Partition function zeros of the black hole thermodynamics

Zhen-Ming Xu (许震明)

西北大学

11-16, 2024 彭桓武高能基础理论研究中心（合肥）

Contents

The 4 Laws of Black Hole Mechanics

- **0th Law** $\kappa = \text{constant}$ — Bardeen/Carter/Hawking CMP **31** (1973) 161

  -- surface Gravity is constant over the event horizon

- **1st Law** $dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ + \cdots$

  -- differences in mass between nearby solutions are equal to differences in area times the surface gravity plus additional work terms

- **2nd Law** $dA \geq 0$ — Bekenstein PRD **7** (1973) 2333

  -- area of the event horizon never decreases in any physical process

- **3rd Law** $\kappa_n > \kappa_{n+1} > 0, n < \infty$ — Israel PRL 57 (1986) 397

  -- No procedure can reduce the surface gravity to 0 in a finite number of steps

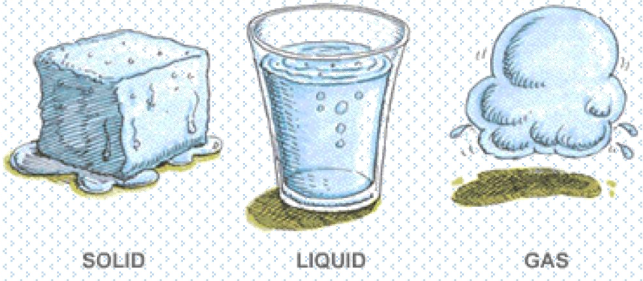
传统黑洞热力学扩展相空间热力学全息扩展相空间热力学约束相空间热力学

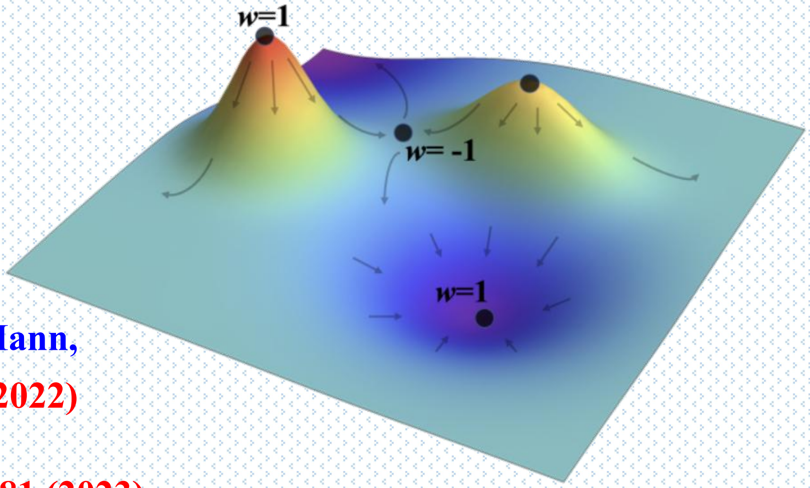
框架 / 内容	传统黑洞热力学	扩展相空间热力学	全息扩展相空间热力学	约束相空间热力学
黑洞质量的热力学对应	内能	焓	内能	内能
第一定律	$dM = TdS + \Phi dQ$	$dM = TdS + \Phi dQ + VdP$	$dM = TdS + \hat{\Phi} d\hat{Q} - \tilde{P} d\tilde{V} + \mu dC$	$dM = TdS + \hat{\Phi} d\hat{Q} + \mu dN$
质量关系	$M = \frac{D-2}{D-3} TS + \Phi Q + \frac{\Lambda \Phi S^2}{\pi^2 (D-1) Q}$	$M = \frac{D-2}{D-3} TS + \Phi Q - \frac{2}{D-3} PV$	$M = TS + \hat{\Phi}\hat{Q} + \mu C$	$M = TS + \hat{\Phi}\hat{Q} + \mu N$
聚焦点	各种黑洞物理量的计算以它们之间满足的等式关系	相变和临界性行为	Euler齐次性问题以及临界性	与第一定律匹配的Euler关系以及相变

Everyday AdS Black Hole Thermodynamics

- Hawking Page Transition — **Hawking, Page, CMP 87, 577 (1983)**
- Van der Waals Fluid, Solid/Liquid/Gas — **Kubiznak, Mann, JHEP 07, 033 (2012)**
- Holographic Heat Engines — **Johnson, CQG 31, 205002 (2014)**
- Thermodynamics geometry — **Ruppeiner, RMP 67, 605 (1995)**
- Free Energy Landscape — **R. Li and J. Wang, PRD 102, 024085 (2020)**
- Black hole topological thermodynamics — **S.-W. Wei, Y.-X. Liu and R. B. Mann, PRL 129, 191101 (2022)**
- Relativistic Stochastic Mechanics — **L. Zhao, et.al., J. Statist. Phys. 190, 193/181 (2023)**
- Extended Iyer-Wald Formalism — **Y. Xiao, Y. Tian, and Y.-X. Liu, PRL 132, 021401 (2024)**

……





Hawking radiation and dynamics the curved spacetime

Scheme

- Mapping the Dirac field to a one-dimensional XY model
- Analog black hole

nature communications

Article https://doi.org/10.1038/s41467-023-39064-6

Quantum simulation of Hawking radiation and curved spacetime with a superconducting on-chip black hole

Received: 17 April 2022

Accepted: 26 May 2023

Published online: 05 June 2023

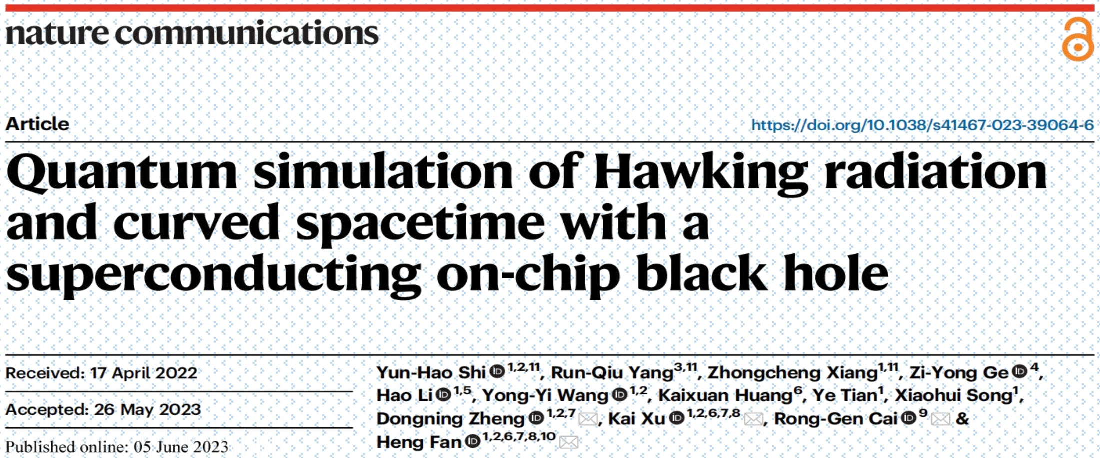
Yun-Hao Shi[1,2,11], Run-Qiu Yang[3,11], Zhongcheng Xiang[1,11], Zi-Yong Ge[4], Hao Li[1,5], Yong-Yi Wang[1,2], Kaixuan Huang[6], Ye Tian[1], Xiaohui Song[1], Dongning Zheng[1,2,7], Kai Xu[1,2,6,7,8], Rong-Gen Cai[9] & Heng Fan[1,2,6,7,8,10]

Testing 01

Superconducting quantum simulation → **Hawking radiation and dynamics the curved spacetime**

Scheme

- **Mapping the Dirac field to a one-dimensional XY model**
- **Analog black hole**


nature communications

Article https://doi.org/10.1038/s41467-023-39064-6

Quantum simulation of Hawking radiation and curved spacetime with a superconducting on-chip black hole

