“2024 引力与宇宙学” 专题研讨会（合肥）

Null energy condition and positivity of black hole mass

Run-Qiu Yang (杨润秋)
Tianjin University

合肥 2024/11/16

Motivation

- In SR and for classical field theory that play a role in physics, the positive total energy is the integral of a positive energy density ρ;

Responsible for ensuring the stability of the ground state

- In gravity, the situation is very different.

There is no satisfactory way to define the local energy density including gravity

- It is an old conjecture that this total energy is in fact always strictly positive, except for flat Minkowski space, which has zero energy

Motivation

- Positive energy theorem:

Under dominant energy condition, the ADM energy of any asymptotically flat spacetime solving the Einstein equations is non-negative, and zero precisely only for Minkowski spacetime .

- The first proof is given by Richard Schoen and Shing-Tung Yau:

Commun. Math. Phys. 65, 45 (1979); Commun. Math. Phys. 79, 231 (1981)

- An alternative proof for Einstein gravity, making crucial use of spinor algebra

Edward Witten, Commun. Math. Phys. 80, 381 (1981)

- Gibbons, Hawking, Horowitz and Perry proved extensions of the theorem to asymptotically anti-de Sitter spacetimes and to Einstein–Maxwell theory.

$M \geq \sqrt{Q^2 + P^2}$ Here Q is charge, P is magnetic charge

Commun. Math. Phys. **88** (3): 295–308

Motivation

- An initial data site has an apparent horizon with area A_H and ADM mass E;
- The system then will evolve and settle down to a Kerr-Newman black hole;
- Assume that the dynamics is dominated by classical physics and null energy condition (NEC) is satisfied, then

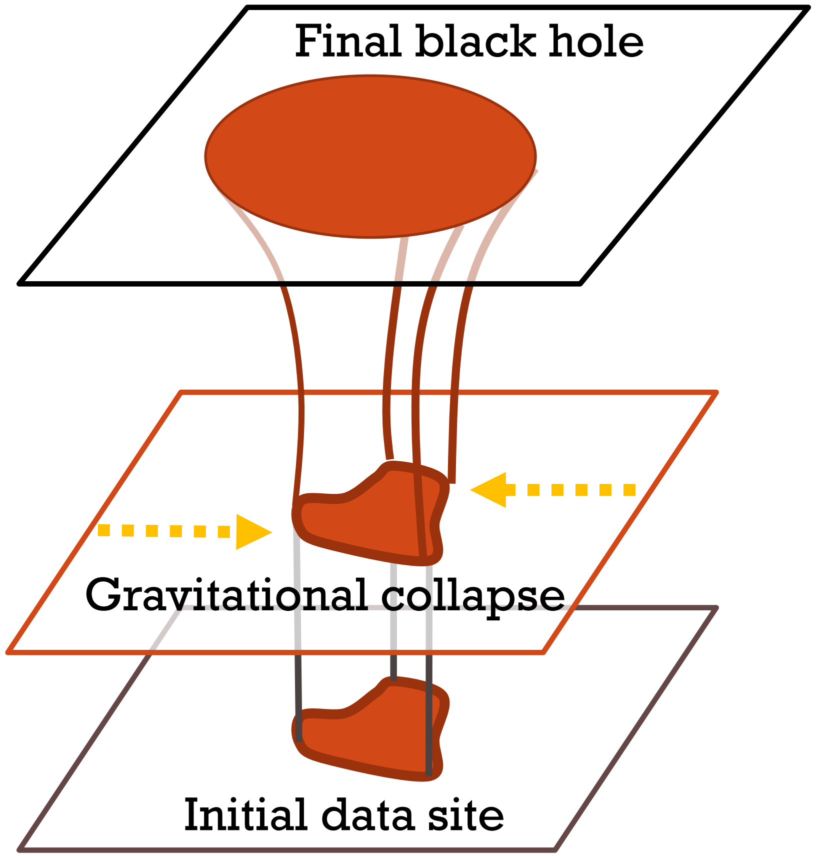
$$A_H \leq A_{H,f}$$

- For the final Kerr-Newman black hole,

$$A_{H,f} = 4\pi\left(r_h^2 + a^2\right), \qquad M_f = \frac{r_h^2 + a^2}{2r_h}$$

