^{(2)}_{\mu\nu}\left[\bar{g}_{\mu\nu}, h_{\mu\nu}\right] + ... \tag{2}$$

- High-frequency equation:

$$G^{(1)}_{\mu\nu} = 8\pi T^{H,(0)}_{\mu\nu}. \tag{3}$$

$\Rightarrow$ Polarization modes and dispersion relationships.

- Low-frequency equation:

$$G^{(0)}_{\mu\nu} = 8\pi\left(T^{L,(0)}_{\mu\nu} + t_{\mu\nu}\right), \quad t_{\mu\nu} = -\frac{1}{8\pi}\left\langle G^{(2)}_{\mu\nu}\right\rangle. \tag{4}$$

$\Rightarrow$ Effective energy-momentum tensor and memory effect.

- The Isaacson picture also applies to modified gravity theories with N-order derivatives, where $N<19$.

3. Perturbation action method

- Consider the following theory:

$$S\left[\phi\right]=\int d^4x\mathcal{L}\left[\phi\right],\quad \delta S=\int d^4x\mathcal{F}\left[\phi\right]\delta\phi. \tag{5}$$

- Dividing ϕ into the background part ϕ_0 and the perturbation part φ:

$$\phi = \phi_0+\varphi, \tag{6}$$

$$S[\phi] = \sum_{i=0}^{\infty} S^{(i)}\left[\phi_0+\varphi\right], \tag{7}$$

$$\mathcal{F}\left[\phi\right] = \sum_{i=0}^{\infty}\mathcal{F}^{(i)}\left[\phi_0+\varphi\right]. \tag{8}$$

- The relationship between perturbation quantities is

$$\mathcal{F}^{(i)}=\frac{\delta S^{(i+1)}}{\delta\varphi}=\frac{\delta S^{(i)}}{\delta\phi_0},\qquad i\in\mathbb{N}. \tag{9}$$

3. Perturbation action method

- Perturbation action method in general relativity

$$\frac{\delta S}{\delta g_{\mu\nu}} = -\frac{1}{16\pi}\sqrt{-g}G^{\mu\nu} = 0. \tag{10}$$

$$\frac{\delta S^{(0)}}{\delta \bar{g}_{\mu\nu}} = \frac{\delta S^{(1)}}{\delta h_{\mu\nu}} = -\frac{1}{16\pi}\sqrt{-g}^{(0)}G^{(0)\mu\nu}, \tag{11}$$

$$\frac{\delta S^{(1)}}{\delta \bar{g}_{\mu\nu}} = \frac{\delta S^{(2)}}{\delta h_{\mu\nu}} = -\frac{1}{16\pi}\left(\sqrt{-g}^{(0)}G^{(1)\mu\nu} + \sqrt{-g}^{(1)}G^{(0)\mu\nu}\right), \tag{12}$$

$$\frac{\delta S^{(2)}}{\delta \bar{g}_{\mu\nu}} = \frac{\delta S^{(3)}}{\delta h_{\mu\nu}} = -\frac{1}{16\pi}\left(\sqrt{-g}^{(0)}G^{(2)\mu\nu} + \sqrt{-g}^{(1)}G^{(1)\mu\nu} + \sqrt{-g}^{(2)}G^{(0)\mu\nu}\right). \tag{13}$$

It can be seen that the relationship between the variation of the perturbed action and the perturbed Einstein tensor is not simply order-by-order correspondence.

- Analyzing GW effects requires knowledge only of $S^{(1)}$ and $S^{(2)}$.