Received: 17 April 2022

Accepted: 26 May 2023

Published online: 05 June 2023

Yun-Hao Shi[1,2,11], Run-Qiu Yang[3,11], Zhongcheng Xiang[1,11], Zi-Yong Ge[4], Hao Li[1,5], Yong-Yi Wang[1,2], Kaixuan Huang[6], Ye Tian[1], Xiaohui Song[1], Dongning Zheng[1,2,7], Kai Xu[1,2,6,7,8], Rong-Gen Cai[9] & Heng Fan[1,2,6,7,8,10]

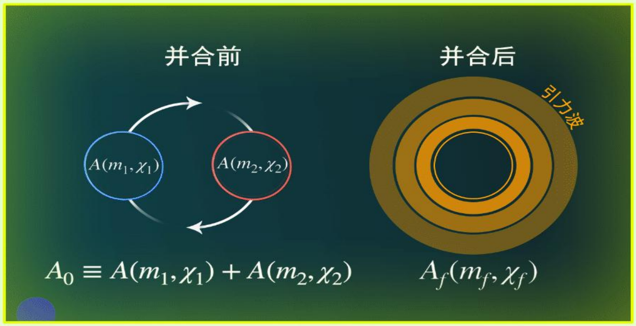
Testing 02

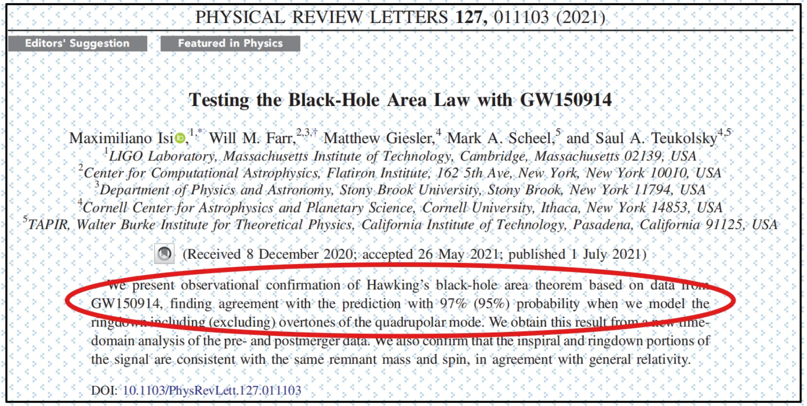
Gravitational wave test → **Hawking black-hole area theorem**

Method.—The horizon area A of a Kerr BH with mass M and spin angular momentum $\vec{J}$ is

$$A(M,\chi) = 8\pi\left(\frac{GM}{c^2}\right)^2\left(1+\sqrt{1-\chi^2}\right),$$

$$\chi \equiv |\vec{J}|c/(GM^2)$$




PHYSICAL REVIEW LETTERS **127,** 011103 (2021)

Editors' Suggestion Featured in Physics

Testing the Black-Hole Area Law with GW150914

Maximiliano Isi,[1,*] Will M. Farr,[2,3,†] Matthew Giesler,[4] Mark A. Scheel,[5] and Saul A. Teukolsky[4,5]

[1]*LIGO Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2]*Center for Computational Astrophysics, Flatiron Institute, 162 5th Ave, New York, New York 10010, USA*
[3]*Department of Physics and Astronomy, Stony Brook University, Stony Brook, New York 11794, USA*
[4]*Cornell Center for Astrophysics and Planetary Science, Cornell University, Ithaca, New York 14853, USA*
[5]*TAPIR, Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, California 91125, USA*

(Received 8 December 2020; accepted 26 May 2021; published 1 July 2021)

We present observational confirmation of Hawking's black-hole area theorem based on data from GW150914, finding agreement with the prediction with 97% (95%) probability when we model the ringdown including (excluding) overtones of the quadrupolar mode. We obtain this result from a new time-domain analysis of the pre- and postmerger data. We also confirm that the inspiral and ringdown portions of the signal are consistent with the same remnant mass and spin, in agreement with general relativity.

DOI: 10.1103/PhysRevLett.127.011103

Landau functional

Landau approximate the free energy of a system

it exhibits the non-analyticity of a phase transition and turns out to capture much of the physics

Landau believed that the order parameter *m* near the critical point T_c is a small amount; thus the free energy function *F(T,m)* can be expanded to the power of *m* near T_c *(second-order phase transition)*

$$F = a(T) + \frac{1}{2}b(T)\mathfrak{m}^2 + \frac{1}{4}c(T)\mathfrak{m}^4 - \mathscr{B}\mathfrak{m} + \cdots .$$ （伊辛模型）

$$a(T) = a_0 + a_1(T - T_c) + \cdots ,$$
$$b(T) = b_0(T - T_c) + \cdots ,$$
$$c(T) = c_0 + c_1(T - T_c) + \cdots .$$

For black holes: **X.-Y. Guo, H.-F. Li, L.-C. Zhang and R. Zhao, PRD 100, 064036 (2019).**
X.-P. Li, Y.-B. Ma, Y. Zhang, L.-C. Zhang, and H.-F. Li, CJP 83, 123 (2023) .

Landau functional Free energy landscape

4D Schwarzschild-AdS BH

$M = \frac{r_+}{2}\left(1 + \frac{r_+^2}{L^2}\right)$ $S = \pi r_+^2$. $T_H = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{L^2}\right)$

Gibbs Free Energy $G = M - T_H S = \frac{r_+}{2}\left(1 + \frac{r_+^2}{L^2}\right) - \frac{r_+}{4}\left(1 + \frac{3r_+^2}{L^2}\right)$

- On-shell Gibbs free energy: $G = M - T_H S$ or calculated directly from the Euclidean action
- Off-shell Gibbs free energy: **replacing the Hawking temperature T_H with the ensemble temperature T**

Free energy landscape $G = M - TS = \frac{r_+}{2}\left(1 + \frac{r_+^2}{L^2}\right) - \pi T r_+^2$

R. Li and J. Wang, PRD 102, 024085 (2020)

Landau functional Thermal potential

Equation of state ⟷ Equation of motion

Thermodynamics — Classical Mechanics

Provided

The process of a system from an unknown state to an equilibrium state:

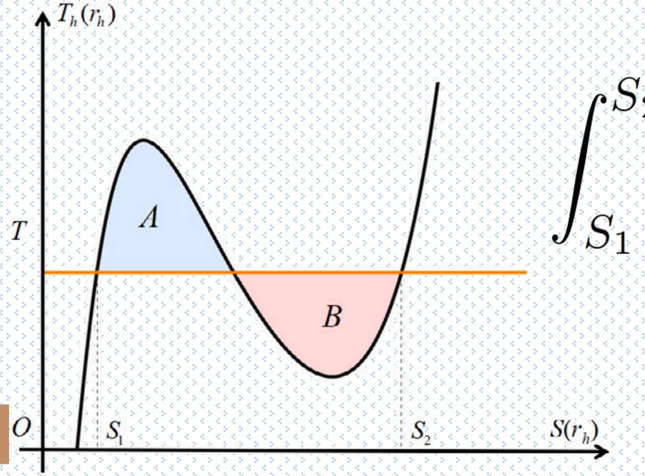
selecting a relation (equation of state in equilibrium) from all possible relations

$$L = \int F(X, T, P) \mathrm{d}X$$

$$F(X, T, P) \equiv P - f(X, T)$$

Z.-M. Xu, B. Wu and W.-L. Yang, CQG (2021)

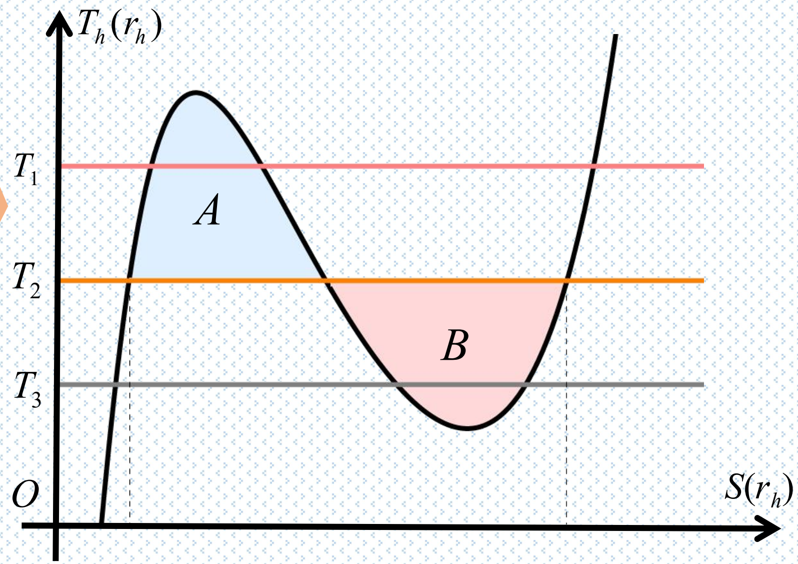
Landau functional **Generalized free energy**



