Hyperboloidal approach to quasinormal modes

——QNM instability and the pseudospectrum

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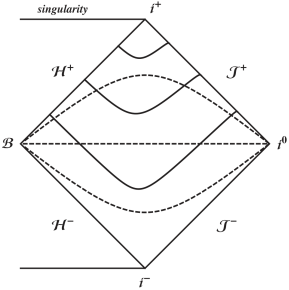
Content

Introduction

- Oscillations of black hole spacetimes (QNMs) exhibit **divergent** behavior toward the **bifurcation sphere** and **spatial infinity**.
- In contrast, QNMs are **regular** when evaluated toward the **event horizon** and **null infinity**.
- Hyperboloidal surfaces connect these regions, providing a geometric regularization.



The traditional approach to QNMs

The Regge-Wheeler-Zerilli wave equation:

$$\Big(-\partial_t^2 + \partial_{r_\star}^2 - V(r_\star)\Big) u(t, r_\star) = 0\,.$$

Time-harmonic solutions:

$$u(t, r_\star) = e^{-i\omega t} R(r_\star)\,.$$

The Helmholtz equation:

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r_\star^2} + \omega^2 - V(r_\star)\Big] R(r_\star) = 0\,.$$

The QNMs are solved by the boundary condition

$$R(r_\star) \sim e^{\pm i\omega r_\star}\,, \quad \text{as} \quad r_\star \to \pm\infty\,.$$

The numerical method are **continued fractions method**, **asymptotic iteration method**, **WKB method** and **pseudospectral method**.

The hyperboloidal approach to QNMs

The construction of hyperboloidal coordinates consists of a **time transformation**, a suitable **spatial compactification**. The new time function is

$$\tau = t + h(r)\,,$$

where h is called the height function. With respect to the new time, we have

$$u(t, r_\star) = e^{-i\omega t} R(r_\star) = e^{-i\omega \tau} e^{i\omega h} R(r_\star) = e^{-i\omega \tau} \bar{R}(r_\star)\,.$$

The rescaled radial function $\bar{R}(r_\star) = e^{i\omega h} R(r_\star)$ is **regular** both near the event horizon and the null infinity.

The hyperboloidal approach to QNMs

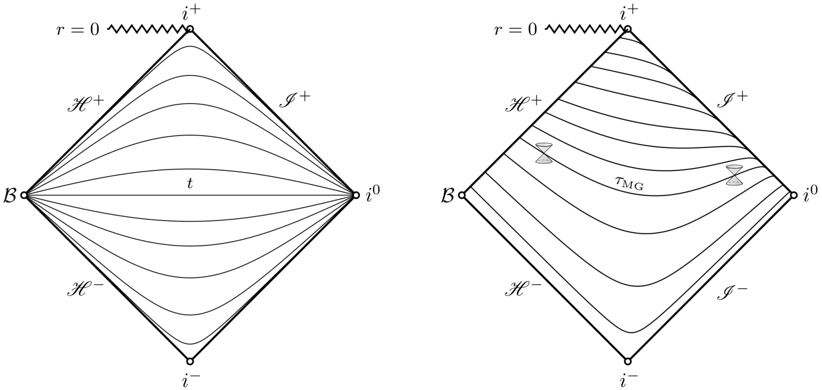


图 1: Left panel: Schwarzchild time slicing. Right panel: hyperboloidal time slicing.

The hyperboloidal approach to QNMs

The wave equations have **short-range** potentials suitable for compactification of the exterior region from the radial coordinate $r \in [r_h, \infty)$ into a compact domain $\sigma \in [\sigma_{\mathscr{I}^+}, \sigma_h]$.

The common compactification strategy is **the minimal gauge** via

$$r = \frac{r_h}{\sigma} .$$

The compact domain $[\sigma_{\mathscr{I}^+}, \sigma_h]$ becomes $[0, 1]$.

The hyperboloidal approach to QNMs

Combine the time transformation with the spatial compactification, the Helmoltz equation has a form

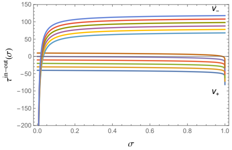
$$\left[\alpha_2(\sigma)\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}+\alpha_1(\sigma)\frac{\mathrm{d}}{\mathrm{d}\sigma}+\alpha_0(\sigma)\right]\bar{R}(\sigma)=0\,,$$

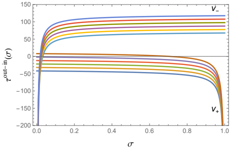
with α_2, α_1, and α_0 depending on the choice of height function. The **most** important property is that

$$\alpha_2\sim(\sigma-\sigma_{\mathscr{I}^+})^2(\sigma-\sigma_h)\,.$$

So the QNMs are **the regular solutions** of the above equation.

From the spacetime perspective, condition $\alpha_2(\sigma_h)=\alpha_2(\sigma_{\mathscr{I}^+})=0$ ensures that the light cones **point outwards** the numerical domain, or that the characteristic speeds of incoming modes **vanished**.





QNM instability and the pseudospectrum

- What the application of the hyperboloidal approach?
- The astrophysical BHs are **not isolated**. The QNM spectrum migrates by the surrounding of BHs.
- The BH spectroscopy program faces the challenge: **QNM spectral instability**. **Small** perturbations can cause **significant** migrations.

First evidence comes from[1].