- We can see that $A_{H,f} \leq 16\pi M_f^2$. Then we see

$$A_H \leq 16\pi M_f^2 \leq 16\pi M^2$$



A gap: Only NEC is involved in above physical heuristic argument

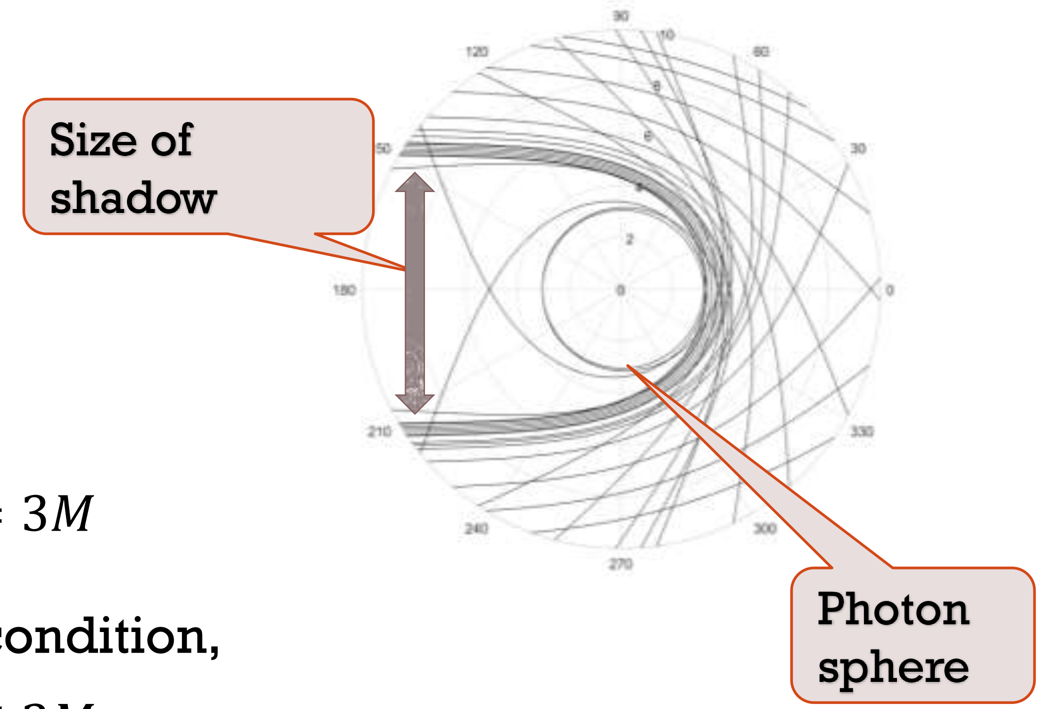
Motivation

- We have four different variables to describe the size of a black hole

  Radius of horizon r_+

  Radius of photon sphere r_{ph}

  Radius of shadow r_{sh}



- In Schwarzschild black hole, we find

$$\frac{3r_+}{2} = r_{ph} = \frac{r_{sh}}{\sqrt{3}} = 3M$$

- Our work shows that with some suitable energy condition,

$$\frac{3r_+}{2} \leq r_{ph} \leq \frac{r_{sh}}{\sqrt{3}} \leq 3M$$

Eur.Phys.J.C 80 (2020) 10, 949

- Particularly, we proved that if NEC is true, then

$$r_h < r_{ph} \leq 3M$$

This suggests NEC is enough to insure positive energy!

New theorem of "positive mass" on black hole

For asymptotically flat stationary black hole, if

(1) Einstein equation and null energy condition are satisfied, and

(2) the cross-section of event horizon has S^2 topology, then

$$3M \geq \frac{\kappa A}{4\pi} + \sqrt{\frac{A}{4\pi}}$$

- Particularly, a regular static spacetime has nonnegative ADM energy.

In static spherically symmetric case

- We consider the spherically symmetric spacetime.

