Hyperfine Structure of Quantum Entanglement

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with L. h. Mo, Y. Zhou and P. Ye, arXiv: 2311.01997

Outline

- Entanglement entropy and contour
- Rényi contour, hyperfine structure
- Application in lattice fermion model
- Holographic dual of Rényi contour
- Conclusions

Entanglement entropy, entanglement contour

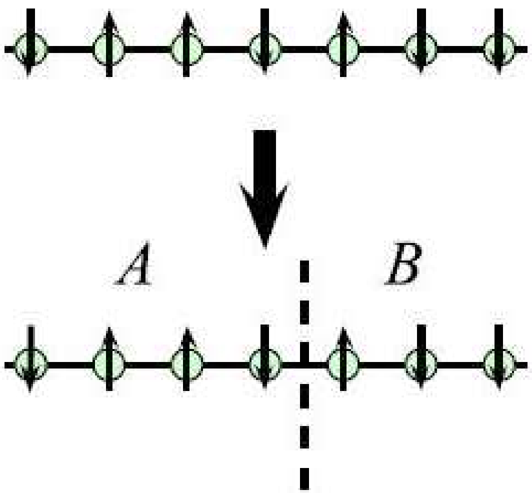
The quantum entanglement is of great importance in characterizing the correlation between different dof in quantum many-body systems, and plays an important role in understanding gravity.

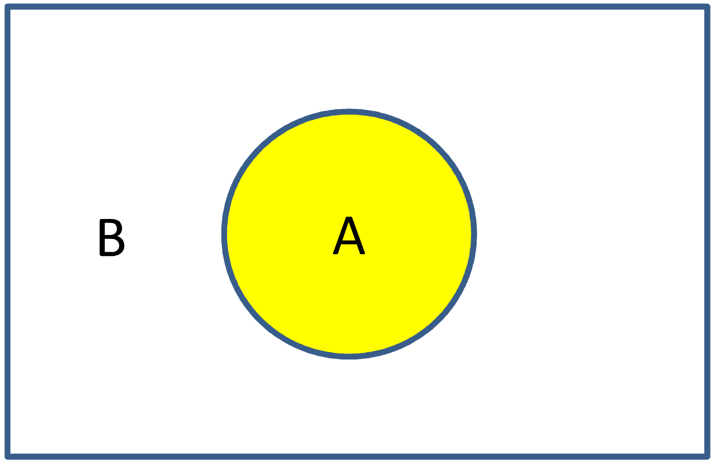
Calabrese and Cardy, hep-th/0405152

Amico, Fazio, Osterloh and Vedral, quant-ph/0703044

For a pure state with density matrix ρ, divide the system into *A*+*B*, the entanglement entropy can be calculated by the **von Neumann entropy** (EE)

$$S_A = -\mathrm{tr}_A(\rho_A \ln \rho_A)$$





There are more general quantity to characterize the entanglement, such as the **Rényi entropy**

$$S_A^{(n)} = \frac{1}{1-n} \ln \mathrm{tr}\, \rho_A^n$$

which can be interpreted as a *n*-replica of the EE, and contains more complete information about the entanglement spectrum. As *n* goes to 1, Rényi entropy reproduces EE.

For a 2d field theory at critical point (zero *T*), the EE is

$$S_{\mathrm{A}} = (c/3)\log((L/\pi a)\sin(\pi\ell/L)) + c_1' \sim (c/3)\log(\ell/a),$$

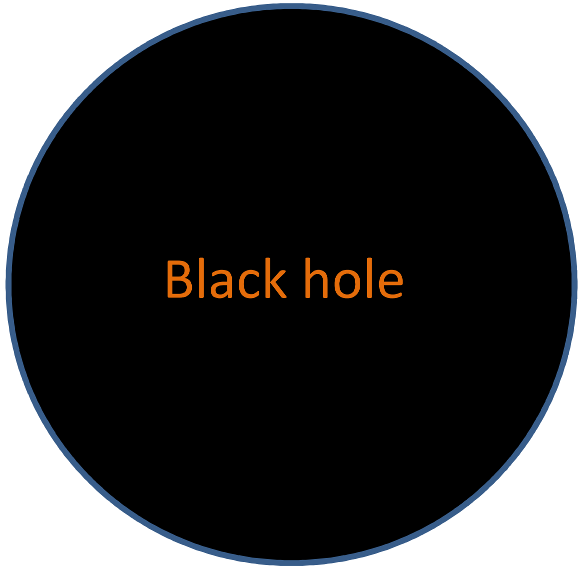
At finite *T*, the EE is $S_A = (c/3)\log\left((\beta/\pi a)\sinh(\pi\ell/\beta)\right) + c_1'$.

For higher dimensional QFT, the EE is proportional to the area of the boundary—*area law+sub-leading corrections.*

Entanglement and gravity--since EE also describes the lack of information for observer in *A* to *B*, it has been used to explain the origin of the black hole Bekenstein-Hawking entropy

Bombelli etc, PRD34, 373 (1986);
Jacobson, gr-qc/9404039

$$S_{\mathrm{EE}} = \frac{A}{a^2} \propto S_{\mathrm{BH}} = \frac{A}{4G_{\mathrm{N}}}$$



Especially, for the braneworld black hole, the entanglement entropy exactly matches with the black hole area entropy