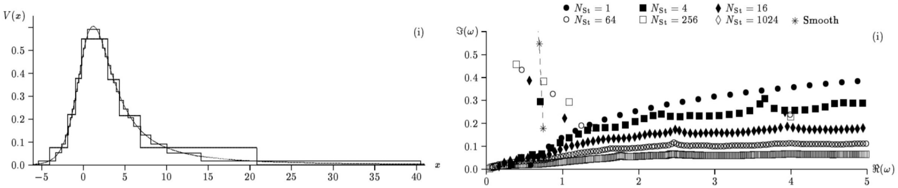


图 2: Left: the various choices of step barriers to approximate the BH potential. Right: the corresponding spectrum resulting from the approximated potential.

[1] H. P. Nollert, Phys. Rev. D **53**, 4397-4402 (1996)

Second evidence comes from[2].

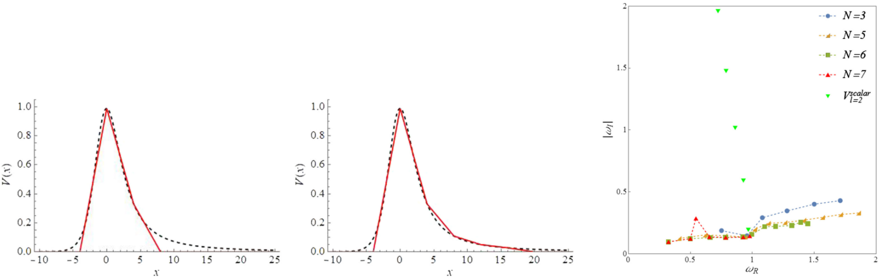


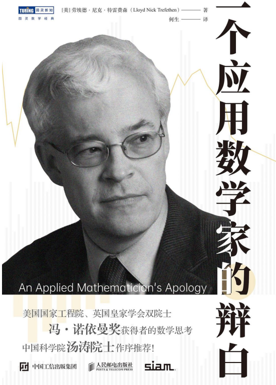
图 3: QNMs for $N = 3, 5, 6, 7$ piecewise linear potentials fitted to $V^{\text{scalar}}_{l=2}$.

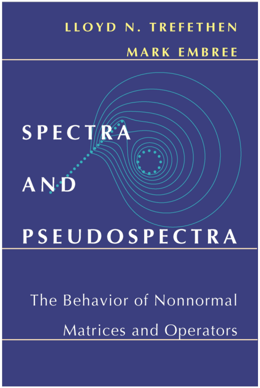
[2] R. G. Daghigh, M. D. Green and J. C. Morey, Phys. Rev. D **101**, no.10, 104009 (2020)

QNM instability and the pseudospectrum

- In terms of BHs, it is a leaky system or an **non-Hermitian** system. Mathematically, the operator is **non-selfadjoint** because of the boundary condition of the QNMs.
- For non-selfadjoint operators, an associated spectral theorem is lacked, which may induce the spectral instability.
- The eigenvalues are not enough for describing the dynamics. The **pseudospectrum** is able to capture the extent to systems controlled by non-selfadjoint operators and tackle the QNM spectral instability.
- The hyperboloidal approach can provide us **the operator expression of the QNM problem** to study the pseudospectrum of QNMs.

QNM instability and the pseudospectrum





- Lloyd N. Trefethen *et al.*, Science 261, 578-584 (1993).
- J. L. Jaramillo, R. Panosso Macedo and L. Al Sheikh, Phys. Rev. X **11**, no.3, 031003 (2021) —— First work of pseudospectrum on gravitational physics.

QNM instability and the pseudospectrum

Given $\epsilon > 0$, the ϵ-pseudospectrum $\sigma^{\epsilon}(A)$ of an operator A is characterized as[3]

$$
\begin{aligned}
\sigma^{\epsilon}(A) &= \{z \in \mathbb{C}, \exists \delta A, \|\delta A\| < \epsilon : z \in \sigma(A + \delta A)\}\,, \\
\sigma^{\epsilon}(A) &= \{z \in \mathbb{C} : \|R_A(z)\| = \|(z\mathbb{I} - A)^{-1}\| > 1/\epsilon\}\,.
\end{aligned}
$$

[3] Trefethen, Lloyd and Embree, Mark. (2005). Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators.

The second definition is more suitable for the **visualization** of the set $\sigma^{\epsilon}(A)$. The ϵ-pseudospectrum is the open subset of the complex plane bounded by the ϵ^{-1} level curve of the norm of the resolvent.

图 4: (a):The geometry of pseudospectra of a self-adjoint operator. (b):The geometry of pseudospectra of a non-selfadjoint operator. The eigenvalues are shown with black dots.

QNM instability and the pseudospectrum

The wave equation becomes **the hyperbolic form**

$$iLu = \partial_\tau u \,,$$

where

$$u = \begin{bmatrix} \phi \\ \psi \end{bmatrix} , \quad L = \frac{1}{i} \begin{bmatrix} 0 & \mathbb{I} \\ L_1 & L_2 \end{bmatrix} ,$$

with

$$\begin{aligned} L_1 &= \frac{p}{w}\partial_\chi^2 + \frac{p'}{w}\partial_\chi - \frac{q}{w} \,, \\ L_2 &= 2\frac{\gamma}{w}\partial_\chi + \frac{\gamma'}{w} \,. \end{aligned}$$