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad f(r) = 1 - \frac{2M}{r} + \cdots, \quad |\chi(r)| \to \mathcal{O}(1/r^{1+\alpha}), \quad r \to \infty$$

- The energy momentum tensor has a form

$$f(r) = 1 - \frac{2M}{r} + \cdots, \quad |\chi(r)| \to \mathcal{O}(1/r^{1+\alpha}), \quad r \to \infty$$

- Surface gravity and area of horizon are given by $\kappa = \frac{f'(r_h)e^{-\chi(r_h)/2}}{2}, \quad \mathcal{A} = \pi r_h^2$

- The Einsteins equation gives the following independent equations

$$f' = (1 - 8\pi r^2\rho - f)/r\,,$$

$$\chi' = -\frac{8\pi r}{f}(\rho + p_r)\,,$$

NEC insures $\chi' \leq 0$ → NEC insures $\chi \geq 0$

$$p_r' = \frac{\rho + 4p_T - 3p_r}{2r} - \frac{(p_r + \rho)(8\pi p_r r^2 + 1)}{2rf}$$

In static spherically symmetric case

- Let us define a new quasi-local masses

$$f(r) = 1 - \frac{2M}{r} + \cdots, \quad |\chi(r)| \to \mathcal{O}(1/r^{1+\alpha}), \quad r \to \infty$$

$$m_n(r) = \frac{r^4 e^{\chi/2}}{6}\left(\frac{fe^{-\chi}}{r^2}\right)' + \frac{r}{3}$$

At infinity, they both give us black hole mass

$$m_s(\infty) = m_n(\infty) = M$$

- At horizon, they are all positive

$$m_n(r_h) = \frac{\kappa \mathcal{A}}{12\pi} + \frac{r_h}{3}$$

Using Einstein equation, one can verify $m_n'(r) = \frac{8\pi}{3} e^{-\chi/2} r^2 (\rho + p_T) + \frac{1}{3}[1 - e^{-\chi(r)/2}]$

NEC gives us theorem on positivity of black hole mass!

Beyond spherically symmetric case

For a static black hole, we can make ADM decomposition

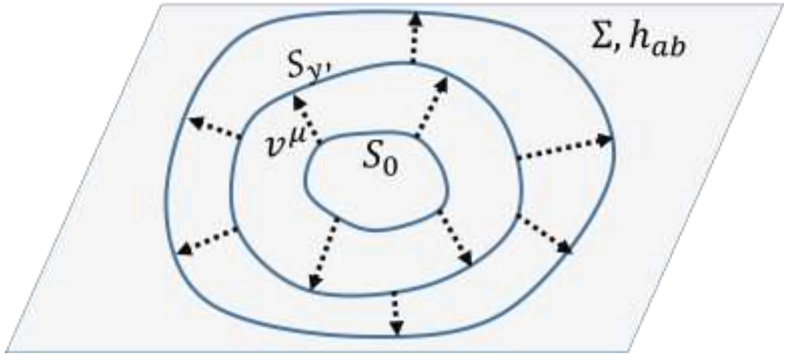
$$ds^2 = -N^2 dt^2 + h_{ab} dx^a dx^b$$

We can foliate the equal t surface by $\{S_y\}$ and find a quasi-local mass m(y) on every surface S_y

(1) $m(0) \leq m(\infty)$;

(2) At the initial surface we have $3m(0) = \sqrt{\mathcal{A}/(4\pi)} + \kappa\mathcal{A}/(4\pi)$;

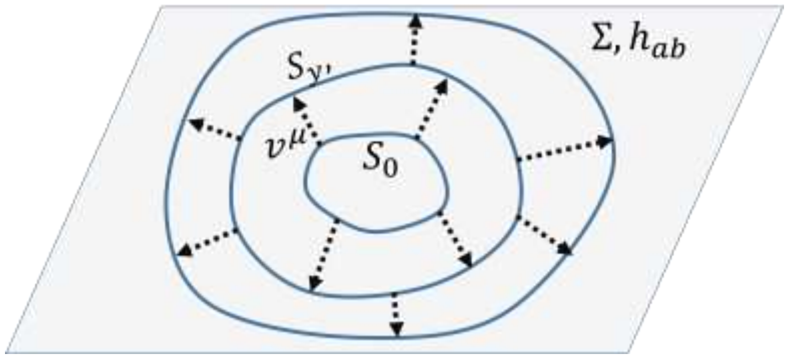
(3) At the infinity $m(\infty) = E$;



Basic idea of proof

- Geroch, Pong Soo Jang and Wald once proposed a very important tool called "inverse mean curvature flow" to solve Penrose inequality;

- For a given initial surface, the flow is generated by the vector field $v^{\mu} = k^{-1} r^{\mu}$ and r^{μ} is the outward unit normal vector of the surface.