PLB 2024
PRD 2021

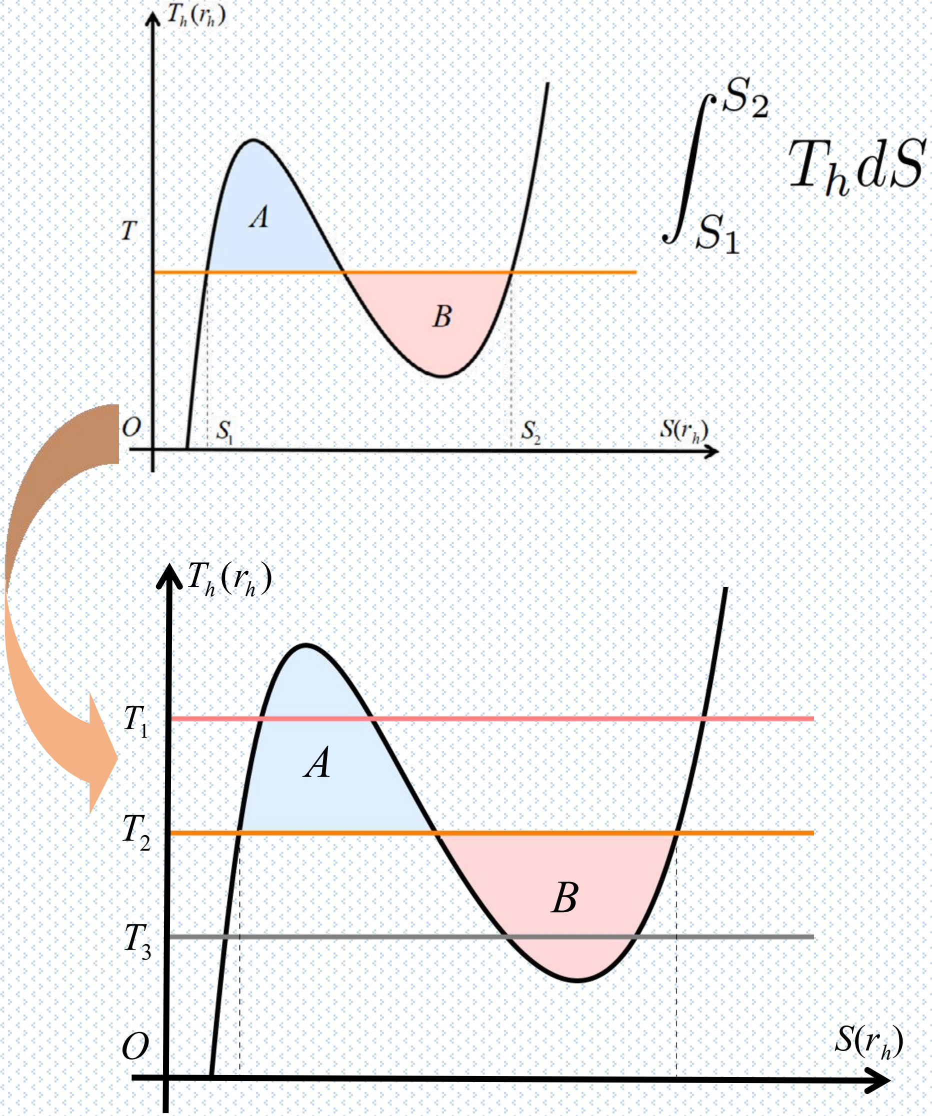
$$\int_{S_1}^{S_2} T_h dS = T \cdot (S_2 - S_1)$$