The QNM frequency spectra are the eigenvalues of the evolution operator L or equivalently as the poles of **the resolvent operator**

$$R_L(\lambda) = (L - \lambda\mathbb{I})^{-1} \,.$$

QNM instability and the pseudospectrum

The smallness of perturbation is defined quantitatively through the energy norm. The overall instability to any perturbation of L is captured by the pseudospectrum in this norm, which is defined as

$$\sigma^{\epsilon}(L) = \{\lambda \in \mathbb{C} : \|R_L(\lambda)\|_E > 1/\epsilon\} ,$$

where $\|\cdot\|_E$ indicates the energy norm of the operator, which is associated with **the inner product**

$$\langle u_1, u_2 \rangle_E = \frac{1}{2} \int_0^1 (w(\chi)\bar{\psi}_1\psi_2 + p(\chi)\partial_\chi\bar{\phi}_1\partial_\chi\phi_2 + q(\chi)\bar{\phi}_1\phi_2)\mathrm{d}\chi .$$

QNM instability and the pseudospectrum

An equivalent definition of the pseudospectrum is

$$\sigma^{\epsilon}(L) = \{\lambda \in \mathbb{C}, \exists \delta L, \|\delta L\| < \epsilon : \lambda \in \sigma(L + \delta L)\}\,.$$

- Note that this definition involves **any perturbation** to L, including ones which can potentially be related to a physical modification of the environment of the black hole, but also ones which completely change the nature of the operator.
- At present, physical studies on $V \to V + \delta V$, which induces that

$$\delta L = \begin{bmatrix} 0 & 0 \\ \delta V & 0 \end{bmatrix}\,.$$

Our works on QNM instability and the pseudospectrum

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The pseudospectrum and spectrum (in)stability of quantum corrected Schwarzschild black hole

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The pseudospectrum and transient of Kaluza–Klein black holes in Einstein–Gauss–Bonnet gravity

Jia-Ning Chen[1,2,3], Liang-Bi Wu[1,2,*] and Zong-Kuan Guo[3,2,1]

work1

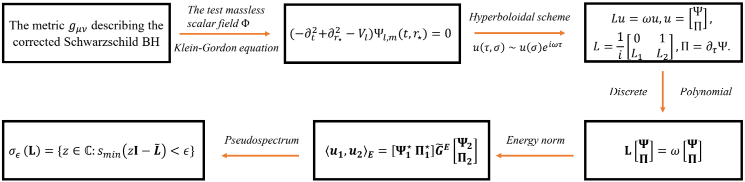
The exterior spacetime of the quantum corrected black hole is given by

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,,$$

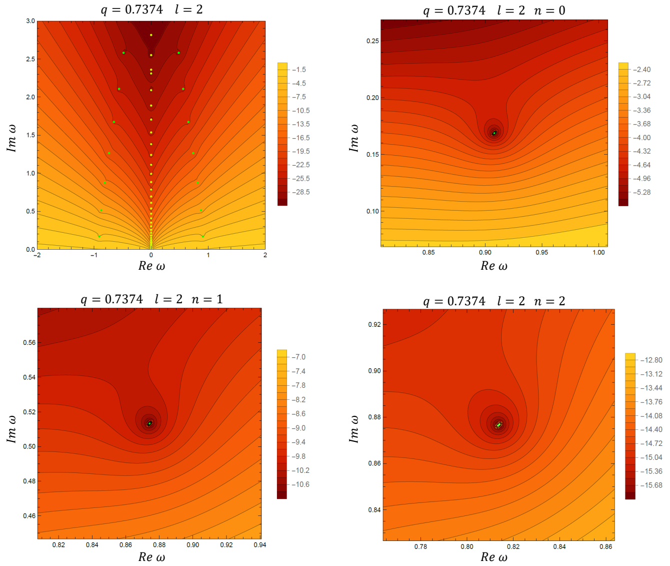
where the metric function is

$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha M^2}{r^4}\,.$$

Calculating the pseudospectrum, there is a flow chart to depict some specific steps.



work1



q = 0.7374 l = 2
Im ω
Re ω
q = 0.7374 l = 2 n = 0
Im ω
Re ω
q = 0.7374 l = 2 n = 1
Im ω
Re ω
q = 0.7374 l = 2 n = 2
Im ω
Re ω

The sensitivity of the QNM spectrum is defined as

$$\delta\omega_n^\epsilon = \frac{|\omega_n^\epsilon - \omega_n^0|}{|\omega_n^0|}\,.$$

Remark: Another formula $\lim_{\epsilon\to 0}\delta\omega_n/\epsilon$ to quantitatively characterize the sensitivity of QNM spectrum (arxiv: 2407.20131).

We choose two perturbations of the effective potential at the event horizon and null infinity as follows

$$\begin{aligned}\delta q_{l1} &= A_1\Big\{1-\tanh\Big[H_1(1-x)\Big]\Big\}\,,\\ \delta q_{l2} &= A_2\Big\{1-\tanh\Big[H_2(1+x)\Big]\Big\}\,.\end{aligned}$$

- Overtaking instability.

