中国科学院理论物理研究所
Institute of Theoretical Physics, Chinese Academy of Sciences

“2024 引力与宇宙学”专题研讨会

Thermodynamics of dyonic black holes in minimal supergravity

Jun-Kun Zhao

ITP, CAS

Work with Prof. Rong-Gen Cai and Prof. Li Li, arXiv:2410.00717

@2024-11-16

Content

The sacred status of thermodynamics:

- *Einstein's view*
  "[Thermodynamics is] the only physical theory of universal content concerning which I am convinced that, within the framework of the applicability of its basic concepts, it will never be overthrown."

- *Eddington*
  "The law that entropy always increases holds, I think, the supreme position among the laws of nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations-then so much the worse for Maxwell's equations. If it is found to be contradicted by observation-well, these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation."

Quantum statistical relation and first law

- The black hole temperature and Bekenstein-Hawking entropy

$$T = \frac{\hbar\kappa}{2\pi}, \quad S_{BH} = \frac{k_B c^3}{4G\hbar} A, \tag{1}$$

- Euclidean path integral [Gibbons and Hawking, 1977]

Quantum statistical relation

$$W = -T \log Z = T S_E \tag{2}$$

1. The Bekenstein-Hawking entropy can be derived in a semiclassical analysis of the gravitational path integral. This QSR is a standard framework for discussing the thermodynamics of black holes in textbooks.

2. Extreme black holes have a large zero-temperature entropy, which violates the Nernst's third law of thermodynamics. Quantum-corrected near-extremal entropy [Turiaci, 2023]

$$\frac{S(T)}{k_B} \approx \frac{A_{ext}}{4 l_P^2} + \frac{4\pi^2 T}{T_{break}} + \frac{3}{2} \ln\left(\frac{T}{T_{break}} \right) + \cdots, \tag{3}$$

- Significant applications:

  1. Hawking-Page transition [Hawking, Page, 1983]
  2. Liquid–gas phase transition [Chamblin, Emparan, Johnson and Myers, 1999; Kubiznak and Mann, 2012]
  3. Important for applying AdS-gravity to strongly coupled QPT
  4. Extended black hole thermodynamics

$$\delta M = T\delta S + V\delta P + \Phi\delta Q + \Omega\delta J\,, \tag{4}$$

$$M = \frac{d-2}{d-3}(TS + \Omega J) + \Phi Q - \frac{2}{d-3}PV\,, \tag{5}$$

- The standard thermodynamic first law [Landau, Lifshitz and Pitaevskii, 1995]

$$\begin{aligned} \delta w &= -s\delta T - \rho\delta\mu - M_B\delta B\,, \\ \delta\epsilon &= T\delta s + \mu\delta\rho - M_B\delta B\,, \end{aligned} \tag{6}$$

where $w = W/V$ and $\epsilon = w + Ts + \mu\rho$.

Dyonic black branes

- The Einstein-Maxwell-Chern-Simons theory [D'Hoker and Kraus, 2012]

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\Big(R+\frac{12}{L^2}-\frac{1}{4}F_{ab}F^{ab}-\frac{k}{24}\epsilon^{abcde}A_aF_{bc}F_{de}\Big), \qquad (7)$$

- $A\wedge F\wedge F$: dual to boundary QPT with chiral anomalous $\partial_\mu J^\mu \propto \kappa \mathbf{E}\cdot\mathbf{B}$.

- Top-down model: $k = k_{susy} = \frac{2}{\sqrt{3}}$, a consistent truncation of Type IIB supergravity or M-theory [Buchel and Liu 2007; Gauntlett, Colgain and Varela 2007].