3. Perturbation action method

- Two definitions of the effective energy-momentum tensor of GWs:

$$t_{\mu\nu} = -2\left\langle \left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\right)^{(2)} \right\rangle = -\frac{1}{8\pi}\left\langle G^{(2)}_{\mu\nu} \right\rangle, \quad \text{by Isaacson,} \tag{14}$$

$$\tilde{t}_{\mu\nu} = -2\left\langle \frac{1}{\sqrt{-\bar{g}}}\frac{\delta S^{(2)}}{\delta \bar{g}^{\mu\nu}} \right\rangle, \quad \text{by Stein and Yunes [PRD 83, 064038 (2011)]}$$

$$= -\frac{1}{8\pi}\left\langle G^{(2)}_{\mu\nu} + \frac{\sqrt{-g}^{(1)}}{\sqrt{-g}^{(0)}} G^{(1)}_{\mu\nu} + \frac{\sqrt{-g}^{(2)}}{\sqrt{-g}^{(0)}} G^{(0)}_{\mu\nu} \right\rangle. \tag{15}$$

- In general, $t_{\mu\nu} \neq \tilde{t}_{\mu\nu}$. However, when considering asymptotic Minkowski spacetime far from the source and on-shell GWs, we have $t_{\mu\nu} = \tilde{t}_{\mu\nu}$.
- The perturbation action method in modified gravity theory is entirely analogous to that in general relativity.

3. Perturbation action method

- Consider the case far from the source in an asymptotic Minkowski spacetime:

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \mathfrak{d}\bar{g}_{\mu\nu}, \quad g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{16}$$

In the Isaacson picture and in a vacuum, high-frequency equation is

$$G^{(1)}_{\mu\nu}\left[\eta_{\mu\nu}, h_{\mu\nu}\right] = 0. \tag{17}$$

and low-frequency equation is

$$G^{(1)}_{\mu\nu}\left[\eta_{\mu\nu}, \mathfrak{d}\bar{g}_{\mu\nu}\right] = 8\pi t_{\mu\nu}, \quad t_{\mu\nu} = -\frac{1}{8\pi}\left\langle G^{(2)}_{\mu\nu}\left[\eta_{\mu\nu}, h_{\mu\nu}\right]\right\rangle. \tag{18}$$

- In an asymptotic Minkowski spacetime far from the source, analyzing GW effects requires knowledge only of $S^{(2)}$.

3. Perturbation action method

- The second-order perturbation action $S^{(2)}_{flat}$, with the Minkowski metric as the background, contains all necessary information to construct the Isaacson picture far from the source:

$$G^{(1)}_{\mu\nu}\left[\eta_{\mu\nu}, h_{\mu\nu}\right] \quad \text{can be derived from} \quad \frac{\delta S^{(2)}_{flat}}{\delta h^{\mu\nu}},$$

$$\left\langle G^{(2)}_{\mu\nu}\left[\eta_{\mu\nu}, h_{\mu\nu}\right]\right\rangle \quad \text{can be derived from} \quad \left\langle \frac{\delta S^{(2)}_{flat}}{\delta \eta^{\mu\nu}} \right\rangle.$$

  The proof can be found in our paper [arXiv: 2409.11838] or in [Heisenberg, Yunes, and Zosso, PRD **108**, 024010 (2023)].

- Constructing a model-independent framework $\Rightarrow$ Constructing the most general $S^{(2)}_{flat}$.

4. Construction of most general 2nd-order actions

- Lovelock's theorem [JMP 12, 498 (1971)]: 4D, GR.

Various methods for modifying general relativity

- Add additional fields
- Higher-order derivative theory
- Use more or fewer than four spacetime dimensions
- Non-Riemannian geometry
- Other methods, etc

4. Construction of most general 2nd-order actions

- We can construct the most general pure metric theory $S\left[g_{\mu\nu}\right]$ and scalar-tensor theory $S\left[g_{\mu\nu},\Psi\right]$ that satisfies the following assumptions in a Minkowski background:

Required assumptions

(1) Spacetime is a 4D Riemannian manifold.

(2) The theory satisfies the least action principle.

(3) The theory is generally covariant ($S^{(2)}_{flat}$ is gauge invariant).

(4) The action of a free particle is $\int ds = \int \sqrt{|g_{\mu\nu}dx^{\mu}dx^{\nu}|}$.

4. Construction of most general 2nd-order actions

- Gravity is described by Riemannian geometry.