$$\Rightarrow \quad \int_{S_1}^{S_2} (T - T_h) dS = 0.$$



$$U^{(T)} = \int (T_h - T) dS,$$

Landau functional Generalized free energy

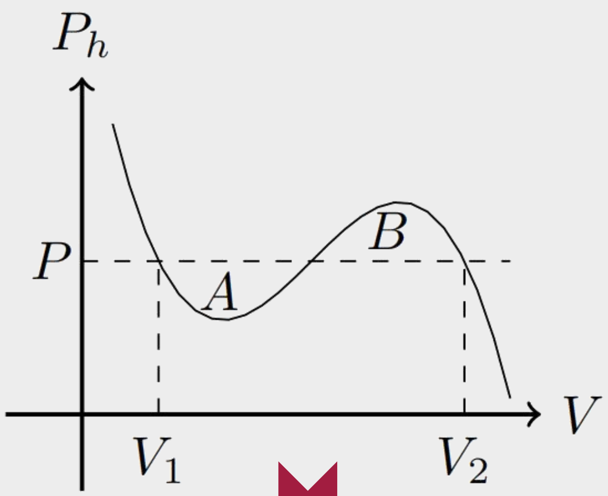


PLB 2024
PRD 2021

$$\int_{S_1}^{S_2} T_h dS = T \cdot (S_2 - S_1)$$

$$\Rightarrow \quad \int_{S_1}^{S_2} (T - T_h) dS = 0.$$

$$U^{(T)} = \int (T_h - T) dS,$$



$$U^{(P)} = \int (P_h - P) dV,$$

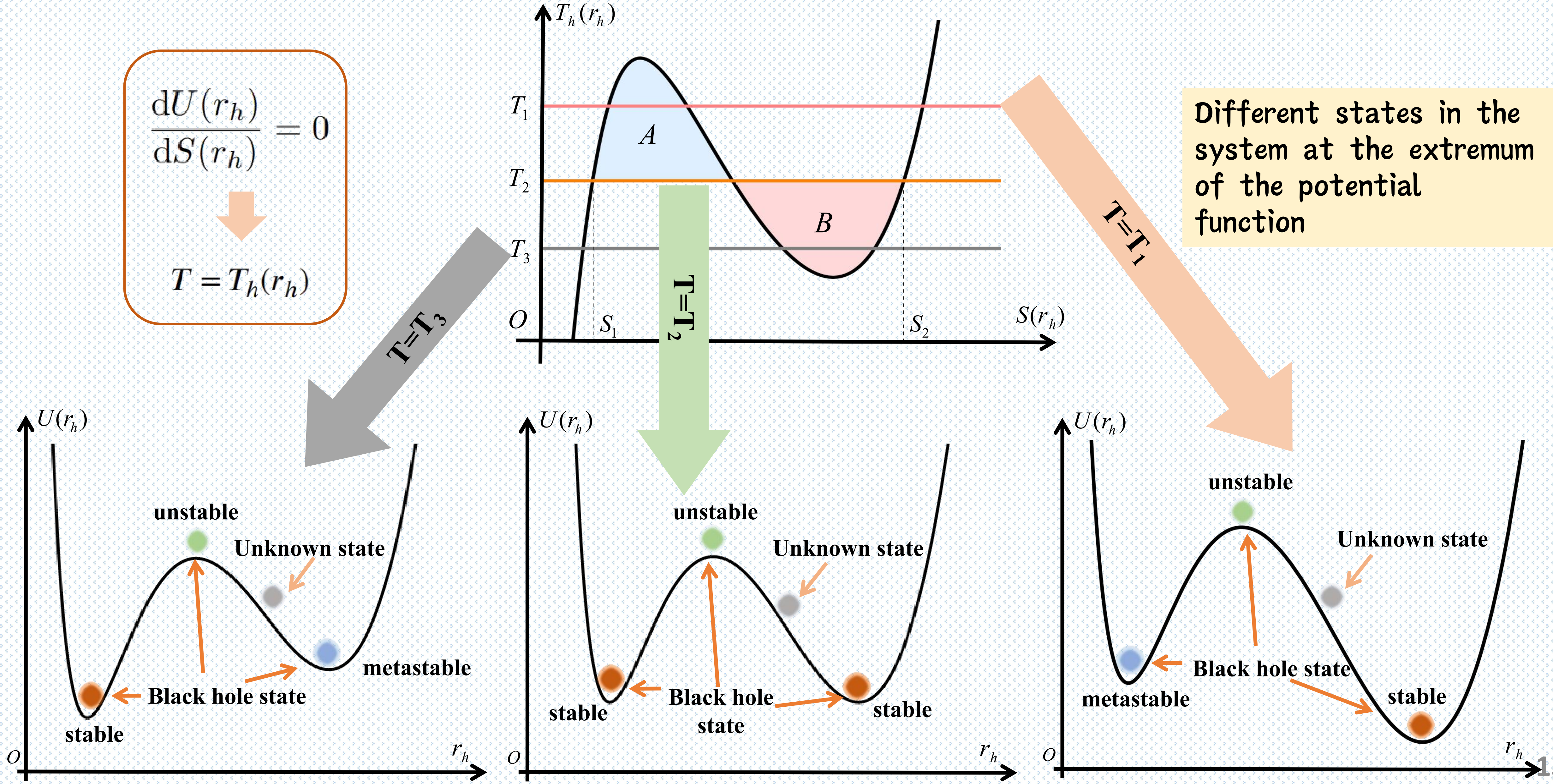
Generalized free energy

$$U(r_h) = \int (T_h(r_h) - T)\mathrm{d}S(r_h).$$

$$\frac{\mathrm{d}U(r_h)}{\mathrm{d}S(r_h)} = 0$$

$$T = T_h(r_h)$$

Different states in the system at the extremum of the potential function

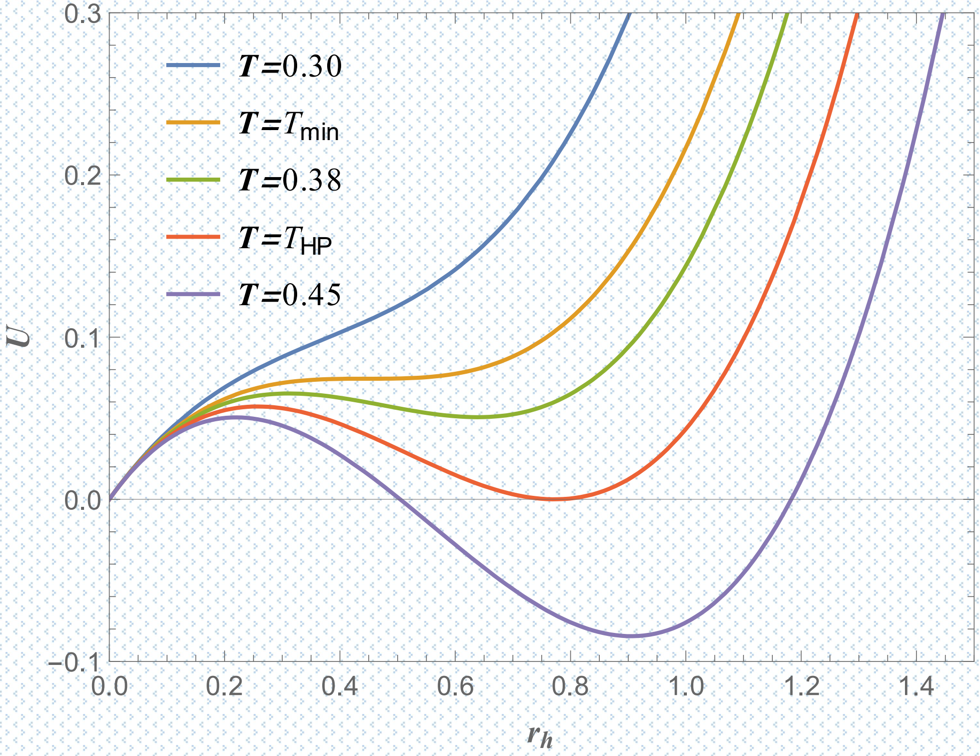