- The background ansatz

$$ds^2 = \frac{1}{r^2}\Big[-(fe^{-\chi}-h^2p^2)dt^2+2ph^2dtdz+dx^2+dy^2+h^2dz^2+\frac{dr^2}{f}\Big], \qquad (8)$$

$$A = A_t dt - \frac{B}{2}ydx + \frac{B}{2}xdy - A_z dz\,, \qquad (9)$$

IR and UV expansions

- At asymptotically AdS_5, one has the the following asymptotic expansion as $r \to 0$,

$$
\begin{aligned}
f(r) &= 1 + \cdots + f_4 r^4 + \cdots , \\
\chi(r) &= \chi_0 + \cdots , \\
h(r) &= 1 + \cdots + h_4 r^4 + \cdots , \\
p(r) &= p_4 r^4 + \cdots , \\
A_t(r) &= e^{-\chi_0/2}\left(\mu - \frac{\rho}{2} r^2 + \cdots\right) , \\
A_z(r) &= A_{z2} r^2 + \cdots .
\end{aligned}
\tag{10}
$$

- The temperature and ekenstein-Hawking entropy density are given by

$$
T = -\frac{e^{\chi_0/2}}{4\pi} f' e^{-\chi/2}\Big|_{r=r_h} , \quad s = \frac{4\pi h}{r^3}\Big|_{r=r_h} . \tag{11}
$$

- Using the quantum statistical relation, the free energy can be obtained from the on-shell Euclidean action, defined as $I_E = I + I_{bdy}$.
  The expectation values of the boundary stress-energy tensor $\langle T_{\mu\nu}\rangle$ and current $\langle J^{\mu}\rangle$ can be obtained from holographic renormalization.

- The free energy density $w = W/V$ can be expressed in three equivalent forms

$$
\begin{aligned}
w &= \epsilon - Ts - \mu\rho - \frac{kB}{3}\int_0^{r_h} A_t A_z' dr\,, & (12)\\
&= -\mathscr{P}_\perp + B\int_0^{r_h}\Big[\frac{B}{r}\Big(e^{-\frac{\chi}{2}}h - 1\Big) + \frac{2k}{3}A_t A_z'\Big]dr + B^2 \ln r_h\,, & (13)\\
&= -\mathscr{P}_\parallel + \frac{kB}{3}\int_0^{r_h} A_t' A_z dr\,, & (14)
\end{aligned}
$$

- Two Smarr-type relations:

$$
\begin{aligned}
\epsilon + \mathscr{P}_\parallel &= Ts + \mu\rho\,, & (15)\\
\mathscr{P}_\parallel &= \mathscr{P}_\perp - \Big(\int_0^{r_h}\Big[\frac{B}{r}\Big(e^{-\frac{\chi}{2}}h - 1\Big) + kA_t A_z'\Big]dr + B\ln r_h\Big)B\,. & (16)
\end{aligned}
$$

- The free energy w in (12) includes a nontrivial bulk integration term, signaling a violation of the standard law of thermodynamics (6).

Hydrodynamics with background field and anomalous

- The hydrodynamic equations for $T^{\mu\nu}$ and J^{μ} are [1701.05565]

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \,, \quad \nabla_\mu J^\mu = \frac{k}{8} E^\mu B_\mu \,, \tag{17}$$

where $E^\mu = F^{\mu\nu} u_\nu$ and $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$ are the electric and magnetic fields.

- The stress-energy tensor $T^{\mu\nu}$ and J_μ can be expressed though constitutive relations in terms of hydrodynamic variables $T(x)$, $\mu(x)$ and $u^\mu(x)$.
To leading order in the derivative expansion:

$$\langle T^{\mu\nu}_{EFT} \rangle = \epsilon_0 u^\mu u^\nu + P_0 \Delta^{\mu\nu} + 2q^{(\mu} u^{\nu)} + M^{\mu\alpha} g_{\alpha\beta} F^{\beta\nu} + u^\mu u^\alpha \left(M_{\alpha\beta} F^{\beta\nu} - F_{\alpha\beta} M^{\beta\nu} \right) + \cdots \,,$$
$$\langle J^{\mu}_{EFT} \rangle = n_0 u^\mu + \xi_B B^\mu + \cdots \,,$$

where $q^\mu = \xi_V B^\mu$ and $M^{\mu\nu} = \chi_{BB} \epsilon^{\mu\nu\alpha\beta} B_\alpha u_\beta$.