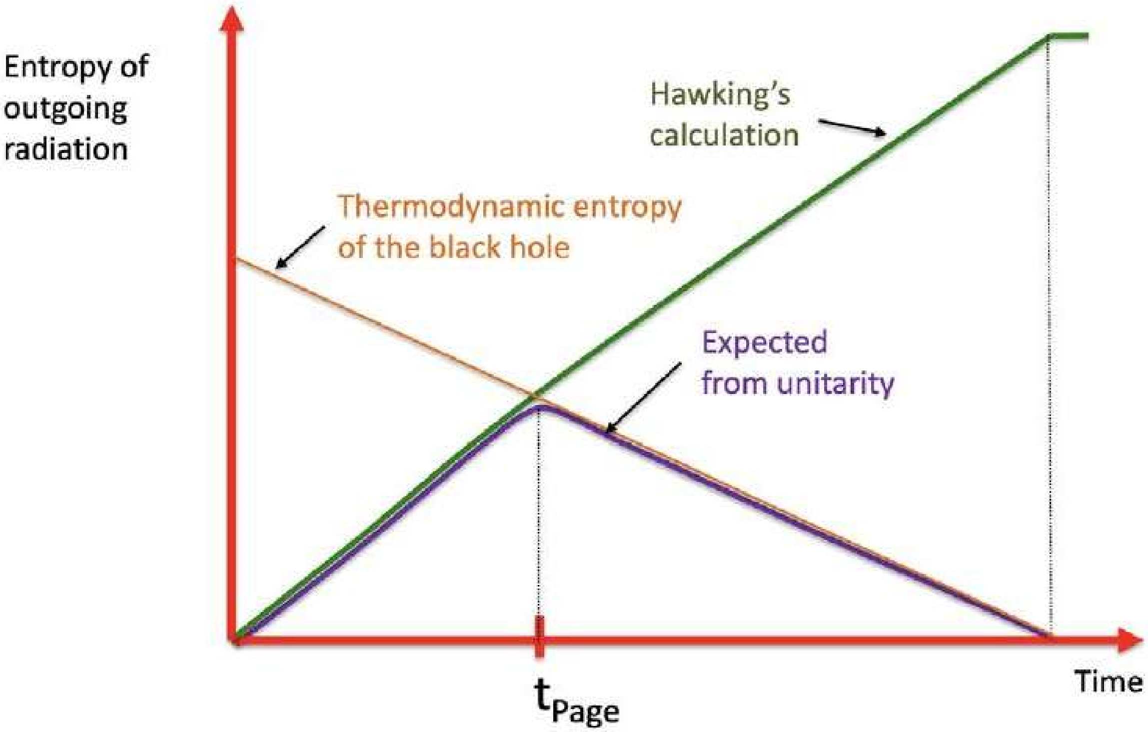
Emparan, JHEP 06, 012 (2006)

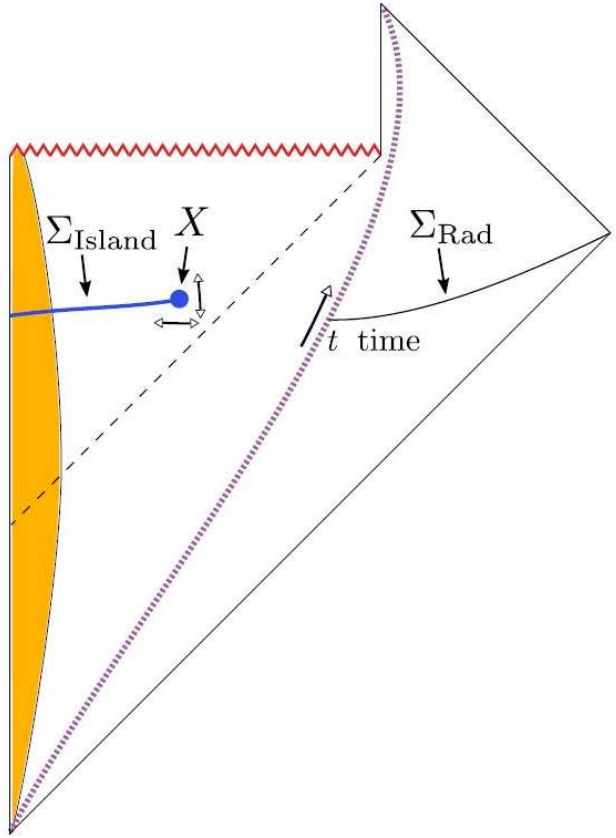
$$S_{\mathrm{EE}} = \frac{A_{\mathrm{ext}}}{4G_{d+1}} = \frac{A}{4G_{d+1}}$$

EE is also essential to solve the black hole information loss paradox

--**Island prescription of Hawking radiation**

Penington, 1905.08255; Almheiri, Engelhardt, Marolf, and Maxfield, 1905.08762;





Bekenstein's generalized entropy

$$S_{\rm gen} = \frac{\text{Area of horizon}}{4\hbar G_N} + S_{\rm outside},$$

Fine-grained entropy

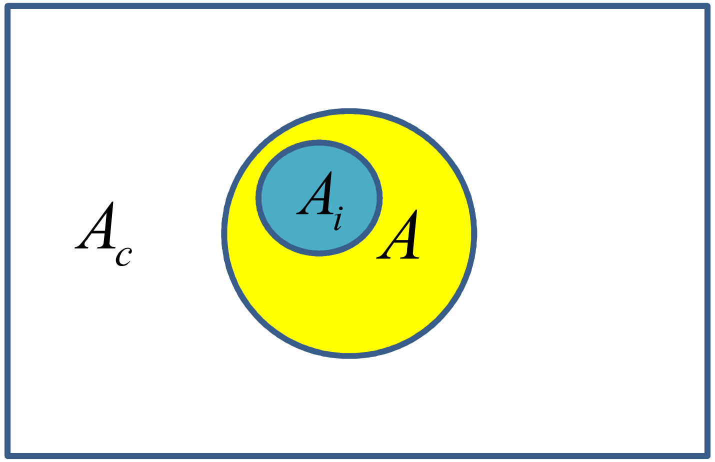
$$S = \min_X \left\{ \text{ext}_X \left[\frac{\text{Area}(X)}{4G_N} + S_{\text{semi-cl}}(\Sigma_X) \right] \right\}$$

Faulkner, Lewkowycz and Maldacena, 2013; Engelhardt and Wall, 2015

Entanglement contour

Chen, Vidal, 1406.1471

Entanglement contour (EC): a local function $s_A(x)$ trying to describe the fine structure the entanglement entropy in real space



$$S_A = \int_A s_A(x) dx$$

Explicit examples of EC: Gaussian states, CFT, partial entanglement entropy (PEE) , e.g., $s_A(A_i)$ of some subsystem of A