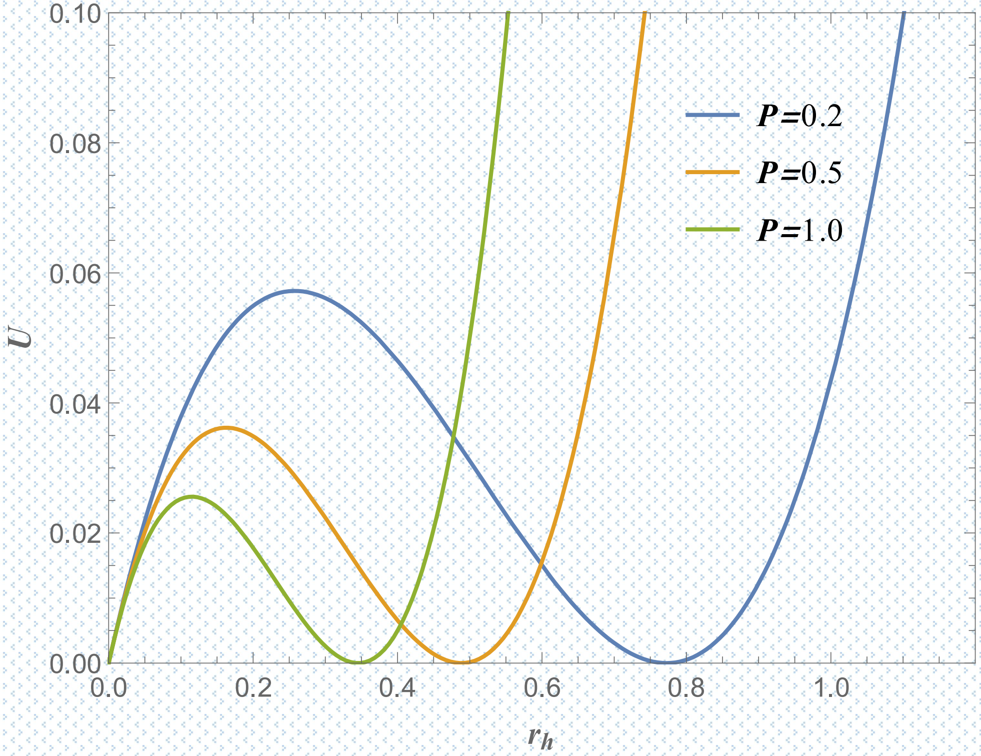
Generalized free energy

$$U(r_h) = \int (T_h(r_h) - T)\mathrm{d}S(r_h).$$

4D Schwarzschild-AdS BH

Hawking-Page phase transition



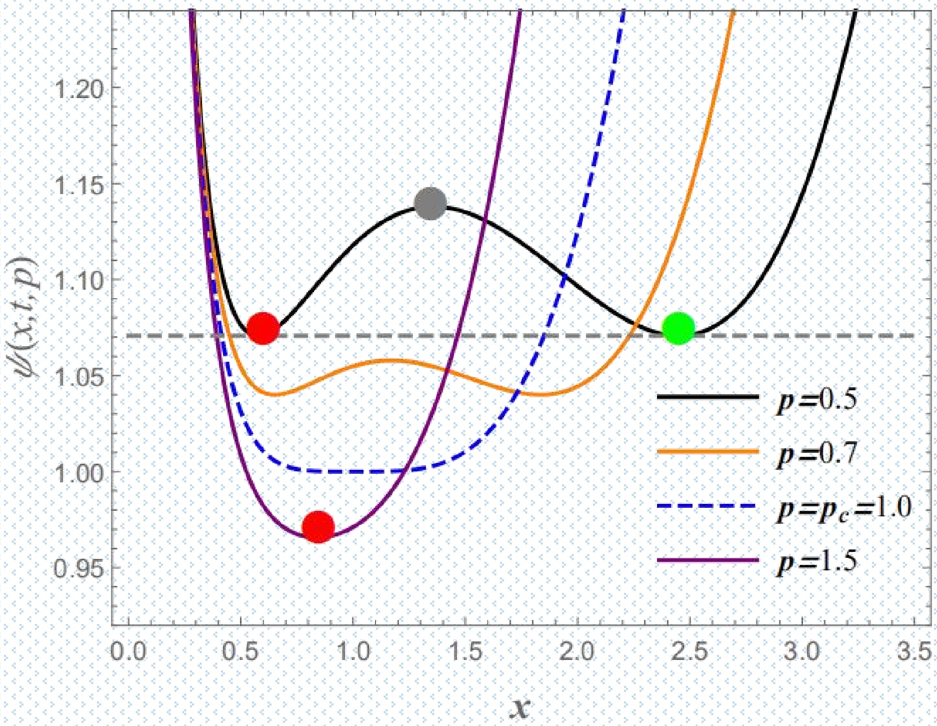


Generalized free energy

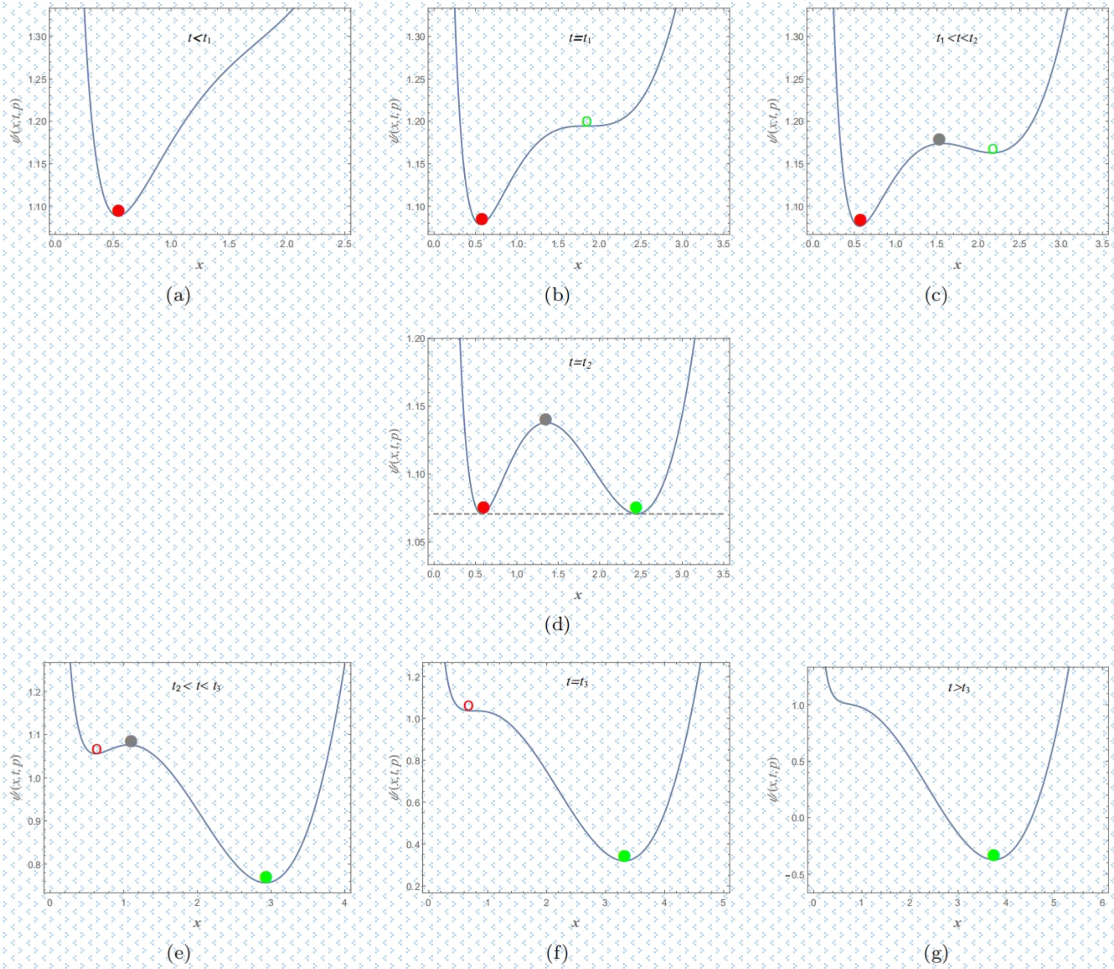
$$U(r_h) = \int (T_h(r_h) - T)\mathrm{d}S(r_h).$$

4D RN-AdS BH

second-order phase transition



first-order phase transition



Z.-M. Xu, B. Wu and W.-L. Yang, CQG (2021)

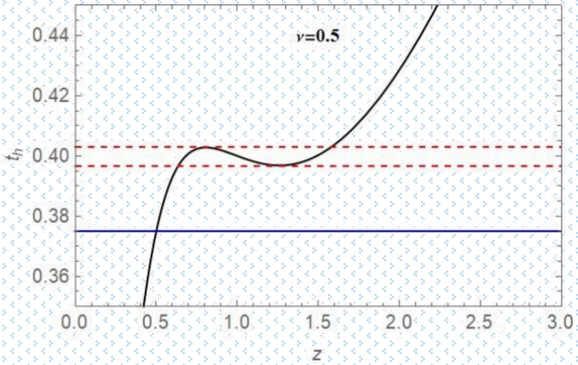
Generalized free energy

$$U(r_h) = \int (T_h(r_h) - T)\mathrm{d}S(r_h).$$

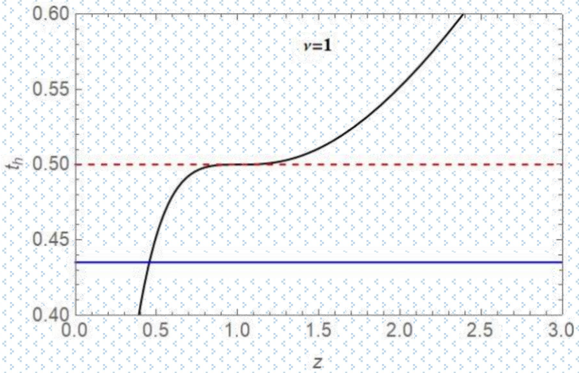
$$U(z) = \int (T_h(z) - T) f(z)\mathrm{d}z \qquad \mathrm{d}S(z) = f(z)\mathrm{d}z$$