Elements required for constructing action

- The metric $g_{\mu\nu}$
- The 4D totally antisymmetric tensor $E_{\mu\nu\lambda\rho}$
- The partial derivative ∂_μ
- The scalar field Ψ (for scalar-tensor theory)

- Each term in $S^{(2)}_{flat}$ can be represented as a combination of $\eta_{\mu\nu}$, $h_{\mu\nu}$, $E^{\mu\nu\lambda\rho}$, ∂_μ, (Ψ_0, ψ for scalar-tensor theory) and the theoretical parameters.

4. Construction of most general 2nd-order actions

- For pure metric theory

$$S^{(2)}_{flat} = S_1 + \sum_{I\geq 2}^{N} S_I, \tag{19}$$

where

$$S_1 = \int d^4x\, a_1 h^{\mu\nu} \left[2\partial_\nu\partial^\lambda h_{\mu\lambda} - \Box h_{\mu\nu} - 2\partial_\mu\partial_\nu h + \eta_{\mu\nu}\Box h\right], \tag{20}$$

$$S_I = \int d^4x\, h^{\mu\nu}\Big[(a_I + b_I)\,\Box^{I-2}\partial_\mu\partial_\nu\partial^\lambda\partial^\rho h_{\lambda\rho} - 2b_I\Box^{I-1}\partial_\nu\partial^\lambda h_{\mu\lambda} + b_I\Box^I h_{\mu\nu} - 2a_I\Box^{I-1}\partial_\mu\partial_\nu h + a_I\eta_{\mu\nu}\Box^I h\Big]. \tag{21}$$

- In order to reduce to general relativity, we should have $a_1 = 1/2$.

4. Construction of most general 2nd-order actions

- For scalar-tensor theory

$$S^{(2)}_{flat} = S_1 + \sum_{I\geq 2}^{N} S_I, \tag{22}$$

where

$$\begin{aligned} S_1 &= \int d^4x\, a_1 h^{\mu\nu}\Big[2\partial_\nu\partial^\lambda h_{\mu\lambda} - \Box h_{\mu\nu} - 2\partial_\mu\partial_\nu h + \eta_{\mu\nu}\Box h\Big] \\ &+ \int d^4x\, \psi\Big[c_0\psi + c_1\Box\psi + d_1\partial_\mu\partial_\nu h^{\mu\nu} - d_1\Box h\Big], \end{aligned} \tag{23}$$

$$\begin{aligned} S_I &= \int d^4x\, h^{\mu\nu}\Big[(a_I + b_I)\,\Box^{I-2}\partial_\mu\partial_\nu\partial^\lambda\partial^\rho h_{\lambda\rho} - 2b_I\Box^{I-1}\partial_\nu\partial^\lambda h_{\mu\lambda} \\ & b_I + \Box^I h_{\mu\nu} - 2a_I\Box^{I-1}\partial_\mu\partial_\nu h + a_I\eta_{\mu\nu}\Box^I h\Big] \\ &+ \int d^4x\, \psi\Big[c_I\Box^I\psi + d_I\Box^{I-1}\partial_\mu\partial_\nu h^{\mu\nu} - d_I\Box^I h\Big]. \end{aligned} \tag{24}$$

4. Construction of most general 2nd-order actions

- We can construct the second-order action of the most general vector-tensor theory that satisfies the following assumptions in a Minkowski background:

Required assumptions

(1) Spacetime is a 4D Riemannian manifold.

(2) The theory satisfies the least action principle.

(3) The theory is generally covariant ($S^{(2)}_{flat}$ is gauge invariant).

(4) The field equations are second-order.

(5) The action of a free particle is $\int ds = \int \sqrt{|g_{\mu\nu}dx^\mu dx^\nu|}$.

- Each term in $S^{(2)}_{flat}$ can be represented as a combination of $\eta_{\mu\nu}$, $\eta^{\mu\nu}$, $h_{\mu\nu}$, A^μ(background vector), B^μ (perturbation), $E^{\mu\nu\lambda\rho}$, ∂_μ, and the theoretical parameters.