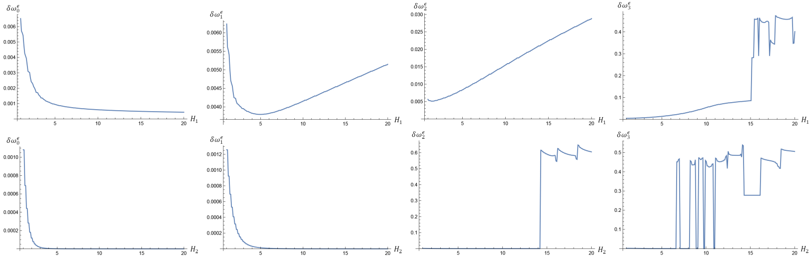


图 5: The energy norm considered are the same $\|\delta L_1\|_E = \|\delta L_2\|_E = 0.02$.

work1

- We investigate the pseudospectrum of quantum corrected Schwarzschild black hole.
- We study the spectrum (in)stability of the quantum corrected black hole by directly adding some particular perturbations into the effective potential. First, perturbations at infinity are more capable of generating spectrum instability than those at the event horizon. Second, we find that the **peak distribution** can lead to the instability of QNM spectrum more efficiently than the **average distribution**.

work2

The so-called Maeda-Dadhich black hole spacetime is locally homeomorphic to $M^4 \times \mathcal{K}^{n-4}$ with the metric reading as

$$g_{MN}\mathrm{d}x^M\mathrm{d}x^N = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + r_0^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\,,$$

where r_0 satisfies

$$r_0^2 = -2K\alpha(n-4)(n-5)\,,$$

and γ_{ij} is unit metric on the constant curvature manifold $\mathcal{K}^{n-4}$ with sectional curvature $K = -1$. The metric function $f(r)$ is written as

$$f(r) = 1 + \frac{r^2}{2(n-4)\alpha}\left\{1 - \left[1 - \frac{2n-11}{3(n-5)} + \frac{4(n-4)^2\alpha^{3/2}\mu}{r^3} - \frac{4(n-4)^2\alpha^2 q}{r^4}\right]^{1/2}\right\}.$$

work2

The master equation of the tensor perturbation[4], after the separation of the variables, can be obtained

$$\Big[\frac{4n-22}{(n-4)(n-5)}g^{ab} - 4\alpha \cdot {}^4G^{ab}\Big] D_a D_b \Phi +$$
$$\Big[\frac{2+\gamma}{(n-4)(n-5)}{}^4R + \frac{3(n-6)(2+\gamma)}{\alpha(n-4)^2(n-5)^2}\Big]\Phi = 0\,,$$

The scalar Φ is the amplitude of the characteristic field $\bar{h}_{ij}$ corresponding to the characteristic value γ. The constant γ is the eigenvalue of the hyperbolic space $\mathcal{K}^{n-4}$.

$$\gamma = -\Big[\zeta^2 + \Big(\frac{n-5}{2}\Big)^2 + 2\Big]\,,$$

with ζ^2 is continuous and positive. The constant γ is limited by $\gamma < -(n-5)^2/4 - 2$.

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[4] L. M. Cao, L. B. Wu, Y. Zhao and Y. S. Zhou, Phys. Rev. D **108** (2023) no.12, 124023

work2

Separating the variable $\Phi(t, r, \theta, \phi)$ as

$$\Phi(t, r, \theta, \phi) = S(r)\varphi(t, r)Y_{lm}(\theta, \phi)\,,$$

where $Y_{lm}(\theta, \phi)$ is the spherical harmonics, we obtain a wave equation with a potential $V(r)$ as follows

$$\Big[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_\star^2} - V(r)\Big]\varphi(t, r) = 0\,.$$

When $r \to +\infty$, we obtain the asymptotic behavior of the effective potential as follows

$$V(r) \sim V_0(\alpha, n, \gamma)r^2\,.$$

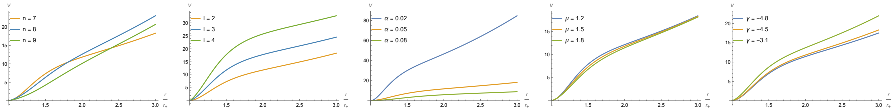
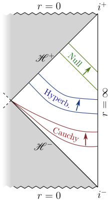


图 6: The effective potential $V(r)$ varying with different parameters n, l, α, μ and γ are depicted by maintaining the event horizon be $r_+ = 1$. The benchmark parameters are $n = 7$, $l = 2$, $\alpha = 0.05$, $\mu = 1.5$ and $\gamma = -4.5$.

work2

We utilize the **ingoing Eddington-Finkelstein (EF) coordinates** to cast the problem of solving for the QNM frequencies as an eigenvalue problem. We can compactify the radial coordinate r by introducing $z = 1/r$. We have the coordinate transformation, which is given by

$$v = t + r_\star \,, \quad \frac{\mathrm{d}r_\star}{\mathrm{d}r} = \frac{1}{f(r)} \,, \quad z = \frac{1}{r} \,.$$



The convergence of the pseudospectrum is more favorable in null slicing[5].

[5] V. Boyanov, V. Cardoso, K. Destounis, J. L. Jaramillo and R. Panosso Macedo, Phys. Rev. D 109 (2024) no.6, 064068

work2

We obtain the equation of evolution in terms of $\psi(v,z)$

$$L_2 \partial_v \psi(v,z) = L_1 \psi(v,z)\,,$$

where the operators L_1 and L_2 are satisfied with

$$\begin{aligned} L_1 &= z^3 f(z)\partial_z^2 + \Big[2(\lambda+1)z^2 f(z) + z^3 f'(z)\Big]\partial_z \\ &\quad + \left[\lambda(\lambda+1)zf(z) + \lambda z^2 f'(z) - \frac{V(z)}{zf(z)}\right], \end{aligned}$$

$$L_2 = 2z\partial_z + 2\lambda\,.$$

In order to obtain QNM frequencies, we perform a Fourier transform like $\psi(v,z) \sim e^{i\omega v}\psi(z)$, then we are going to get a **generalised eigenvalue problem**

$$(L_1 - i\omega L_2)\psi(z) = 0\,.$$

In order to quantify the size of perturbations, we should define an appropriate norm $\|\cdot\|_E$, in which the energy norm is more natural. A wide class of perturbations around a black hole can be reduced into such form like

$$\eta^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi - V\varphi = 0\,,$$