qBTZ BH

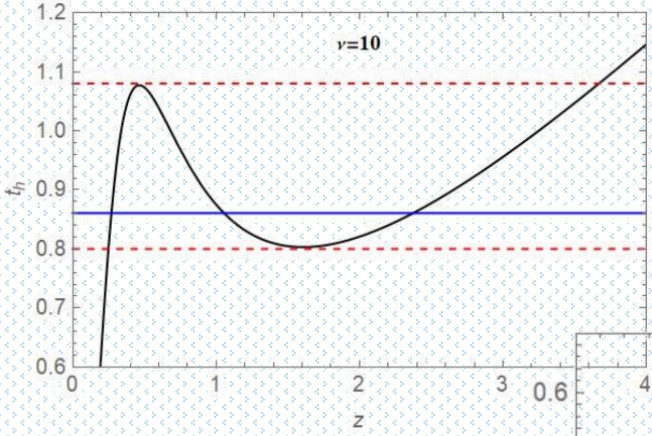
$$(T_h(z) - T) f(z) = 0$$



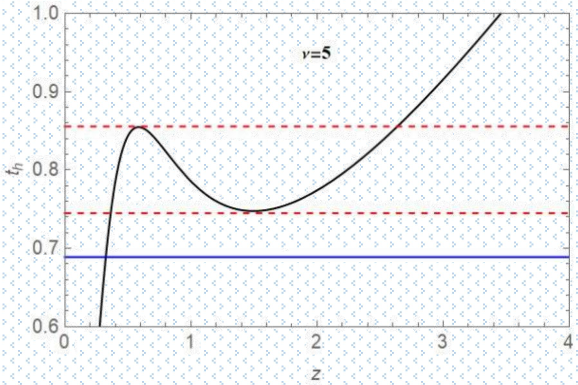
(a)$\nu = 0.5$



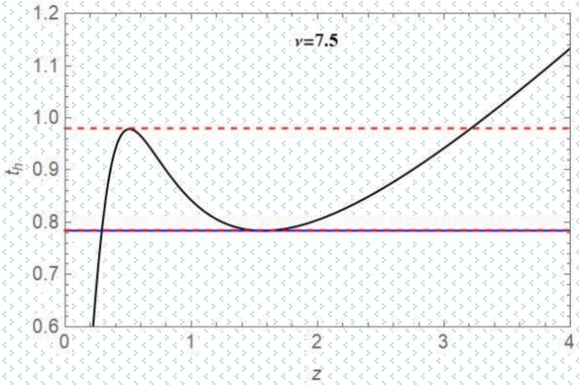
(b)$\nu = 1.0$



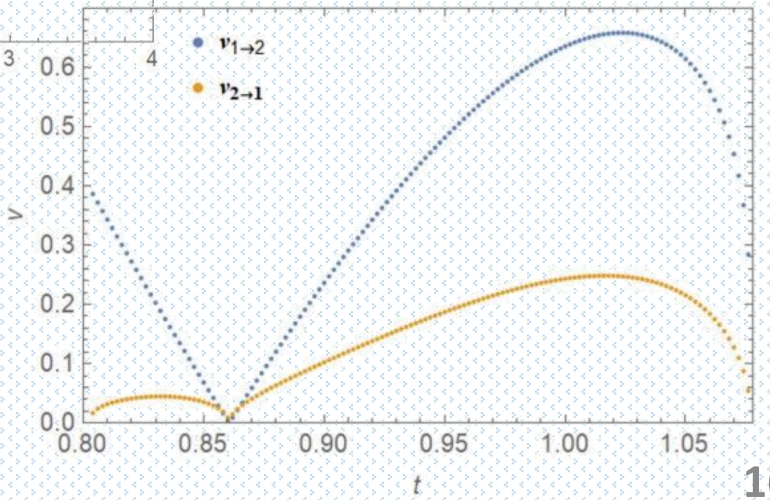
(e)$\nu = 10$



(c)$\nu = 5.0$

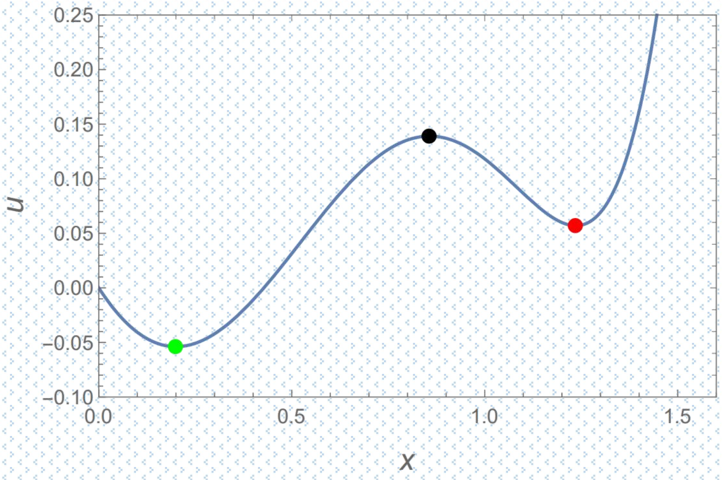


(d)$\nu = 7.5$



Complex correspondence

$$U(r_h) = \int (T_h(r_h) - T)\mathrm{d}S(r_h) \quad \Rightarrow \quad \boldsymbol{U(z)} = u(x,y) + iv(x,y)$$

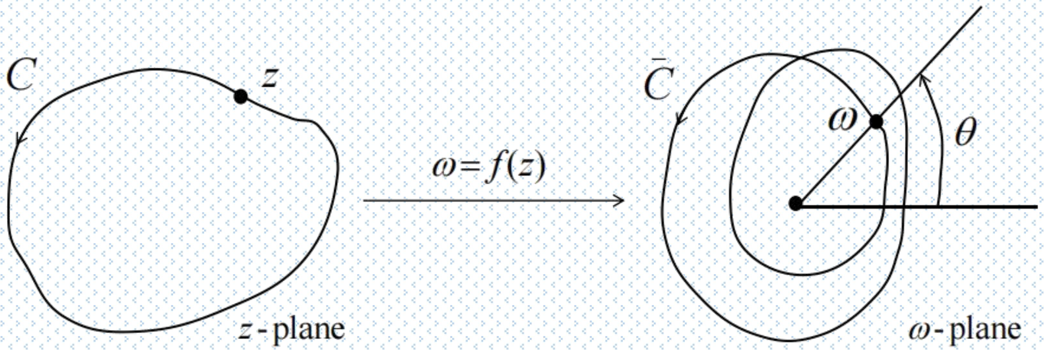


Black hole state at $f(z) = \frac{dU}{dz}$ ⟺ **Zeroes of Analytic Function**

PLB 2024

Complex correspondence

Argument Principle



Let C be a simple closed contour lying entirely within a domain D. Suppose f is analytic in D except at a finite number of poles inside C, and that $f(z) \neq 0$ on C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N_0 - N_p,$$

where N_0 is the total number of zeros of f inside C and N_p is the total number of poles of f inside C. In determining N_0 and N_p, zeros and poles are counted according to their order or multiplicities.

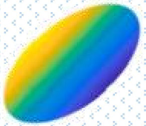
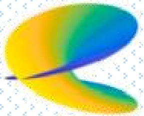
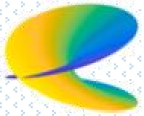
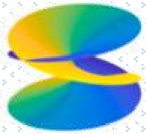
Winding number

$$W := \frac{1}{2\pi i}\oint_{\bar{C}} \frac{\mathrm{d}\omega}{\omega} = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\mathrm{d}z$$

$$N_0 - N_p = \frac{1}{2\pi}\,[\text{change in arg } (f(z)) \text{ as } z \text{ traverses } C \text{ once in the positive direction}].$$

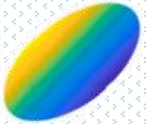
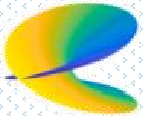
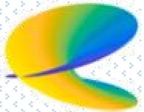
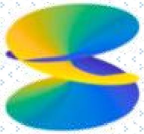
Complex correspondence

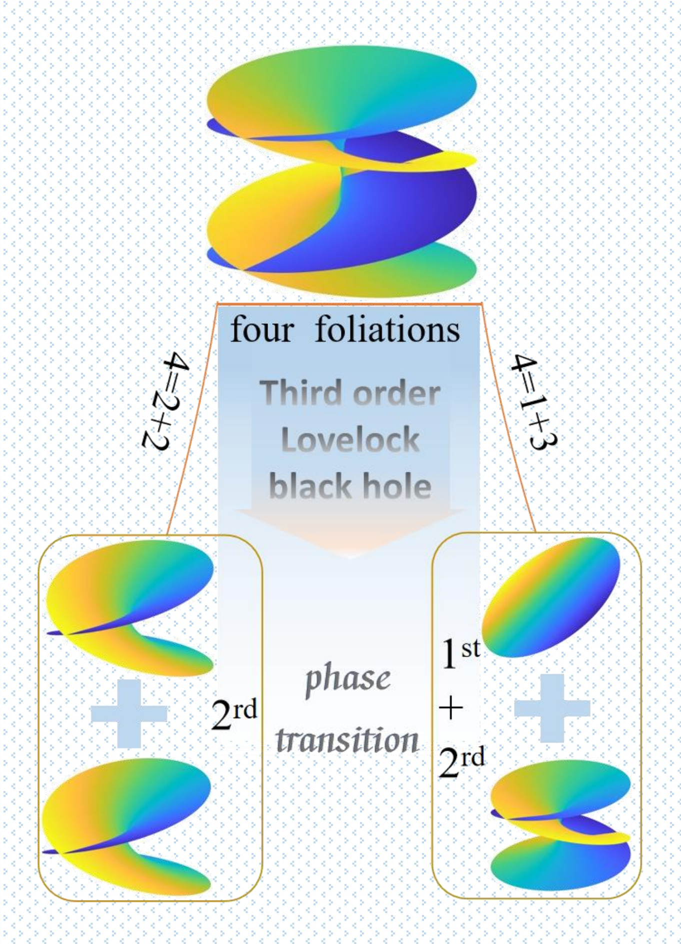
Complex correspondence

Black holes	Phase transition	Winding number		Complex counterpart (Riemann surface)
		local-max	global	
Schwarzschild	No	1	0	one foliation 
Reissner-Nordström	2^{nd}	2	0	two foliations 
Schwarzschild-AdS	2^{nd}	2	1	two foliations 
Charged AdS	1^{st} and 2^{nd}	3	1	three foliations 
6D charged Gauss-Bonnet	1^{st} and 2^{nd}	5	1	five foliations 