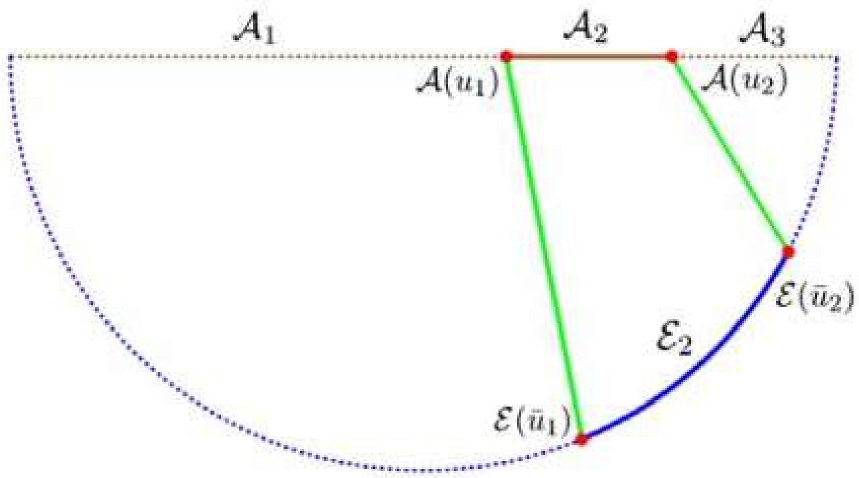
$$s_A(A_i) = \int_{A_i} s_A(x) dx$$

However, the bove requirements are not sufficient to uniquely determine the PEE in general.

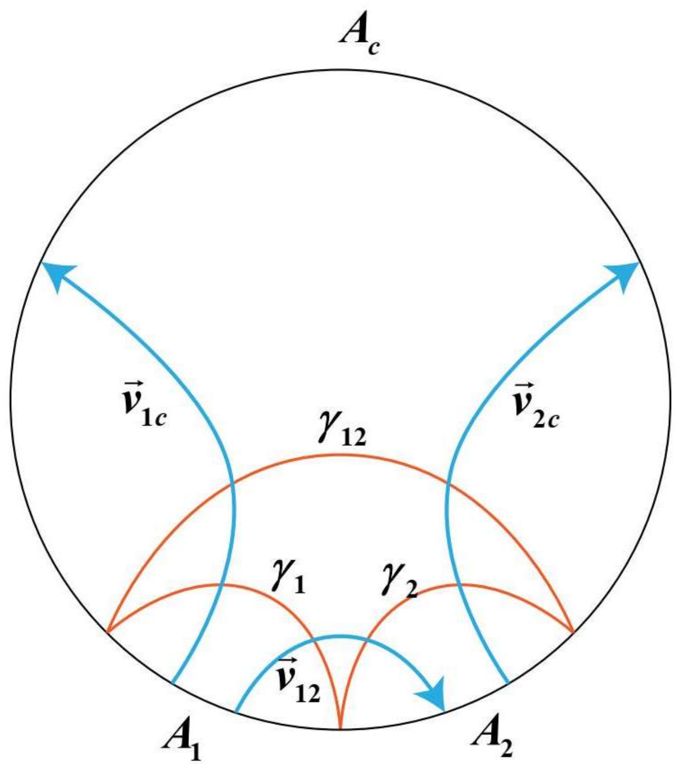
PEE proposal Q Wen, 1902.06905, 1803.05552; Kudler-Flam, MacCormack, Ryu, 1902.04654

$$s_A(A_2) = \frac{1}{2}(S_{12} + S_{23} - S_1 - S_3)$$

Holographically, PEE can be described by combination of **extremal surfaces** the **bit threads**



Q Wen, 1803.05552



Y-y Lin, **JRS**, J Zhang, 2105.09176

Rényi contour, hyperfine structure

L H Mo, Y Zhou, **JRS**, P Ye, 2311.01997

A natural question is to ask what is the entanglement contour for the Rényi entropy?

We introduced a hyperfine structure for entanglement by exactly decomposing the Rényi contour into the contributions from particle-number cumulants in free fermion system.

$$\begin{array}{ccccc} S(A) & \Longrightarrow & s(i) & \Longrightarrow & h_{1;k}(i) \\ \uparrow n=1 & & \uparrow n=1 & & \uparrow n=1 \\ S_n(A)(\tilde{S}_n(A)) & \Longrightarrow & s_n(i)(\tilde{s}_n(i)) & \Longrightarrow & h_{n;k}(i)(\tilde{h}_{n;k}(i)) \end{array}$$

$$s_n(j) \equiv \sum_{k=1}^{\infty} s_{n;k}(j) = \sum_{k=1}^{\infty} \Big(\beta_k(n) C_k(j) \Big),$$

$$\beta_k(n) = \frac{2}{n-1}\frac{1}{k!}\left(\frac{2\pi i}{n}\right)^k \zeta\left(-k, \frac{n+1}{2}\right)$$ is nonzero for even *k*,

$C_k(j)$ is the density of cumulant on site *j*, it is a 2*k*-point function

$$C_k(j) \equiv (-i\partial_\lambda)^{k-1} \frac{\langle \exp(i\lambda \hat{N}_A)\hat{n}_j \rangle}{\langle \exp(i\lambda \hat{N}_A) \rangle}\Bigg|_{\lambda=0},$$

$\hat{n}_j$ is particle number operator on site *j*, and $\hat{N}_A$ is the particle number operator of *A*. λ is a real number.

The first nonzero term is

$$C_2(j) = \sum_i \langle \hat{n}_i \hat{n}_j \rangle - \langle \hat{n}_i \rangle \langle \hat{n}_j \rangle$$

Properties:

- additivity $s_{n;k}(i) + s_{n;k}(j) = s_{n;k}(i \cup j)$
- normalization
- exchange symmetry $s_{n;k}(i) = s_{n;k}(j)$
- invariance under local unitary transformation
- post-measurement state entanglement

$$S_{n;k} = \beta_n(k) C_k$$

Entanglement spectrum reconstruction from RC

The n-th power of reduced density matrix is

$$T_n = \mathrm{tr}(\hat{\rho}_A^n) = e^{(1-n)S_n}.$$

Defining a D corss D matrix as

$$U = \begin{pmatrix} 1 & 1 & 0 & \dots & \\ T_2 & 1 & 2 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ T_{D-1} & T_{D-2} & \dots & T_2 & D-1 \\ T_D & T_{D-1} & \dots & T_2 & 1 \end{pmatrix}$$

a polynomial can be constructed

$$P(x) = \sum_{n=0}^{D} \frac{(-1)^n}{n!} (\det U_n) x^{D-n} = \det(xI - \hat{\rho}_A).$$

Application in lattice fermion model

We consider a Chern insulator model called Qi-Wu-Zhang model

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}} H(\mathbf{k}) \hat{c}_{\mathbf{k}},$$

$$H(\mathbf{k}) = (m + \cos k_x + \cos k_y)\sigma_z + \lambda(\sin k_x \sigma_x + \sin k_y \sigma_y)$$