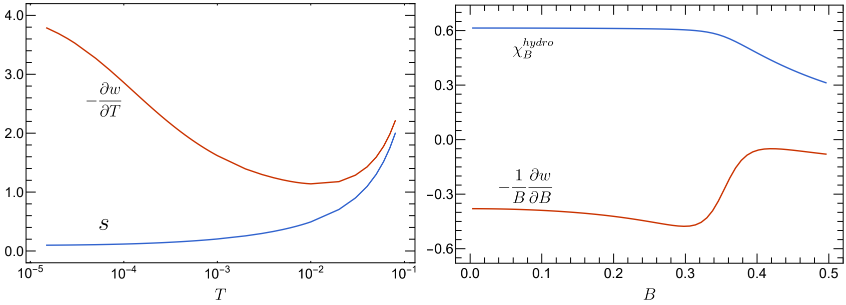
- In thermal equilibrium, we can choose $u^\mu = (1,0,0,0), B^\mu \propto \vec{z}$ and then obtain the constitutive relations for $T^{\mu\nu}, J^\mu$. By comparing with the holographic results, the magnetic susceptibility reads

$$\chi_B^{Hydro} = \frac{1}{B^2} \left(\langle T^{zz}_{EFT} \rangle - \langle T^{xx}_{EFT} \rangle \right) = \frac{1}{4} + \frac{8h_4}{B^2} \,, \tag{18}$$

Violation of thermodynamics first law

A direct calculation of the entropy and magnetic susceptibility $\chi_B \equiv M_B/B$ from the free energy shows

$$-\left(\frac{\partial w}{\partial T}\right)_{B,\mu} \neq s, \qquad -\frac{1}{B}\left(\frac{\partial w}{\partial B}\right)_{T,\mu} \neq \chi_B^{hydro}, \tag{19}$$



Test of first law for dyonic black holes obtained by the QSR (Left $B = 0.33$, Right $T = 0.005$; All plots are for minimal supergravity with $k = k_{\text{susy}} = \frac{2}{\sqrt{3}}$ and $\mu = 1$)

Euclidean action

- The Euclidean action is given by

$$I_E = I + I_{bdy}\,, \tag{20}$$

where $I = -iS = -iS_{eff}$ and I_{bdy}. The effective action of the system

$$\begin{aligned} S_{eff} = \int d^5x \frac{e^{-\chi/2}h}{r^3}\Big\{ & -f'' - \frac{2fh''}{h} + f\chi'' + f'\left(\frac{8}{r} - \frac{2h'}{h} + \frac{3\chi'}{2}\right) \\ & + \left(\frac{h'}{rh} - \frac{\chi'}{2r}\right)\left(8f + rf\chi'\right) + \frac{1}{2}e^{\chi}\left[h^2p'^2 + r^2(A_t' + pA_z')^2\right] \\ & + \frac{12-20f}{r^2} - \frac{r^2}{2}\left(B^2 + \frac{fA_z'^2}{h^2}\right)\Big\} + \int d^5x \frac{kB}{3}\left(A_t'A_z - A_tA_z'\right). \end{aligned} \tag{21}$$

- Consequently, the variation of the total on-shell Euclidean action can be separated into two parts:

$$\delta I_E = \delta I + \delta I_{bdy}\,, \tag{22}$$

- In Euclidean signature, the black hole horizon is smooth when the Euclidean time $\tau = it$ is periodic, with a period given by $\Delta\tau = 4\pi/(f' e^{-\chi/2})|_{r=r_h}$. Consequently, the only remaining boundary is the AdS boundary.

  The variation of the total on-shell Euclidean action is given by

$$\delta I_E = \Delta\tau V \left\{ e^{-\frac{\chi_0}{2}} \left[-\rho\delta\mu - (e^{\chi_0}\epsilon - \mu\rho)\frac{\delta\chi_0}{2} \right] - k\delta Q_{cs} - e^{-\chi_0/2} M_B \delta B \right\}, \qquad (23)$$

- The quantity Q_{cs}, arising from the Chern-Simons term of (7), is given by

$$Q_{cs} = \frac{B}{6}\int_0^{r_h} (A_z' A_t - A_z A_t')dr\,, \qquad (24)$$

  and M_B is defined as

$$M_B = -\left(\int_0^{r_h} \left[\frac{B}{r}\left(e^{-\chi/2}h - 1\right) + \frac{k}{2}(A_z' A_t - A_z A_t') \right] dr + B\ln r_h \right). \qquad (25)$$