4. Construction of most general 2nd-order actions

- For vector-tensor theory

$$S_{flat}^{(2)} = S_0^{(2)} + S_1^{(2)} + S_2^{(2)} = \int d^4x\sqrt{-\eta}\left(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2\right), \tag{25}$$

where

$$\begin{aligned} \mathcal{L}_0 &= A_{(0)}A^\mu A^\nu A^\lambda A^\rho h_{\mu\nu}h_{\lambda\rho} + 4A_{(0)}A^\mu A^\nu A^\lambda h_{\mu\nu}B_\lambda \\ &+ 4A_{(0)}A_\mu A_\nu B^\mu B^\nu, \end{aligned} \tag{26}$$

$$\begin{aligned} \mathcal{L}_1 &= A_{(1)}\left(E^{\mu\lambda\sigma\gamma}A^\nu A^\rho A_\sigma \partial_\gamma h_{\mu\nu}\right)h_{\lambda\rho} + B_{(1)}\left(\left(A\cdot\partial\right)A_\mu A_\nu h\right)h^{\mu\nu} \\ &- 2B_{(1)}\left(A^\mu A^\nu \partial^\lambda h_{\mu\nu}\right)B_\lambda + 2A_{(1)}\left(E^{\mu\lambda\sigma\gamma}A_\sigma\partial_\gamma A^\nu h_{\mu\nu}\right)B_\lambda \\ &+ 2B_{(1)}\left(\left(A\cdot\partial\right)A^\mu h\right)B_\mu \\ &- 4B_{(1)}\left(A_\mu\partial_\nu B^\mu\right)B^\nu - A_{(1)}\left(E^{\mu\nu\lambda\rho}\partial_\lambda A_\rho B_\mu\right)B_\nu. \end{aligned} \tag{27}$$