Complex correspondence

Complex correspondence

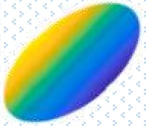
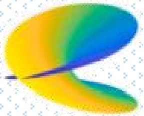
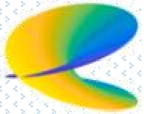
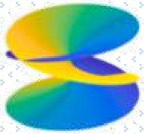
Black holes	Phase transition	Winding number: local-max	Winding number: global	Complex counterpart (Riemann surface)
Schwarzschild	No	1	0	one foliation 
Reissner-Nordström	2^{nd}	2	0	two foliations 
Schwarzschild-AdS	2^{nd}	2	1	two foliations 
Charged AdS	1^{st} and 2^{nd}	3	1	three foliations 
6D charged Gauss-Bonnet	1^{st} and 2^{nd}	5	1	five foliations 

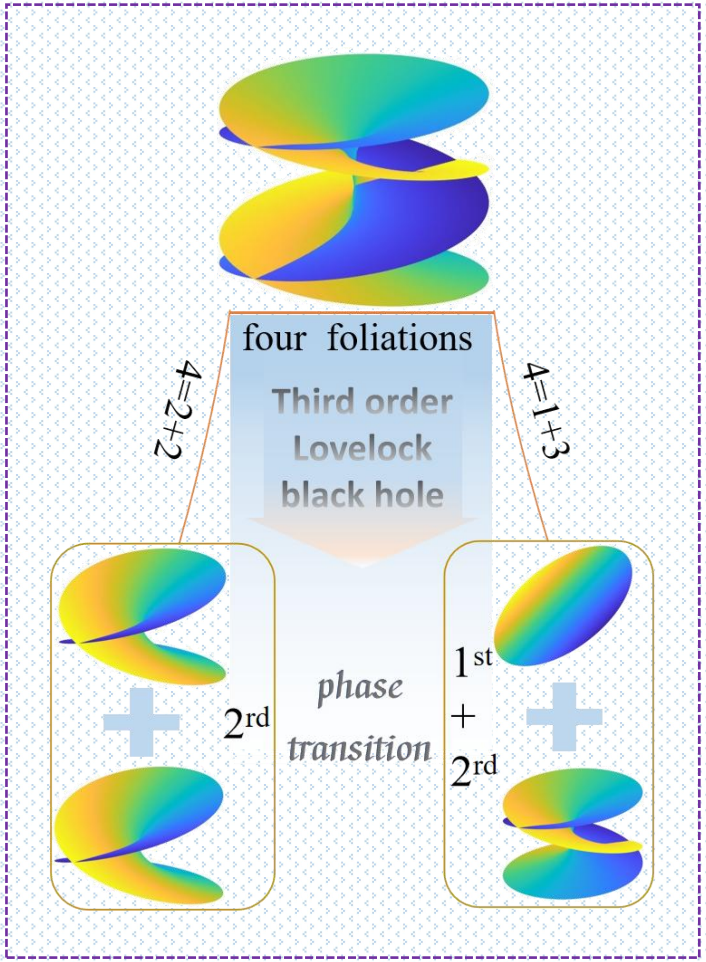


Complex correspondence

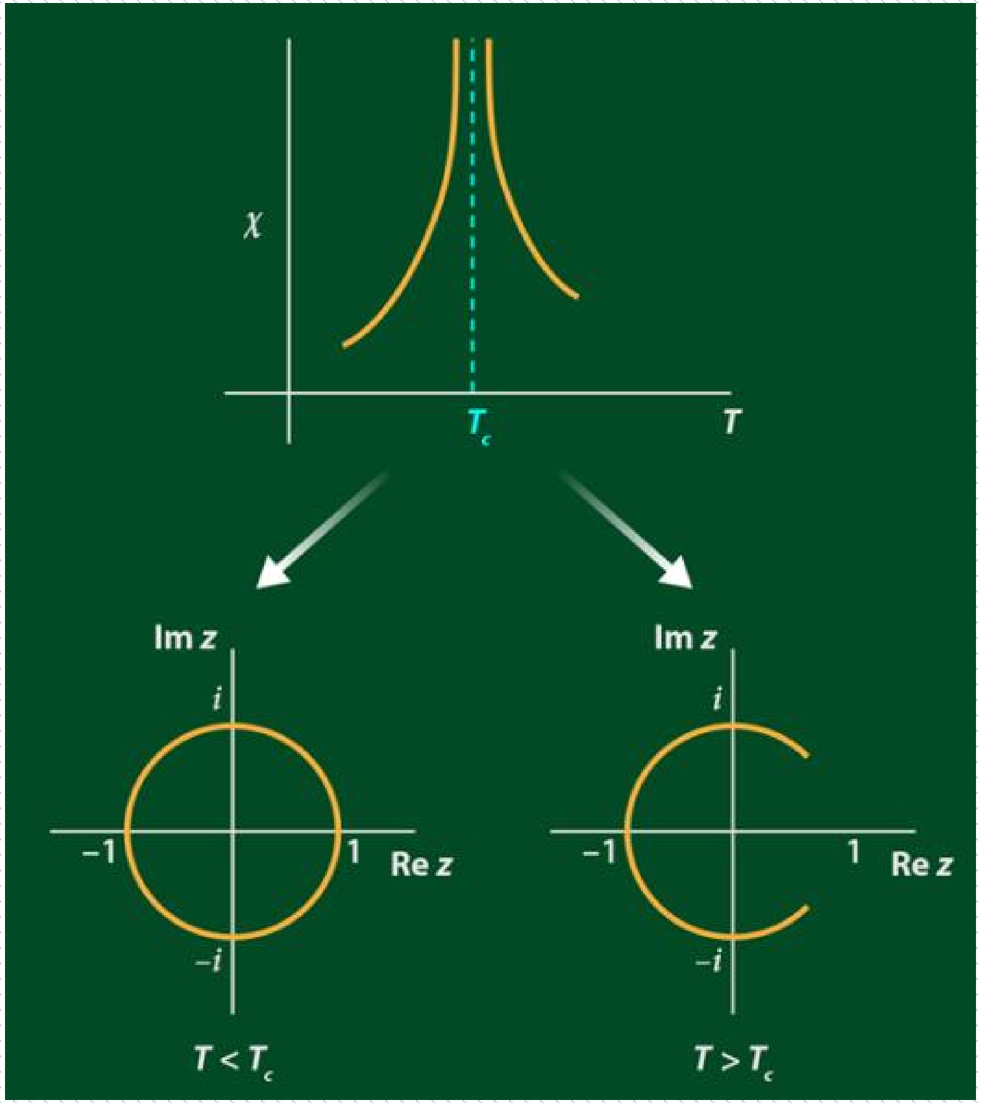
Complex correspondence

Riemann surface = Phase transition?