On-shell variation

- Note that $T = \frac{e^{\chi_0/2}}{\Delta\tau}$, thus the variation of χ_0 gives $\delta\chi_0 = 2\frac{\delta T}{T}$ with $\Delta\tau$ held fixed [Donos and Gauntlett 2013]. Using QSR, we obtain

$$\delta w = -\left(s + e^{\frac{\chi_0}{2}} \frac{kQ_{cs}}{T}\right)\delta T - \rho\delta\mu - e^{\frac{\chi_0}{2}} k\delta Q_{cs} - M_B \delta B\,. \tag{26}$$

  This indicates the deviation from the standard form of the first law is attributed to the bulk integration Q_{cs}.

- However, after the Legendre transformation $\tilde{w} = w + e^{\chi_0/2} kQ_{cs}$, we arrive at the thermodynamic ensemble that gives the standard thermodynamic relation

$$\tilde{w} = \epsilon - Ts - \mu\langle J^t\rangle\,, \tag{27}$$

$$\delta w = -s\delta T - \rho\delta\mu - M_B\delta B\,, \tag{28}$$

  where M_B is the magnetization of the system.

- The results (27-28) can be definitively proven by the Iyer-Wald formalism!

Iyer-Wald formalism

- A powerful framework for studying black hole thermodynamics:
  1. Extended black hole thermodynamics [Xiao, Tian, and Liu 2024]
  2. Thermodynamics of black holes with scalar hair [Li 2021]
- The general variation of the Lagrangian n-form $\mathbf{L} = \mathscr{L}\boldsymbol{\epsilon}$ can be expressed as

$$\delta \mathbf{L} = \mathbf{E}(\phi)\delta\phi + d\boldsymbol{\Theta}, \tag{29}$$

  where $\phi = (g_{ab}, A_a)$ is the dynamical field and $\boldsymbol{\Theta}$ is the symplectic potential form.
- Considering an infinitesimal diffeomorphism variation $\delta_\xi x^a = \xi^a(x)$, one can associate to ξ^a a Noether current $(n-1)$ form, defined by

$$\mathbf{J} \equiv \boldsymbol{\Theta}(\phi, \mathscr{L}_\xi \phi) - \xi \cdot \mathbf{L}, \tag{30}$$

  Then, we have $d\mathbf{J} = -\mathbf{E}\mathscr{L}_\xi\phi$. There exists a Noether charge $(n-2)$ form $\mathbf{Q}$ locally constructed from ϕ and ξ^a satisfying $\mathbf{J} = d\mathbf{Q}$ once $\mathbf{E}(\phi)$.
- The variation of $\mathbf{J}$ gives the fundamental identity

$$\boldsymbol{\omega}(\delta\phi, \mathscr{L}_\xi\phi) = \delta\mathbf{J} - d(\xi\cdot\boldsymbol{\Theta}) = d\,(\delta\mathbf{Q} - \xi\cdot\boldsymbol{\Theta}) = 0, \tag{31}$$

  which holds if $\mathbf{E}(\phi) = 0$, $\mathscr{L}_\xi\phi = 0$ and $\mathbf{E}(\delta\phi) = 0$.

- Choose ξ^a to be the time-like Killing vector $\xi^a = (\partial_t)^a = \delta^a_t$ and Σ be a t =constant space-like hypersurface. Thus

$$\partial\Sigma = S_{r=r_h} \cup S_{r=0} \cup S_{x=L/2} \cup S_{x=-L/2} \cup S_{y=L/2} \cup S_{y=-L/2} \cup S_{z=L/2} \cup S_{z=-L/2}$$

- Integrating (31) over the hypersurface Σ gives

$$\delta\epsilon = T\delta s + \mu\delta\rho - M_B \delta B \tag{32}$$

where the expression for M_B is the same as (25).

- In addition to the fundamental identity, we have

$$d\mathbf{Q} = -\xi \cdot \mathbf{L}, \tag{33}$$

when $\mathbf{E}(\phi)$ and $\mathscr{L}_\xi \phi = 0$. Integrating over the hypersurface Σ, we can obtain

$$\tilde{w} = \epsilon - Ts - \mu\rho = w + kQ_{cs}, \tag{34}$$

Therefore, combining (34) and (32), we are arriving at

$$\delta\tilde{w} = -s\delta T - \rho\delta\mu - M_B \delta B, \tag{35}$$

- Exactly the same results from the on-shell variation of the Euclidean action.