4. Construction of most general 2nd-order actions

$$
\begin{aligned}
\mathcal{L}_2 = &\, A_{(2)} \left(\Box A^\mu A^\nu A^\lambda A^\rho h_{\mu\nu}\right) h_{\lambda\rho} + B_{(2)} \left((A\cdot\partial)^2 A^\mu A^\nu A^\lambda A^\rho h_{\mu\nu}\right) h_{\lambda\rho} \\
&+ C_{(2)} \left((A\cdot\partial) A^\mu A^\nu A^\lambda \partial^\rho h_{\mu\nu}\right) h_{\lambda\rho} + D_{(2)} \left(A^\mu A^\nu \partial^\lambda \partial^\rho h_{\mu\nu}\right) h_{\lambda\rho} \\
&+ E_{(2)} \left(A^\mu A^\lambda \partial^\nu \partial^\rho h_{\mu\nu}\right) h_{\lambda\rho} - E_{(2)} \left(\Box A^\lambda A_\rho h_{\mu\lambda}\right) h^{\mu\rho} \\
&+ F_{(2)} \left((A\cdot\partial)^2 A^\lambda A_\rho h_{\mu\lambda}\right) h^{\mu\rho} + G_{(2)} \left((A\cdot\partial) A^\lambda \partial_\rho h_{\mu\lambda}\right) h^{\mu\rho} \\
&- 2H_{(2)} \left(\partial^\lambda \partial_\rho h_{\mu\lambda}\right) h^{\mu\rho} + H_{(2)} \left(\Box h_{\mu\nu}\right) h^{\mu\nu} + 2H_{(2)} \left(\partial_\mu \partial_\nu h\right) h^{\mu\nu} - H_{(2)} \left(\Box h\right) h \\
&+ I_{(2)} \left((A\cdot\partial)^2 h_{\mu\nu}\right) h^{\mu\nu} - D_{(2)} \left(\Box A_\mu A_\nu h\right) h^{\mu\nu} + J_{(2)} \left((A\cdot\partial)^2 A_\mu A_\nu h\right) h^{\mu\nu} \\
&- G_{(2)} \left((A\cdot\partial) A_\mu \partial_\nu h\right) h^{\mu\nu} + K_{(2)} \left((A\cdot\partial)^2 h\right) h \\
&+ \left(4A_{(2)} + C_{(2)}\right) \left(\Box A^\mu A^\nu A^\lambda h_{\mu\nu}\right) B_\lambda + 4B_{(2)} \left((A\cdot\partial)^2 A^\mu A^\nu A^\lambda h_{\mu\nu}\right) B_\lambda \\
&+ \left(C_{(2)} + 2J_{(2)}\right) \left((A\cdot\partial) A^\mu A^\nu \partial^\lambda h_{\mu\nu}\right) B_\lambda + 2\left(C_{(2)} + F_{(2)}\right) \left((A\cdot\partial) A^\mu \partial^\nu A^\lambda h_{\mu\nu}\right) B_\lambda \\
&+ \left(2E_{(2)} - G_{(2)}\right) \left(A^\mu \partial^\nu \partial^\lambda h_{\mu\nu}\right) B_\lambda + \left(2D_{(2)} + G_{(2)}\right) \left(\partial^\mu \partial^\nu A^\lambda h_{\mu\nu}\right) B_\lambda \\
&+ \left(-2E_{(2)} + G_{(2)}\right) \left(\Box A^\mu h_{\mu\lambda}\right) B^\lambda + 2F_{(2)} \left((A\cdot\partial)^2 A^\mu h_{\mu\lambda}\right) B^\lambda \\
&+ \left(G_{(2)} + 4I_{(2)}\right) \left((A\cdot\partial) \partial^\mu h_{\mu\lambda}\right) B^\lambda + \left(-2D_{(2)} - G_{(2)}\right) \left(\Box A^\mu h\right) B_\mu \\
&+ 2J_{(2)} \left((A\cdot\partial)^2 A^\mu h\right) B_\mu + \left(-G_{(2)} + 4K_{(2)}\right) \left((A\cdot\partial) \partial^\mu h\right) B_\mu \\
&+ \left(4A_{(2)} + 2C_{(2)} + F_{(2)}\right) \left(\Box A_\mu A_\nu B^\mu\right) B^\nu + 4B_{(2)} \left((A\cdot\partial)^2 A_\mu A_\nu B^\mu\right) B^\nu \\
&+ \left(2C_{(2)} + 2F_{(2)} + 4J_{(2)}\right) \left((A\cdot\partial) A_\mu \partial_\nu B^\mu\right) B^\nu \\
&+ \left(E_{(2)} + 2I_{(2)} - G_{(2)} + 4K_{(2)}\right) \left(\partial_\mu \partial_\nu B^\mu\right) B^\nu \\
&+ \left(-E_{(2)} + G_{(2)} + 2I_{(2)}\right) \left(\Box B_\mu\right) B^\mu + F_{(2)} \left((A\cdot\partial)^2 B_\mu\right) B^\mu,
\end{aligned}
$$

5. Polarization modes of gravitational waves

Consider

- GWs are weak
- Minkowski background
- The action of a free particle is $\int ds = \int \sqrt{|g_{\mu\nu}dx^\mu dx^\nu|}$

- Relative displacement η_i of two free test particles

$$\frac{d^2\eta_i}{dt^2} = -R_{i0j0}\eta^j. \tag{28}$$

- The polarization modes of GWs can be defined by R_{i0j0}.