Such equation comes from the action

$$S \sim -\frac{1}{2}\int \mathrm{d}^2 y\sqrt{-\eta}\Big(\eta^{\alpha\beta}\nabla_\alpha\bar{\varphi}\nabla_\beta\varphi + V\bar{\varphi}\varphi\Big)\,.$$

The energy momentum tensor reads as

$$T_{\alpha\beta} = \frac{1}{2}\nabla_\alpha\bar{\varphi}\nabla_\beta\varphi + \frac{1}{2}\nabla_\alpha\varphi\nabla_\beta\bar{\varphi} - \frac{\eta_{\alpha\beta}}{2}(\nabla_\mu\varphi\nabla^\mu\bar{\varphi} + V\varphi\bar{\varphi})\,,$$

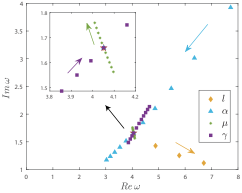
work2

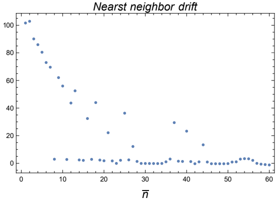
The energy norm can be written as

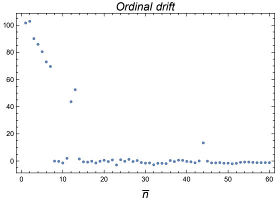
$$\begin{aligned}\|\psi\|_E^2 \equiv \langle\psi,\psi\rangle_E &= \int_{\Sigma_v} T_{\alpha\beta}\xi^\alpha\kappa^\beta \mathrm{d}\Sigma_v = \frac{1}{2}\int_{\Sigma_v}\Big(\partial_{r_\star}\bar{\varphi}\partial_{r_\star}\varphi + V\bar{\varphi}\varphi\Big)\mathrm{d}r_\star \\ &= \frac{1}{2}\int_0^1\Big[z^{2\lambda+2}f(z)(\partial_z\bar{\psi})(\partial_z\psi) \\ &\quad +\lambda z^{2\lambda+1}f(z)\psi\partial_z\bar{\psi} + \lambda z^{2\lambda+1}f(z)\bar{\psi}\partial_z\psi \\ &\quad +z^{2\lambda-2}\Big(\lambda^2 z^2 f(z) + \frac{V(z)}{f(z)}\Big)\bar{\psi}\psi\Big]\mathrm{d}z\,.\end{aligned}$$

The definition of the inner product

$$\begin{aligned}\langle\psi_1,\psi_2\rangle_E &= \frac{1}{2}\int_0^1\Big[z^{2\lambda+2}f(z)(\partial_z\bar{\psi}_1)(\partial_z\psi_2) + \lambda z^{2\lambda+1}f(z)(\psi_2\partial_z\bar{\psi}_1 + \bar{\psi}_1\partial_z\psi_2) \\ &\quad +z^{2\lambda-2}\Big(\lambda^2 z^2 f(z) + \frac{V(z)}{f(z)}\Big)\bar{\psi}_1\psi_2\Big]\mathrm{d}z\,.\end{aligned}$$







$$r_{\bar{n},\mathrm{ordinal}} = \frac{\min(|\omega_{\bar{n}}|, \sigma_{\bar{n}})}{\delta_{\bar{n},\mathrm{ordinal}}}, \quad r_{\bar{n},\mathrm{nearest}} = \frac{\min(|\omega_{\bar{n}}|, \sigma_{\bar{n}})}{\delta_{\bar{n},\mathrm{nearest}}}.$$

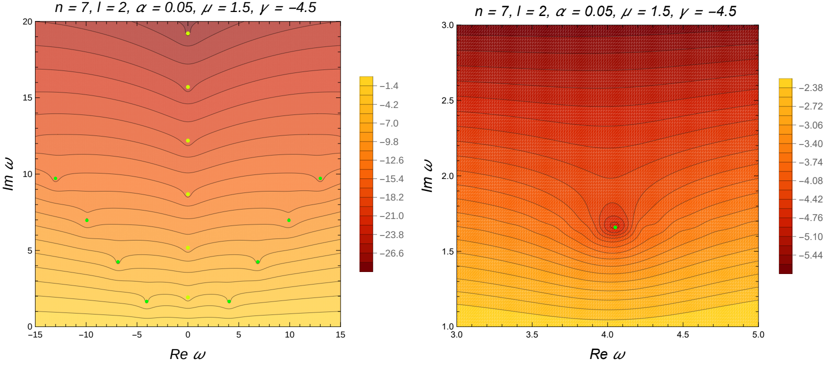


图 7: The results of spectrum and ϵ-pseudospectrum. The contour lines in both panels denote the boundary of ϵ-pseudospectrum sets given the perturbation size ϵ, whose values are $\log_{10}\epsilon$ and are shown in the legends.