- Under the scale transformation $r \to \lambda \hat{r}$ with λ a positive constant, we have

$$s = \lambda^3 \hat{s}, \quad \rho = \lambda^3 \hat{\rho}, \quad B = \lambda^2 \hat{B}, \tag{36}$$

while the energy density ϵ acquires an anomalous scaling

$$\epsilon = \lambda^4 \hat{\epsilon} - \frac{\hat{B}^2}{2} \lambda^4 \ln \lambda . \tag{37}$$

- Thus, when we express ϵ as a function of s, ρ and B, we have

$$\epsilon(\lambda^3 \hat{s}, \lambda^3 \hat{\rho}, \lambda^2 \hat{B}) = \lambda^4 \hat{\epsilon}(\hat{s}, \hat{\rho}, \hat{B}) - \frac{\hat{B}^2}{2} \lambda^4 \ln \lambda . \tag{38}$$

Taking derivative with respect to λ

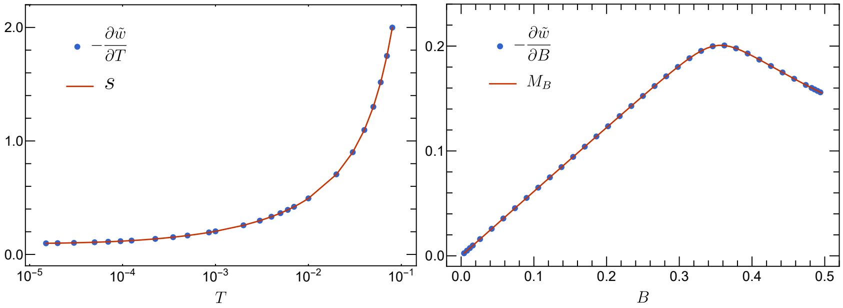
$$\left(\frac{\partial \epsilon}{\partial s}\right)_{\rho,B} (3s) + \left(\frac{\partial \epsilon}{\partial \rho}\right)_{s,B} (3\rho) + \left(\frac{\partial \epsilon}{\partial B}\right)_{s,\rho} (2B) = 4\epsilon - \frac{B^2}{2} . \tag{39}$$

Taking advantage of the first law (32), the (generalized) Smarr relation reads

$$-4\epsilon + 3(Ts + \mu\rho) - 2M_B B = -\frac{B^2}{2} . \tag{40}$$

- The right hand side indicates a derivation from the standard Smarr formula due to the chiral anomaly. According to (34), this Smarr relation effectively expresses the trace anomaly of $\langle T_{\mu\nu} \rangle$ *i.e.* $T^{\mu}_{\ \mu} = -B^2/2$.

Numerical verification



Numerical verification of the thermodynamics of dyonic black holes in minimal supergravity: $\delta\tilde{w} = -s\delta T - \rho\delta\mu - M_B\delta B$. (Left $B = 0.33$, Right $T = 0.005$; All plots are for minimal supergravity with $k = k_{\text{susy}} = \frac{2}{\sqrt{3}}$ and $\mu = 1$)

Summary and discussion

- We have found that the textbook results for the first law of dyonic black hole in 5D EMCS break down when applying the quantum statistical relation.
- Using the on-shell variation method and the Iyer-Wald formalism, we have resolved this issue and established the standard first law of thermodynamics, which agrees with field theory and hydrodynamics, and has been validated through numerical tests.
- The free energy should be $\tilde{w} = w + kQ_{cs}$, rather than w derived from QSR.

1. A deeper understanding of BH thermodynamics is necessary, especially for $B \neq 0$. Important for studying the magnetocaloric effect and magnetic field driven QPT.
2. The methods employed here are applicable to any geometric theory of gravity. Similar results may arise in other systems, e.g. [D'Hoker and Kraus 2012]].
3. A microscopic interpretation for our findings from both the supergravity perspective and the dual boundary QFT?