5. Polarization modes of gravitational waves

- For a plane GW

$$R_{i0j0} = A E_{ij} e^{ikx}. \tag{29}$$

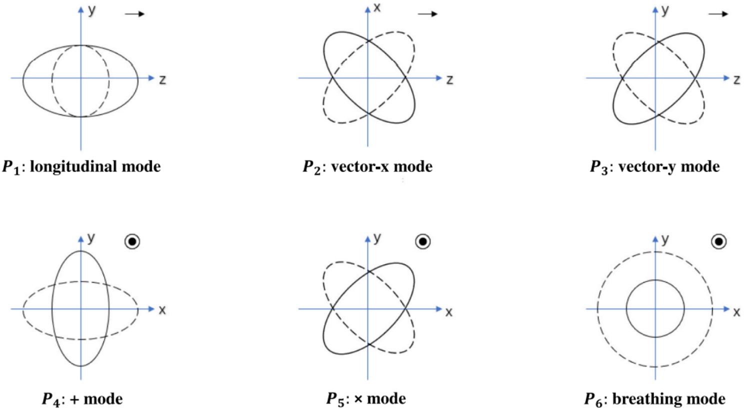
- E_{ij} is a 3×3 symmetric matrix with only 6 independent components. So GWs have at most 6 independent polarization modes in 4-dimensional space-time.

- Define $P_1, ..., P_6$

$$E_{ij} = \begin{pmatrix} P_4 + P_6 & P_5 & P_2 \\ P_5 & -P_4 + P_6 & P_3 \\ P_2 & P_3 & P_1 \end{pmatrix}. \tag{30}$$

5. Polarization modes of gravitational waves

- Six polarization modes of GWs.[1]



[1] D. M. Eardley, D. L. Lee, and A. P. Lightman, Phys. Rev. D 8, 3308 (1973)

5. Polarization modes of gravitational waves

- We can uniquely decompose $h_{\mu\nu}$ and B^μ [2]:

$$\begin{aligned} B^0 &= B^0, \quad B^i = \partial^i\omega + \mu^i, \\ h_{00} &= h_{00}, \quad h_{0i} = \partial_i\gamma + \beta_i, \\ h_{ij} &= h_{ij}^{TT} + \partial_i\epsilon_j + \partial_j\epsilon_i + \frac{1}{3}\delta_{ij}H + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta)\zeta. \end{aligned} \tag{31}$$

- Gauge invariants include

$$\begin{aligned} h_{ij}^{TT} &= h_{ij}^{TT}, \quad \Xi_i = \beta_i - \partial_0\epsilon_i, \quad \Sigma_i = \mu_i + A\partial_0\epsilon_i, \\ \Theta &= \frac{1}{3}(H - \Delta\zeta), \quad \Psi = \omega + \frac{1}{2}A\partial_0\zeta, \\ \phi &= -\frac{1}{2}h_{00} + \partial_0\gamma - \frac{1}{2}\partial_0\partial_0\zeta, \\ \Omega &= B^0 - A\partial_0\gamma + \frac{1}{2}A\partial_0\partial_0\zeta. \end{aligned} \tag{32}$$

$$\overset{(1)}{R}_{i0j0} = -\frac{1}{2}\partial_0\partial_0 h_{ij}^{TT} + \frac{1}{2}\partial_0\partial_i\Xi_j + \frac{1}{2}\partial_0\partial_j\Xi_i + \partial_i\partial_j\phi - \frac{1}{2}\delta_{ij}\partial_0\partial_0\Theta. \tag{33}$$

[2] R. Bluhm, S. H. Fung, and V. A. Kostelecky, Phys. Rev. D 77, 065020 (2008)

5. Polarization modes of gravitational waves

- For pure metric theory and scalar-tensor theory
- The equation describing the tensor mode GWs is

$$\Box\Big(-2a_1+\sum_{I\geq 2}^{N}2b_I\Box^{I-1}\Big)h_{ij}^{TT}=0. \tag{34}$$

- We can write the operator [H. Lu and C.N. Pope, PRL 106, 181302 (2011)]

$$-2a_1+\sum_{I\geq 2}^{N}2b_I\Box^{I-1}=\prod_{k=1}^{M}\left(\Box-m_k^2\right)^{n_k}. \tag{35}$$

- The equation of tensor modes can be rewritten as

$$\Box\prod_{k=1}^{M}\left(\Box-m_k^2\right)^{n_k}h_{ij}^{TT}=0. \tag{36}$$

- The equation of vector modes is

$$\prod_{k=1}^{M}\left(\Box-m_k^2\right)^{n_k}\Xi_i=0. \tag{37}$$

5. Polarization modes of gravitational waves

- Scalar modes under various cases in most general pure metric theory.