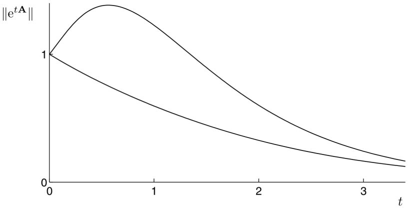
work2

- The properties of non-normal operators encompass various aspects, notably their ability to generate **the transient effect** in time-dependent dynamical systems.
- The fact is that eigenvalues are unable to accurately describe all features of the transient effect, while pseudospectra are adept at detecting and measuring the transients that eigenvalues overlook.

Consider the system $\mathrm{d}u/\mathrm{d}t = \mathbf{A}u$, where $\mathbf{A} = \mathbf{A}_1$ or $\mathbf{A} = \mathbf{A}_2$. The solution is $u(t) = \exp(\mathbf{A}t)u(0)$.

Here is a plot of the 2-norms $\|e^{t\mathbf{A}}\|$ for the two matrices

$$\mathbf{A}_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} -1 & 5 \\ 0 & -2 \end{pmatrix}.$$



Which curve is which?

Two matrices are both non-normal. In fact, the curve above corresponds to $\mathbf{A}_2$ and the curve below corresponds to $\mathbf{A}_1$.

We begin the following linear evolution system

$$\frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}v} = \mathbf{H}\boldsymbol{\Psi}\,,$$

where $\mathbf{H} = \mathbf{L}_2^{-1}\mathbf{L}_1$ is a matrix acting on the function $\boldsymbol{\Psi}(v)$. Therefore, the formal solution is

$$\boldsymbol{\Psi}(v) = e^{v\mathbf{H}}\boldsymbol{\Psi}(0)\,,$$

where $\boldsymbol{\Psi}(0)$ is the initial condition, and we call $e^{v\mathbf{H}}$ **the evolution operator**. The definition of $\|e^{v\mathbf{H}}\|_E$, i.e.,

$$G(v) \equiv \|e^{v\mathbf{H}}\|_E = \sup_{\boldsymbol{\Psi}(0)\neq\mathbf{0}} \frac{\|e^{v\mathbf{H}}\boldsymbol{\Psi}(0)\|_E}{\|\boldsymbol{\Psi}(0)\|_E} = \|e^{v\tilde{\mathbf{H}}}\|_2\,,$$

where $\tilde{\mathbf{H}} = \mathbf{W}\mathbf{H}\mathbf{W}^{-1}$. The matrix $\mathbf{W}$ is derived from the **Cholesky decomposition** of Gram matrix $\tilde{\mathbf{G}}^E$.

work2

In order to describe the dynamic behavior of the function $G(v)$, three abscissas are supposed to introduced, i.e., **spectral abscissa** $\bar{\alpha}(\mathbf{H})$, ϵ**-pseudospectral abscissa** $\bar{\alpha}_\epsilon(\mathbf{H})$ and **numerical abscissa** $\bar{\omega}(\mathbf{H})$.

The spectral abscissa $\bar{\alpha}(\mathbf{H})$ is defined by

$$\bar{\alpha}(\mathbf{H}) = \sup \mathrm{Re}(\sigma(\mathbf{H})) = \sup\{\mathrm{Re}(\lambda), \text{with } \lambda \in \sigma(\mathbf{H})\} ,$$

where $\sigma(\mathbf{H})$ represents the spectra of $\mathbf{H}$, i.e., the QNM frequencies. The lower bound of $G(v)$ is controlled by the spectral abscissa,

$$G(v) \geq e^{v\bar{\alpha}(\mathbf{H})} , \quad \forall v \geq 0 .$$

work2

In order to estimate **the maximum of the possible dynamical transient**, we are supposed to use the pseudospectral abscissa. For each $\epsilon > 0$, the ϵ-pseudospectral abscissa $\bar{\alpha}_\epsilon(\mathbf{H})$ is defined by

$$\bar{\alpha}_\epsilon(\mathbf{H}) = \sup \mathrm{Re}(\sigma_\epsilon(\mathbf{H})) = \sup\{\mathrm{Re}(\lambda), \text{with } \lambda \in \sigma_\epsilon(\mathbf{H})\} \,,$$

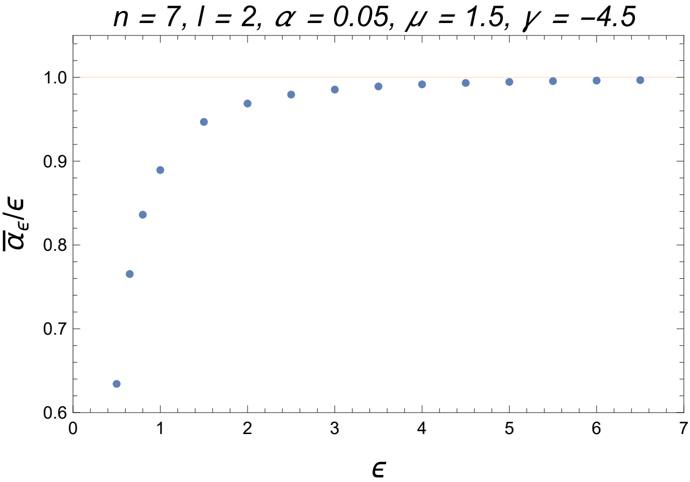
where $\sigma_\epsilon(\mathbf{H})$ is the pseudospectrum of $\mathbf{H}$. The supremum of $G(v)$ can be estimated, i.e.,

$$\sup_{v \geq 0} G(v) \geq \frac{\bar{\alpha}_\epsilon(\mathbf{H})}{\epsilon} \,, \quad \forall \epsilon > 0 \,.$$