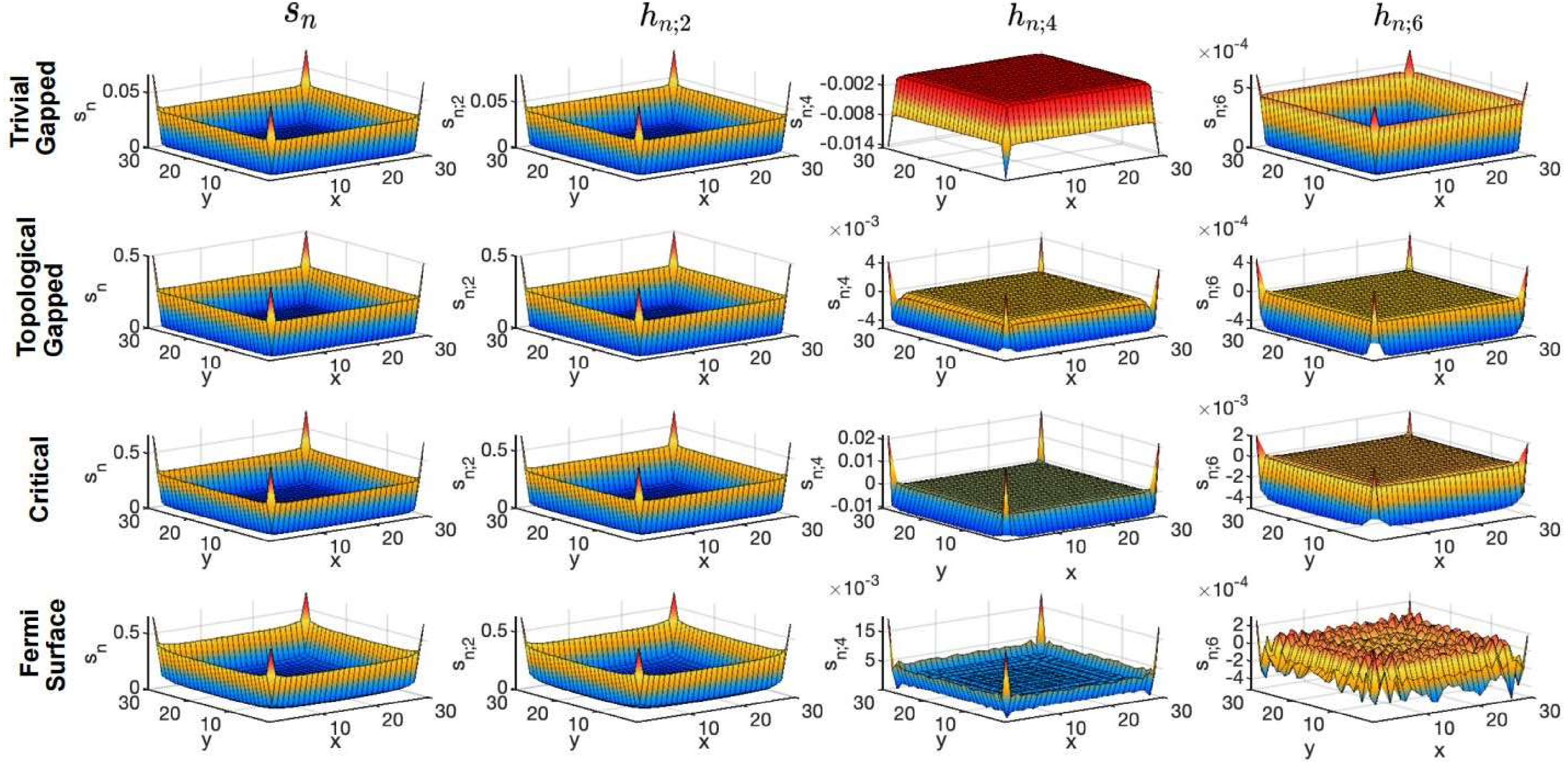
The energy gap closes at $m = \pm 2,$ forming Dirac point at

$$k_x = k_y = 0 \text{ and } k_x = k_y = \pi.$$

The energy gap closes at $m = 0,$ forming Dirac point at

$$k_x = 0, k_y = \pi \text{ and } \quad k_x = \pi, k_y = 0.$$

The topological properties of the electronic band structure are characterized by the Chern number.