Black holes	Phase transition	Winding number: local-max	Winding number: global	Complex counterpart (Riemann surface)
Schwarzschild	No	1	0	one foliation 
Reissner-Nordström	2^{nd}	2	0	two foliations 
Schwarzschild-AdS	2^{nd}	2	1	two foliations 
Charged AdS	1^{st} and 2^{nd}	3	1	three foliations 
6D charged Gauss-Bonnet	1^{st} and 2^{nd}	5	1	five foliations 



Complex phase diagram



配分函数的零点分布

$F=-T\ln Z=TI_E$

欧式作用量 I_E

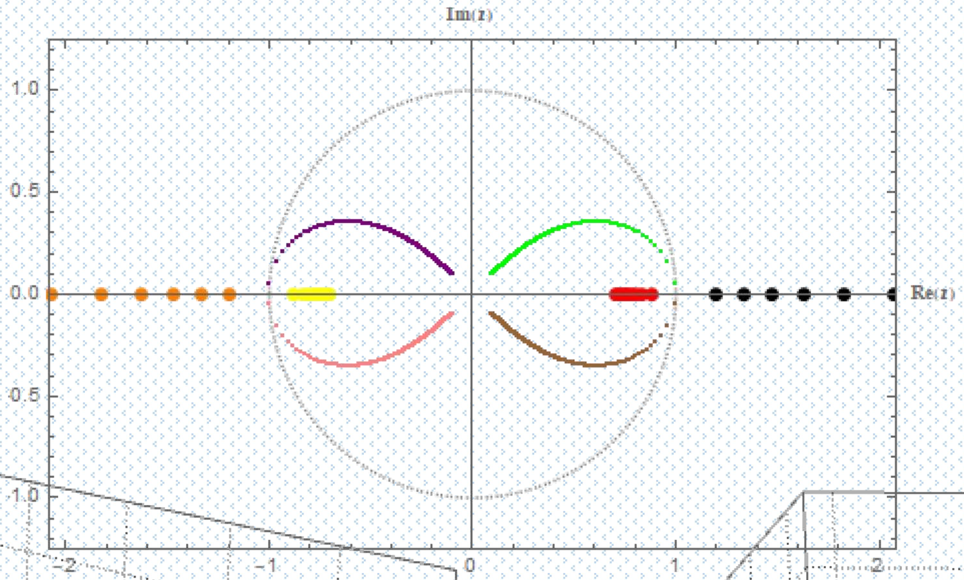
黑洞系统配分函数 Z

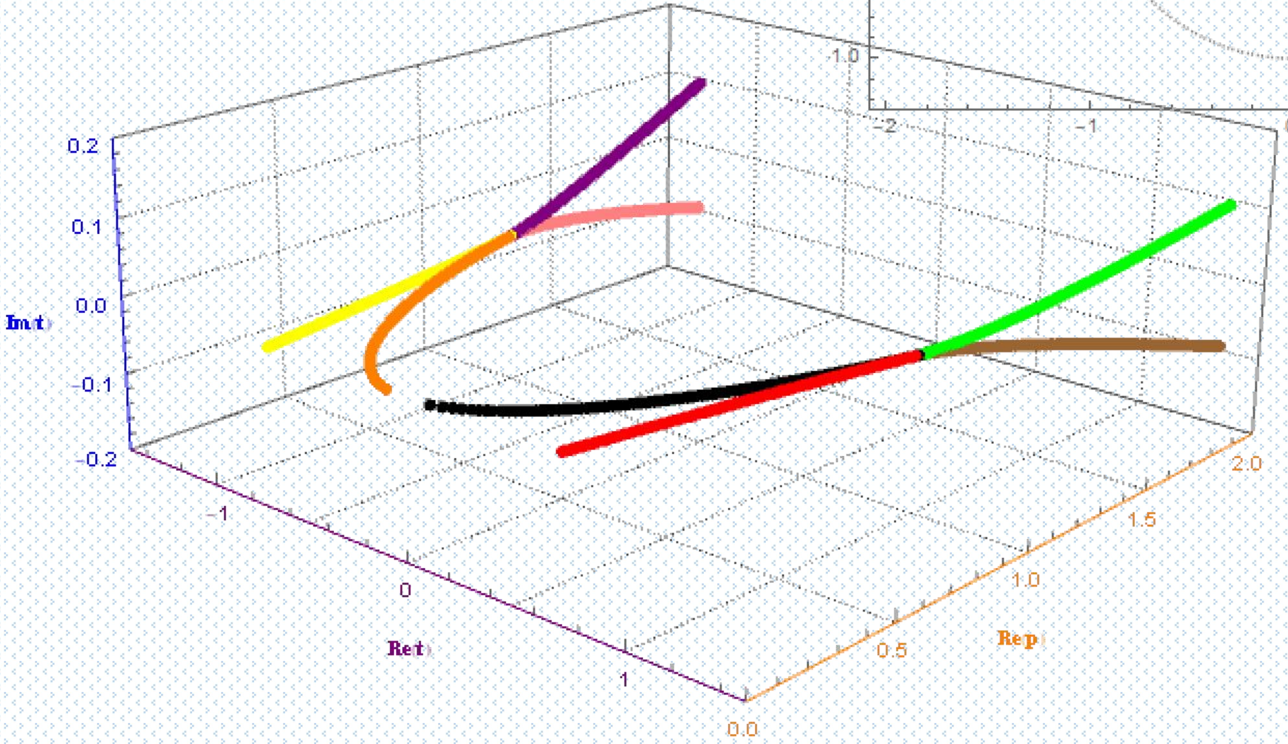
黑洞系统自由能 F

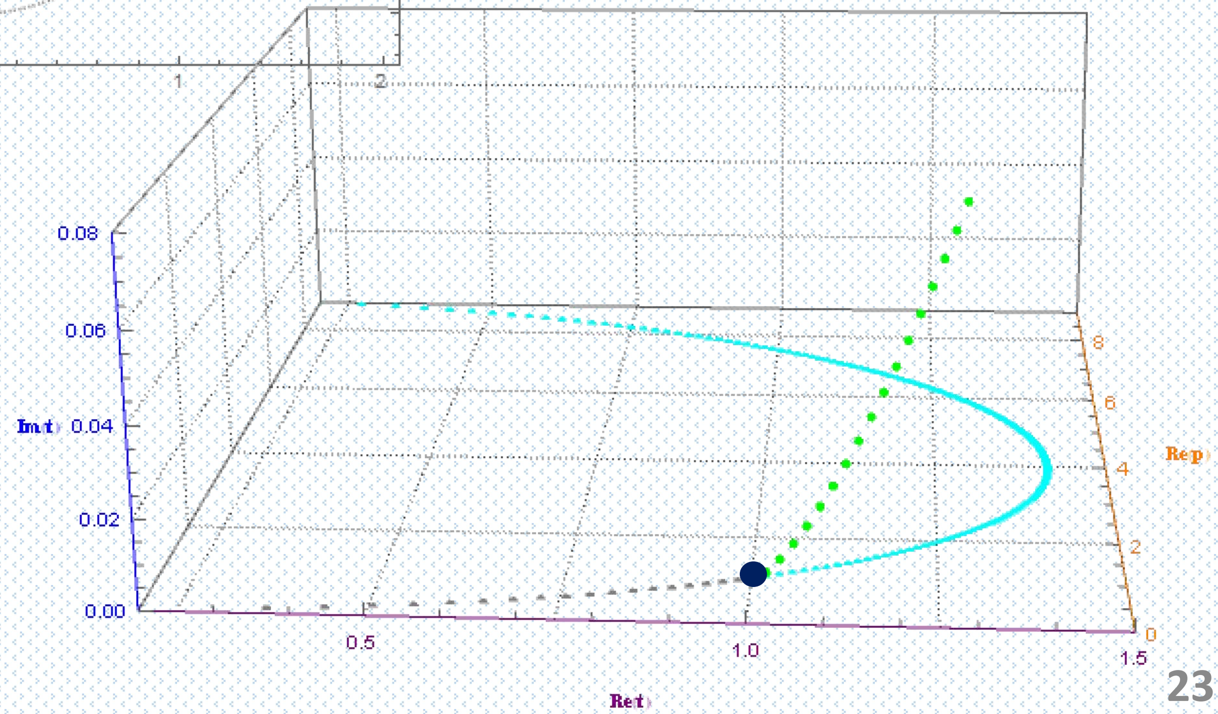
黑洞自由能的奇点分布

Complex phase diagram

4D RN-AdS BH







SUMMARY

- We have constructed the generalized free energy of black hole thermodynamics, and demonstrated the phase transition picture.
- It is possible to establish complex correspondences in black hole thermodynamics.
- Furthermore, by utilizing the rich properties of zeros of the analytical function, a deeper level of correspondence for the thermodynamic phase transition of black holes can be constructed.

THANK YOU