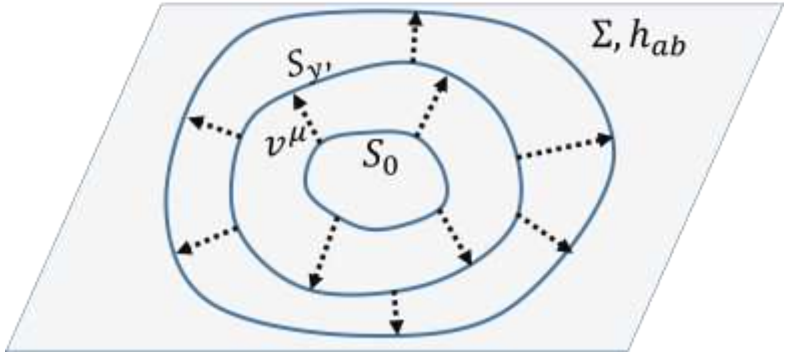
- The Hawking-Geroch mass of surface S_y is defined as

$$m_H(S_y) = \frac{\sqrt{A(S_y)}}{64\pi^{3/2}} \int_{S_y} (2\mathcal{R} - k^2) dS$$

- Hawking-Geroch mass satisfies

$$m_H(0) = \frac{\sqrt{A(S_0)}}{64\pi^{3/2}} \int_{S_0} (2\mathfrak{R} - k^2) \mathrm{d}S = \sqrt{\frac{\mathcal{A}}{16\pi}} \qquad m_H(\infty) = E$$

Geroch, Wald and Jang found that



$$m_H(y) - m_H(0) \geq \int_{\Sigma_t} \rho Q \mathrm{d}V$$

Here Q is a positive function and will approach to 1 rapidly.

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + h_{ab} \mathrm{d}x^a \mathrm{d}x^b$$

We now introduce a scalar field W on according to following elliptic equation

$$D^2(NW) = 8\pi Q(\rho + T/2) \qquad W|_{\partial\Sigma_t} = 1$$

$$\partial_a (WN) \mathrm{d}S^a |_{\partial\Sigma_t} = \partial_a N \mathrm{d}S^a |_{\partial\Sigma_t}$$

We consider a new quasi-local mass

$$m_K(y) = \frac{1}{4\pi} \int_{S_y} \partial_a (NW) \mathrm{d}S^a \longrightarrow m_K(\infty) - m_K(0) = 2 \int_{\Sigma_t} Q(\rho + T/2) \mathrm{d}V$$

$$\lim_{y \to 0} \frac{1}{4\pi} \int_{S_y} \partial_a N \mathrm{d}S^a = \frac{\kappa \mathcal{A}}{4\pi}, \quad \lim_{y \to \infty} \frac{1}{4\pi} \int_{S_y} \partial_a N \mathrm{d}S^a = E \longrightarrow m_K(0) = \frac{\kappa \mathcal{A}}{4\pi}, \quad m_K(\infty) = E$$

Define the combination $m(r) := \frac{2}{3}m_H(r) + \frac{1}{3}m_K(r)$ → $3m(\infty) - 3m(0) = 3E - \left(\frac{\kappa A}{4\pi} + \sqrt{\frac{A}{4\pi}}\right)$

$$m_H(y) - m_H(0) \geq \int_{\Sigma_t} \rho Q \mathrm{d}V \qquad m_K(\infty) - m_K(0) = 2\int_{\Sigma_t} Q(\rho + T/2)\mathrm{d}V$$

→ $$m(\infty) - m(0) \geq \frac{1}{3}\int_{\Sigma_t} (4\rho + T)Q\mathrm{d}V$$

NEC insures it nonnegative

→ $$3E \geq \left(\frac{\kappa A}{4\pi} + \sqrt{\frac{A}{4\pi}}\right)$$

We then conclude that NEC can insures the nonnegativity of ADM energy for static black hole!

Summary

- I conjecture there may be a universal inequality by only requiring NEC;

$$3E \geq \frac{\kappa A}{4\pi} + \sqrt{\frac{A}{4\pi}}, \qquad 2E \geq \sqrt{\frac{A}{4\pi}}$$

- A proof for static black hole that contains one connect compact horizon is given;

- This begs a new challenge for a famous settled question of general relativity:

In what general case can NEC replace DEC to insure nonnegative ADM energy?