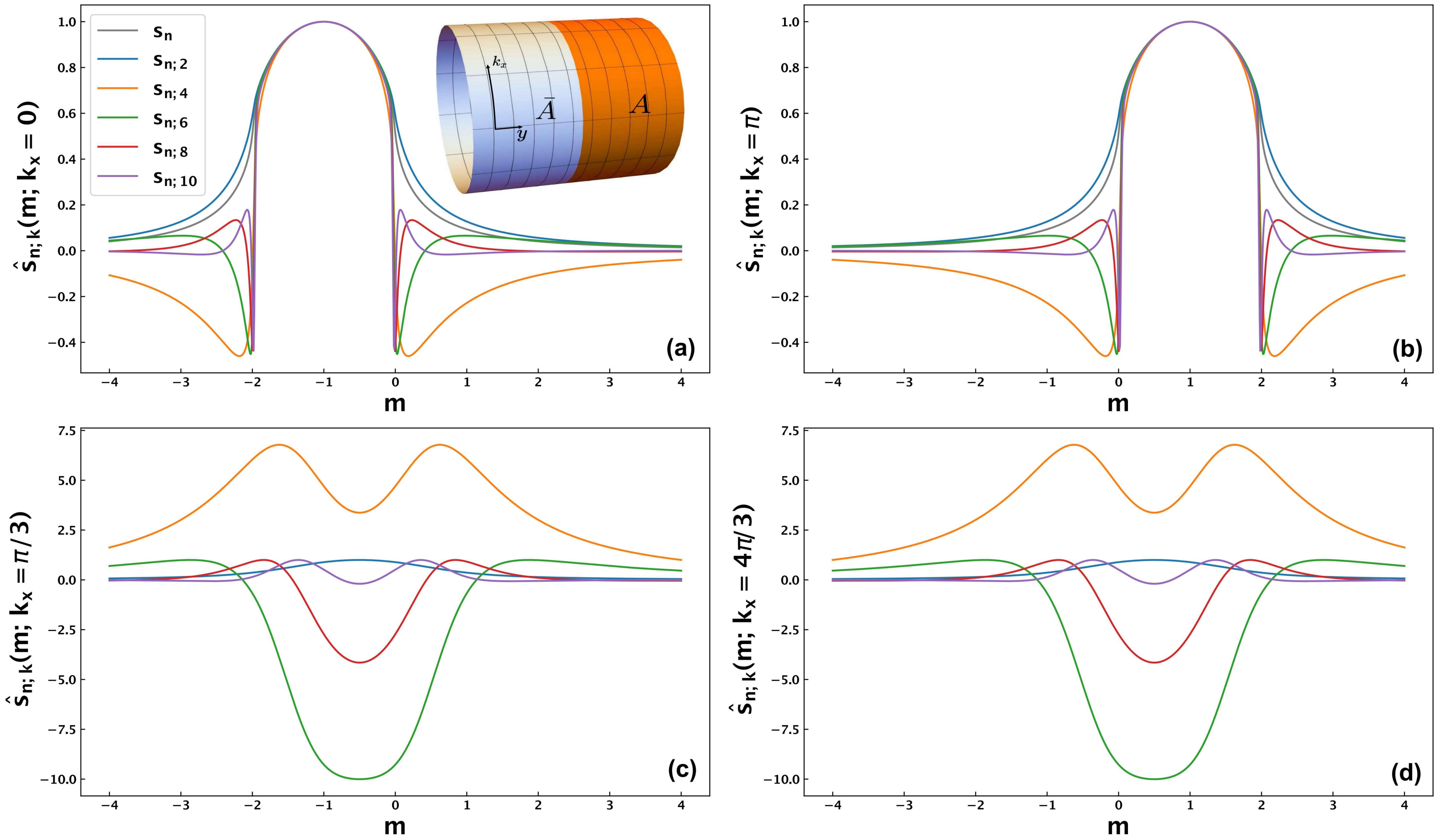


Distributions of RC and hyperfine RC for different energy spectrum:

k=2: dominant contribution; k>2

(trivial)gapped: ‘bowl’ shape; critical: corner vs hinge sites

Fermi surface: oscillation; topological: same as critical



$$\hat{s}_{n;k}(m; k_x) \equiv s_{n;k}(m; k_x) / \max s_{n;k}(m; k_x)$$

$k_x = 0, k_x = \pi$ emergence of scaling law

other momentum: randomly distrubted.

The emergence of scaling law within topological gap regions $m \in (-2,0) \cup (0,2)$, strongly suggest the existence of critical edge states and a fundamental 1/2 mode in the entanglement spectrum, which is the most entangled and correlated mode --**boundary EPR pair**.

The distribution properties of **hyperfine RC highlight the different features** of a mass gap, a critical Dirac cone, and a Fermi surface, and they **reveal an universal scaling behavior in the presence of topological edge states**.

Holographic dual of Rényi contour

The AdS/CFT correspondence and the more general holographic duality provide a novel connection between different theories, one is a higher dimensional gravitational theory, another is a quantum field theory without gravity on the boundary.

The key equation in the AdS/CFT correspondence is

$$Z_{\mathrm{AdS}}\left[\phi_0(\vec{x})\right] = Z_{\mathrm{CFT}}\left[\phi_0(\vec{x})\right] = \left\langle \exp \int d^4x O(\vec{x}) \phi_0(\vec{x}) \right\rangle$$

Important properties:

field/operator duality, **strong/weak duality**.

From the bulk to boundary--studying the strongly coupled systems

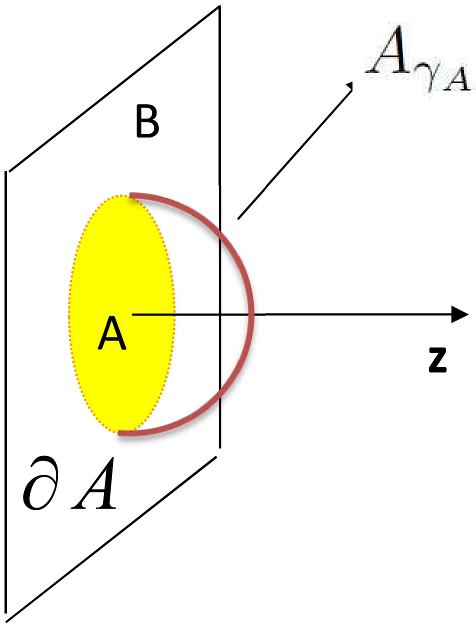
From the boundary to the bulk--an emergent picture of gravity

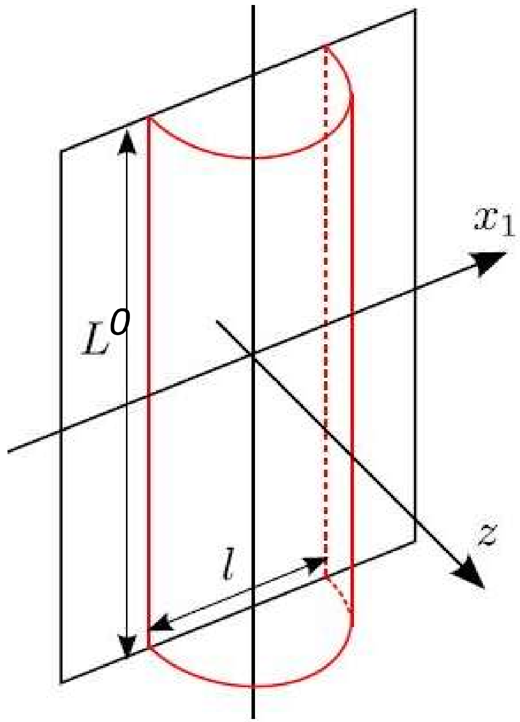
Holographic entanglement entropy

Ryu and Takayanagi 2006

$$S_A = \frac{A_{\gamma_A}}{4G_{d+1}},$$

A_{γ_A} is the *d* dimensional static minimal surface in AdS with boundary ∂A . To calculate entanglement entropy of CFT from the bulk dual gravity.