The constant $\mathcal{K}(\mathbf{H})$ is called the Kreiss constant of $\mathbf{H}$, which is defined by $\sup_{\epsilon > 0} \bar{\alpha}_\epsilon / \epsilon$. This results in the definitive lower bound,

$$\sup_{v \geq 0} G(v) \geq \mathcal{K}(\mathbf{H}) \,.$$

work2



$n = 7,\ l = 2,\ \alpha = 0.05,\ \mu = 1.5,\ \gamma = -4.5$
1.0
0.9
0.8
0.7
0.6
$\overline{\alpha}_\epsilon/\epsilon$
0
1
2
3
4
5
6
7
ϵ

work2

The numerical abscissa $\bar{\omega}(\mathbf{H})$ can be obtained by

$$\bar{\omega}(\mathbf{H}) = \sup \sigma\left(\frac{1}{2}(\mathbf{H} + \mathbf{H}^\dagger)\right).$$

The discrete version $\mathbf{H}^\dagger$ can be obtained from

$$\mathbf{H}^\dagger = (\tilde{\mathbf{G}}^E)^{-1} \cdot \mathbf{H}^\star \cdot \tilde{\mathbf{G}}^E.$$

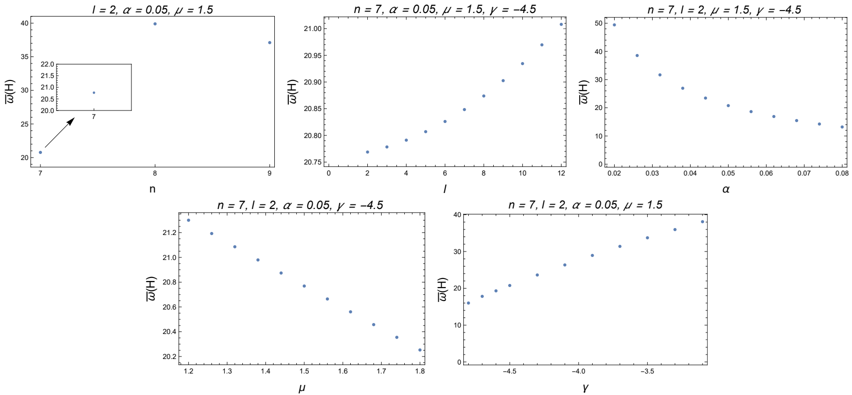
The numerical abscissa characterizes the initial slope of the norm of the evolution operator $e^{v\mathbf{H}}$, which means that

$$\bar{\omega}(\mathbf{H}) = \frac{\mathrm{d}}{\mathrm{d}v} \|e^{v\mathbf{H}}\|_E \Big|_{v=0}.$$

Therefore, if the value $\bar{\omega}(\mathbf{H}) > 0$, there will exist **an initial growth** in the evolution. The upper bound of $G(v)$ is controlled by the numerical abscissa,

$$G(v) \leq e^{v\bar{\omega}(\mathbf{H})}, \quad \forall v \geq 0.$$

work2



l = 2, α = 0.05, μ = 1.5
n = 7, α = 0.05, μ = 1.5, γ = −4.5
n = 7, l = 2, μ = 1.5, γ = −4.5
n = 7, l = 2, α = 0.05, γ = −4.5
n = 7, l = 2, α = 0.05, μ = 1.5
ϖ(H)
n
l
α
μ
γ

work2

How much external perturbation would lead to the spectrum migrating to the dynamically unstable region? Therefore, it is natural to investigate the distance to dynamical instability of $\tilde{\mathbf{H}} = \mathbf{W}\mathbf{H}\mathbf{W}^{-1}$. Its definition called **complex stability radius** is given by

$$\min\left\{\|\mathbf{E}\|_2 : \tilde{\mathbf{H}} + \mathbf{E} \text{ is unstable}\right\}.$$

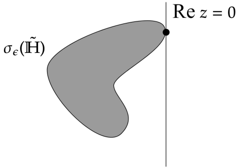
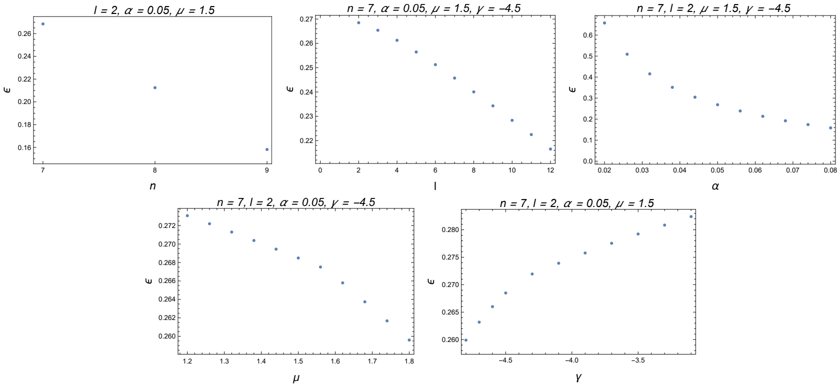


图 8: The instability threshold for $\tilde{\mathbf{H}}$ corresponds to the minimal value of ϵ at which the boundary of set $\sigma_\epsilon(\tilde{\mathbf{H}})$ intersects the imaginary axis.

work2



l = 2, α = 0.05, μ = 1.5
ϵ
n
n = 7, α = 0.05, μ = 1.5, γ = −4.5
ϵ
l
n = 7, l = 2, μ = 1.5, γ = −4.5
ϵ
α
n = 7, l = 2, α = 0.05, γ = −4.5
ϵ
μ
n = 7, l = 2, α = 0.05, μ = 1.5
ϵ
γ

work2

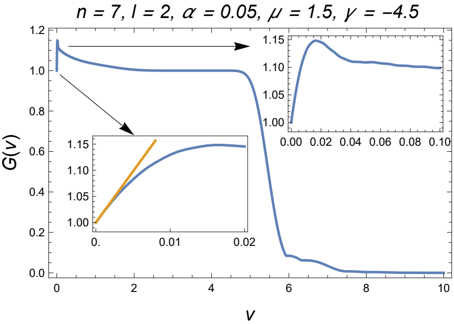


图 9: The relationship between the norm $G(v)$ of the evolution operator $e^{v\mathbf{H}}$ and the time v within the benchmark parameters.