Cases	**Conditions**	Breathing mode	Longitudinal mode	Dependent or not	$\mathcal{R}=m^2/\omega^2$
case 1.1	$m^2 \notin \{m_k^2\}, m^2 \notin \{\tilde{m}_k^2\}$	×	×	-	-
case 1.2	$m^2 \notin \{m_k^2\}, m^2 \in \{\tilde{m}_k^2\}$	✓	✓	✓	✓
case 2.1	$m^2 \in \{m_k^2\}, m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	✓	×
case 2.2	$m^2 \in \{m_k^2\}, m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	×	✓	-	-
case 2.3	$m^2 \in \{m_k^2\}, m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	✓	✓
case 2.4	$m^2 \in \{m_k^2\}, m^2 \in \{\tilde{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	✓	✓	×	-

5. Polarization modes of gravitational waves

- Scalar modes under various cases in most general scalar-tensor theory.

Cases	Conditions	Breathing mode	Longitudinal mode	Dependent or not	$\mathcal{R}=m^2/\omega^2$
case 1.1	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) \neq 0$	×	×	-	-
case 1.2	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) = 0, m^2 \in \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 1.3	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) = 0, m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 1.4	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) = 0, m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}$	×	×	-	-
case 1.5	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) = 0, m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 2.1	$m^2 \in \{m_k^2\}, det(\mathcal{A}) \neq 0, \gamma = 0$	×	✓	-	-
case 2.2	$m^2 \in \{m_k^2\}, det(\mathcal{A}) \neq 0, \gamma \neq 0$	✓	✓	✓	×
case 3.1.1	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly independent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 3.1.2	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly independent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 3.1.3	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly independent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}$	×	×	-	-
case 3.1.4	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly independent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 3.2.1	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	✓	✓	×	-
case 3.2.2	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	×	✓	-	-
case 3.2.3	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	✓	✓	×	-
case 3.2.4	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	✓	✓
case 3.2.5	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	✓	✓	×	-
case 3.2.6	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	✓	×
case 3.2.7	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	×	-
case 3.2.8	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	×	-

5. Polarization modes of gravitational waves

- Some universal inferences in pure metric theory and scalar-tensor theory.

Inference 1

- If there exist vector modes of GWs with mass m in this theory, then tensor modes with the same mass m must be present in the theory.
- Correspondingly, if there exist tensor modes with mass $m \neq 0$ in this theory, then vector modes with mass m must be present.

Inference 2

- If there exist only tensor modes propagating at the speed of light, then vector modes must not be present.

5. Polarization modes of gravitational waves

- Some universal inferences in pure metric theory and scalar-tensor theory.

Inference 3

- If there exist only tensor modes propagating at the speed of light, then for a scalar mode with mass m and frequency ω, it must be a mixed mode of the longitudinal mode and breathing mode. And their amplitude ratio is m^2/ω^2.

Inference 4

- If the amplitude ratio of the longitudinal mode to the breathing mode of a scalar GW is not m^2/ω^2, then the theory must have a field equation higher than the second derivative and must have tensor and vector GWs with velocities less than the speed of light.

5. Polarization modes of gravitational waves

For vector-tensor theory

- The properties of polarization modes are different from pure metric theory and scalar-tensor theory.
- Tensor modes generally deviate from the speed of light.
- The $E^{\mu\nu\lambda\rho}$ term will break the parity symmetry of vector modes.
  In some cases, there may even be only one of the left-handed or right-handed waves.
  However, $E^{\mu\nu\lambda\rho}$ can also lead to superluminal problem.
- Scalar modes satisfy one of the following three cases:
  - (1) no scalar mode;
  - (2) two independent modes: breathing and longitudinal ;
  - (3) a mixture of two modes.

More details can be found in [arXiv: 2409.11838].

6. Conclusion

We have constructed a model-independent general framework, which can be used to uniformly analyze the GW effects of various theories.

Thanks for your listening!