Lewkowycz and Maldacena 2013

$$S_A = -n\partial_n \left[\ln Z[n] - n \ln Z[1]\right]\Big|_{n=1} = \frac{A_{\gamma_A}}{4G}$$

Bulk reconstruction in AdS/CFT

bulk matter fields:

using the boundary operators to construct the bulk matter fields.

Banks, Douglas, Horowitz, Martinec, th/9808016;
Hamilton, Kabat, Lifschytz, and Lowe, th/0606141.

$$\phi(z,x) = \int dx' K(x'|z,x)\phi_0(x').$$

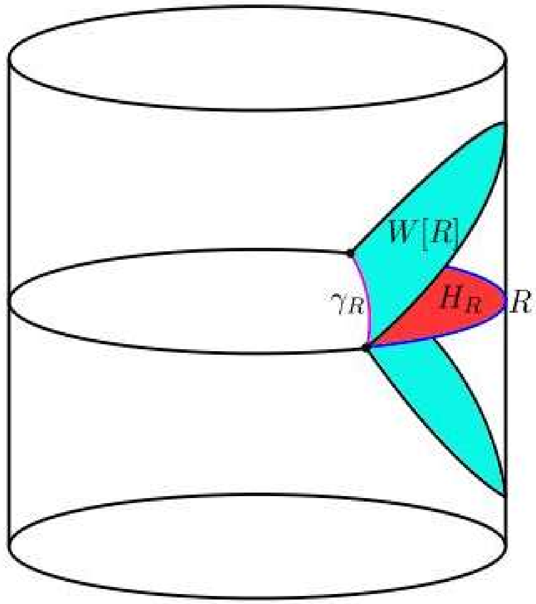
bulk local field ⟺ *boundary nonlocal operators*

Entanglement wedge reconstruction

Headrick, Hubeny, Lawrence, Rangamani, 2014;
Dong, Harlow, Wall, 2016

$$\mathcal{W}_{\mathcal{E}}[\mathcal{A}] := \tilde{D}[\mathcal{R}_{\mathcal{A}}]\,.$$

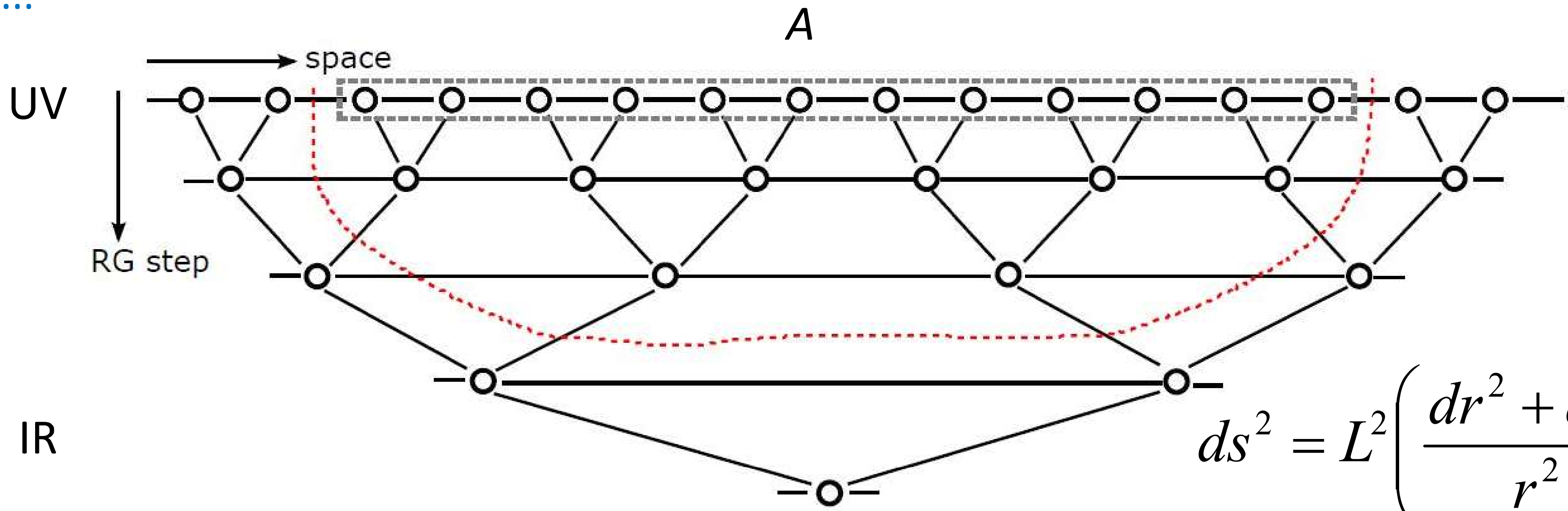
subregion-subregion duality.



It's more difficult to construct the bulk geometry and the gravitational dynamics from the boundary CFT.

Emergence of AdS geometry from MERA tensor networks

Swingle, 0905.1317; 1209.3304;
Qi, 1309.6282;
Almheiri, Dong, Harlow, 1411.7041;
Pastawski, Yoshida, Harlow, Preskill, 1503.06237;
Hayden, Nezami, Qi, Thomas, Walter, Yang, 1601.01694;
Bhattacharyya, Gao, Hung, Liu, 1606.00621;
Gan and Shu, 1705.05750;
Ling, Xiao and Wu, 1907.01215
...



$$ds^2 = L^2\left(\frac{dr^2 + dx^2}{r^2}\right)$$

$S_A \propto$ # of external legs of the tensor networks $\quad s \sim L \ln \frac{x_0}{a}$