In order to illustrate the relation between **the transient growth of energy norm** and the analysis of $G(v)$, we depict the evolution of normalised energy norm as it varies with different initial positions and initial widths of the wave packet.

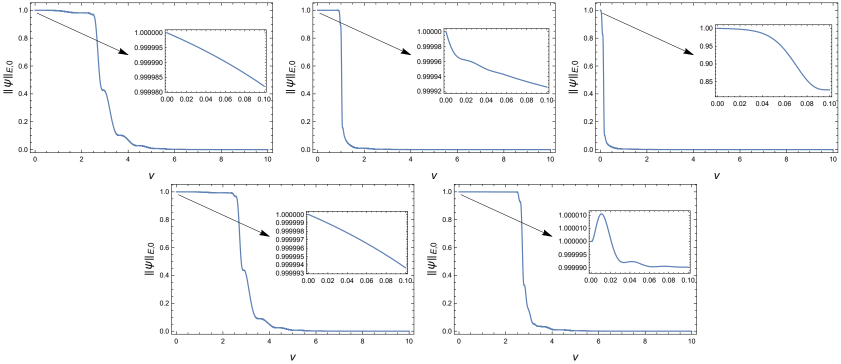
$$\|\mathbf{\Psi}(v)\|_{E,0} = \frac{\sqrt{\mathbf{\Psi}^{\star}(v) \cdot \tilde{\mathbf{G}}^{E} \cdot \mathbf{\Psi}(v)}}{\sqrt{\mathbf{\Psi}^{\star}(0) \cdot \tilde{\mathbf{G}}^{E} \cdot \mathbf{\Psi}(0)}} .$$

The initial condition of both cases $\psi(0,z)$ is set as a Gaussian wave packet

$$\psi(0,z) = h \exp\left[-\frac{(z-z_0)^2}{2\lambda^2}\right]$$

We use pseudo-spectral method to solve $\mathbf{\Psi}(v)$.

work2



work2

- The spectrum and dynamical instability, as well as the transient effect of the tensor perturbation for the so-called Maeda-Dadhich black hole.
- In terms of **spectrum instability**, the QNM spectrum and ϵ-pseudospectrum has been studied, while the open structure of ϵ-pseudospectrum caused by the non-normality of operator indicates the spectrum instability.
- In terms of **dynamical instability**, we introduce the concept of the distance to dynamical instability.
- Finally, we show the behaviour of the energy norm of the evolution operator, and find the **transient growth** of the energy norm of the evolution operator.

Summary and future work

- The flow of GWs into the BH and out into the wave zone places BH perturbation theory within the framework of non-selfadjoint operators.
- The successful application of non-selfadjoint operator theory to gravitational systems was by the hyperboloidal approach to black-hole perturbations.
- One can use the pseudospectra to study the QNM spectral instability and perform a non-modal analysis.
- QNM spectra are unstable but in time domain QNMs are stable. Greybody factors are also stable (arxiv: 2411.07734).

Summary and future work

- The wave equation with **the source term**

$$\Big(-\partial_t^2 + \partial_{r_\star}^2 - V(r_\star) \Big) u(t, r_\star) = S(t, r)\,.$$

If $S(t,r) = G(t,r)\delta(r - r_p) + F(t,r)\delta'(r - r_p)$, one can study the EMRIs and self-force.
If $S(t,r)$ comes from second order perturbation, one can study the quadratic QNMs (QQNMs).

Summary and future work

- The pseudospectrum of the Kerr black hole. Teukolsky equation is written as a hyperbolic form associated with the boundary conditions of QNMs

$$\partial_\tau u_m = iLu_m\,, \quad L = \frac{1}{i}\begin{bmatrix} 0 & 1 \\ L_1 & L_2 \end{bmatrix}\,,$$

$$\begin{aligned} L_1 &= {}^{[x]}L_1^2(\sigma,x)\frac{\partial^2}{\partial x^2} + {}^{[\sigma]}L_1^2(\sigma,x)\frac{\partial^2}{\partial \sigma^2} \\ &\quad + {}^{[x]}L_1^1(\sigma,x)\frac{\partial}{\partial x} + {}^{[\sigma]}L_1^1(\sigma,x)\frac{\partial}{\partial \sigma} + L_1^0(\sigma,x)\,, \\ L_2 &= {}^{[\sigma]}L_2^1(\sigma,x)\frac{\partial}{\partial \sigma} + L_2^0(\sigma,x)\,. \end{aligned}$$

Two dimensional pseudo-spectral method to solve u_m. The spectrum of the Kerr black hole is the eigenvalues of the operator L. The pseudospectrum of Kerr black hole is based on such operator L (in preparation).

Thank you!