Holographic realization of RC

Consider the boundary CFT is a *d*+1 dim Fermi gas,

$$\frac{S_n}{C_2} = \frac{(1+n^{-1})\pi^2}{6} + o(1)$$

then the hyperfine RC can be simplified as

$$\frac{s_n(x)}{h_{n;2}(x)} = 1 + o(1),$$

where

$$h_{n;2}(x) = \beta_k(n) \int_{y \in A} [\langle \hat{n}(x)\hat{n}(y)\rangle - \langle \hat{n}(x)\rangle\langle \hat{n}(y)\rangle] dy.$$

$$I_2(A_1, A_2) = 2\left(S(A_1) - \int_{x \in A_1} h_{1;2}(x) dx \right),$$

for Dirac fermion

$$
\begin{aligned}
&\langle \hat{n}(x)|\hat{n}(y)\rangle - \langle \hat{n}(x)\rangle\langle \hat{n}(y)\rangle \\
&= - \langle \psi_R^\dagger(x)\psi_R(y)\rangle\langle \psi_R^\dagger(y)\psi_R(x)\rangle - \langle \psi_L^\dagger(x)\psi_L(y)\rangle\langle \psi_L^\dagger(y)\psi_L(y)\rangle \\
&= \frac{1}{2\pi^2}\frac{1}{(x-y)^2}.
\end{aligned}
$$

the dominant hyperfine structure is expressed as

$$
\begin{aligned}
h_{n;2}(x) &= \frac{(1+n^{-1})\pi^2}{6}\int_{-R+\epsilon}^{R-\epsilon}\frac{1}{2\pi^2}\frac{1}{(x-y)^2}dy \\
&= \frac{(1+n^{-1})}{12}\left(\frac{1}{R-x}+\frac{1}{R+x}\right).
\end{aligned}
$$

$$
S_n = \int_{-R+\epsilon}^{R-\epsilon} dx\, s_n(x) \approx \left[\left(1+n^{-1}\right)/6\right]\ln\frac{2R}{\epsilon}.
$$

which gives the central charge c=1.

To find the holographic duality of the hyperfine RC, it's convenient to use the **refined Rényi entropy**

$$\widetilde{S}_n \equiv n^2 \partial_n \left(\frac{(n-1)}{n} S_n \right)$$

which is dual to the cosmic brane in AdS spacetime, and the tension of the brane is $T_n = (n-1)/(4nG)$

Dong, 1601.06788

Interesting, **the refined Rényi entropy is equivalent to the von Neumann entropy of a new density matrix** $\widetilde{\rho}_A = \hat{\rho}_A^n / \mathrm{tr}\, \hat{\rho}_A^n$

$$\begin{aligned}
\tilde{S}_n(\hat{\rho}_A) &= -n^2 \partial_n \left(\frac{1}{n} \log \mathrm{tr}\, \hat{\rho}_A^n\right) \\
&= \log \mathrm{tr}\, \hat{\rho}_A^n - n\partial_n \log \mathrm{tr}\, \hat{\rho}_A^n \\
&= -\mathrm{tr} \left(\frac{\hat{\rho}_A^n}{\mathrm{tr}\, \hat{\rho}_A^n}\right) \log \left(\frac{\hat{\rho}_A^n}{\mathrm{tr}\, \hat{\rho}_A^n}\right) \\
&= S\left(\frac{\hat{\rho}_A^n}{\alpha_n}\right)
\end{aligned}$$

Then we obtain **the refined Rényi contour**

$$\widetilde{s}_n(x) = n^2 \partial_n \left(\frac{(n-1)}{n} s_n(x) \right)$$

Furthermore, using the entanglement Hamiltonian, **the refined Rényi contour** can be expressed from the particle number fluctuation

$$\hat{\rho}_A = \sum_p a_p^2 |\psi_A^p\rangle\langle\psi_A^p| = e^{-\hat{K}_A}$$

Entanglement Hamiltonian: $\hat{K}_A = \sum_p - \ln a_p^2 |\psi_A^p\rangle\langle\psi_A^p|$

$$\tilde{\rho}_A^{(n)} = \sum_p \frac{a_p^{2n}}{\alpha_n} |\psi_A^p\rangle\langle\psi_A^p| = \frac{e^{-K_A^{(n)}}}{Z^{(n)}}$$

Entanglement Hamiltonian: $\hat{K}_A^{(n)} = n\hat{K}_A$ with $T = \frac{1}{n}$.

Then we obtain

$$\tilde{s}_n(x) \approx \tilde{h}_{n;2}(x) = \frac{\pi^2}{3n} \int_{y \in A} dy [\langle \hat{n}(x) \hat{n}(y) \rangle - \langle \hat{n}(x) \rangle \langle \hat{n}(y) \rangle].$$

Now using the HEE, the holographic duality for **Rényi entropy** is just the bulk extremal surface (RT surface) for **the refined Rényi entropy.**

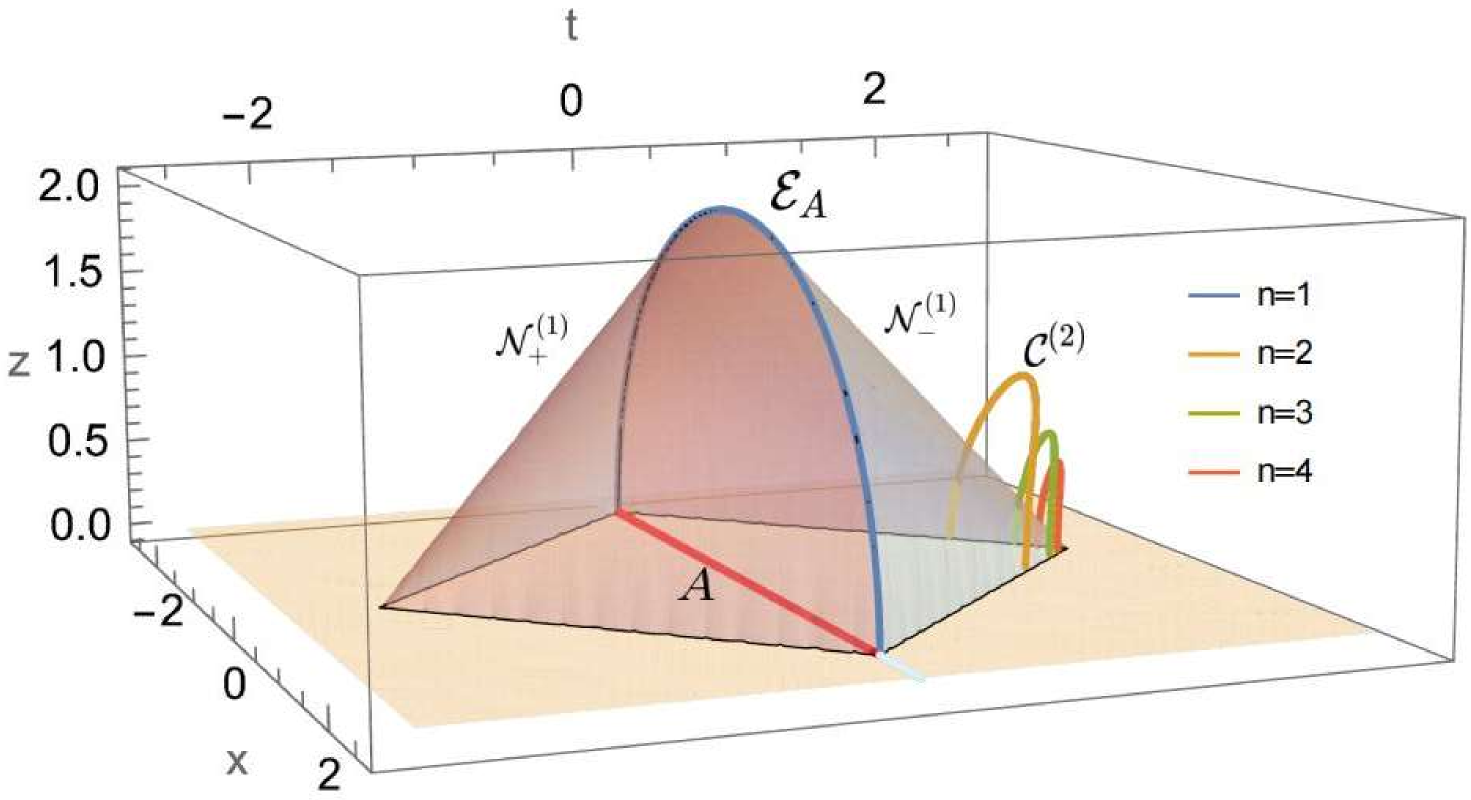
For AdS_3 case, using the Rindler method to map the extremal surface of subregion *A* to the horizon area entropy

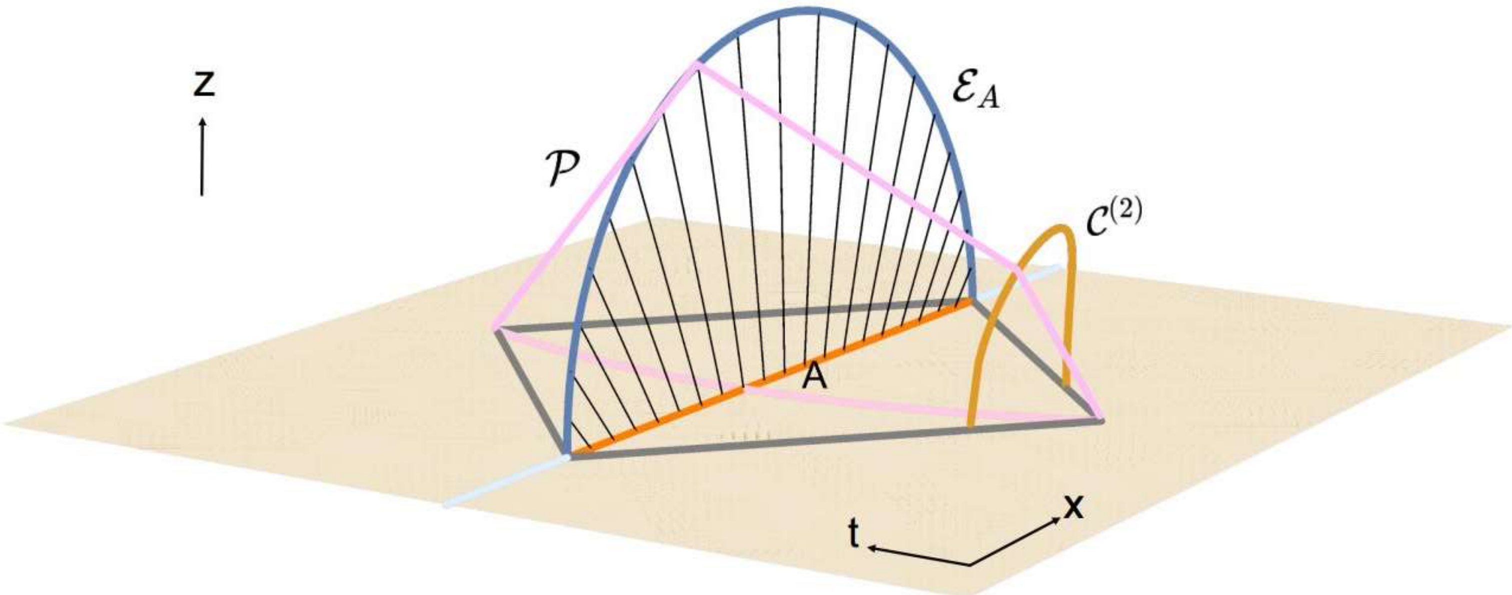
$$ds^2 = 2r du dv + \frac{dr^2}{4r^2}.$$

$$ds^2 = du^{*2} + r^{*2} du^* dv^* + dv^{*2} + \frac{dr^{*2}}{4(r^{*2} - 1)},$$

$$\mathcal{N}_+^{(n)} : \quad r = \frac{-2n^2}{l_u^2(n^2 - 2) + 4n^2 uv + 2l_u \sqrt{l_u^2(1 - n^2) - 4n^2 uv + n^4 (u + v)^2)}}$$

$$\mathcal{N}_-^{(n)} : \quad r = \frac{-2n^2}{l_u^2(n^2 - 2) + 4n^2 uv + 2l_u \sqrt{l_u^2(1 - n^2) - 4n^2 uv + n^4 (u + v)^2)}}$$





As n increases, $\mathcal{C}^{(n)}$ decreases.

For n>1, $\mathcal{C}^{(n)}$ are outside the entanglement wedge of the n=1 extremal surface (EE), which means the Rényi entanglement wedge can probe more information of the bulk spacetime.

Conclusions and Discussions

*We derive the hyperfine structure of Rényi contour from particle number cumulants for free fermions;

*The hyperfine Rényi contour shows many interesting features: such as can be used to characterized the topological edge states;

*The holographic duality of the hyperfine Rényi contour, the Rényi entanglement wedge give new tool to study the bulk reconstruction and more refined description for subregion-subregion duality.

*We also proposed an experiment to probe the hyperfine Rényi contour.

*Future works: application in quantum information, more general systems, higher dimensional holographic duality...

A main motivation of study entanglement contour is to explore the fine structure of entanglement.

Recently we proposed a new surface growth approach for bulk reconstruction, which also aims to probe the fine structure of entanglement in subregions, it may connect with the entanglement contour description.