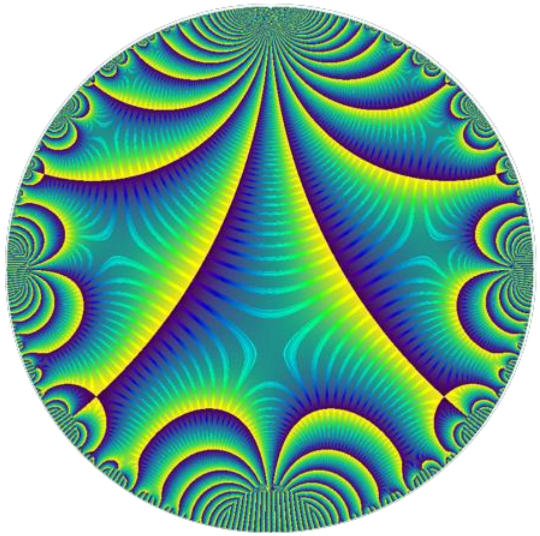
Modular Cosmology



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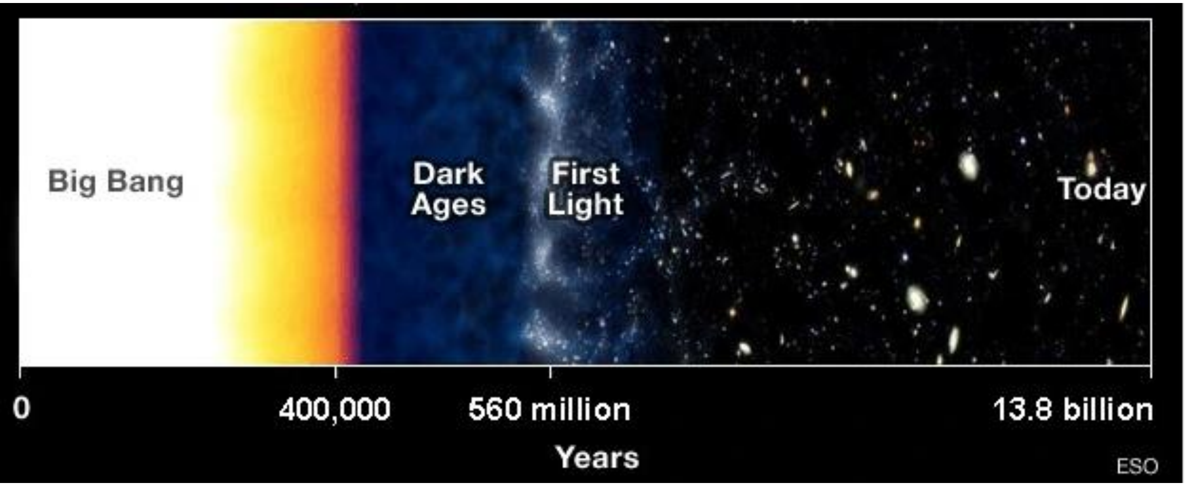
“2024引力与宇宙学”专题研讨会,
彭桓武高能基础理论研究中心（合肥），2024年11月16日

based on: Gui-Jun Ding, Si-Yi Jiang, Wenbin Zhao, arXiv:2405.06497, JCAP 10 (2024) 016；
Gui-Jun Ding, Si-Yi Jiang, Yong Xu, Wenbin Zhao, arXiv:2411.xxxxx (in preparation)

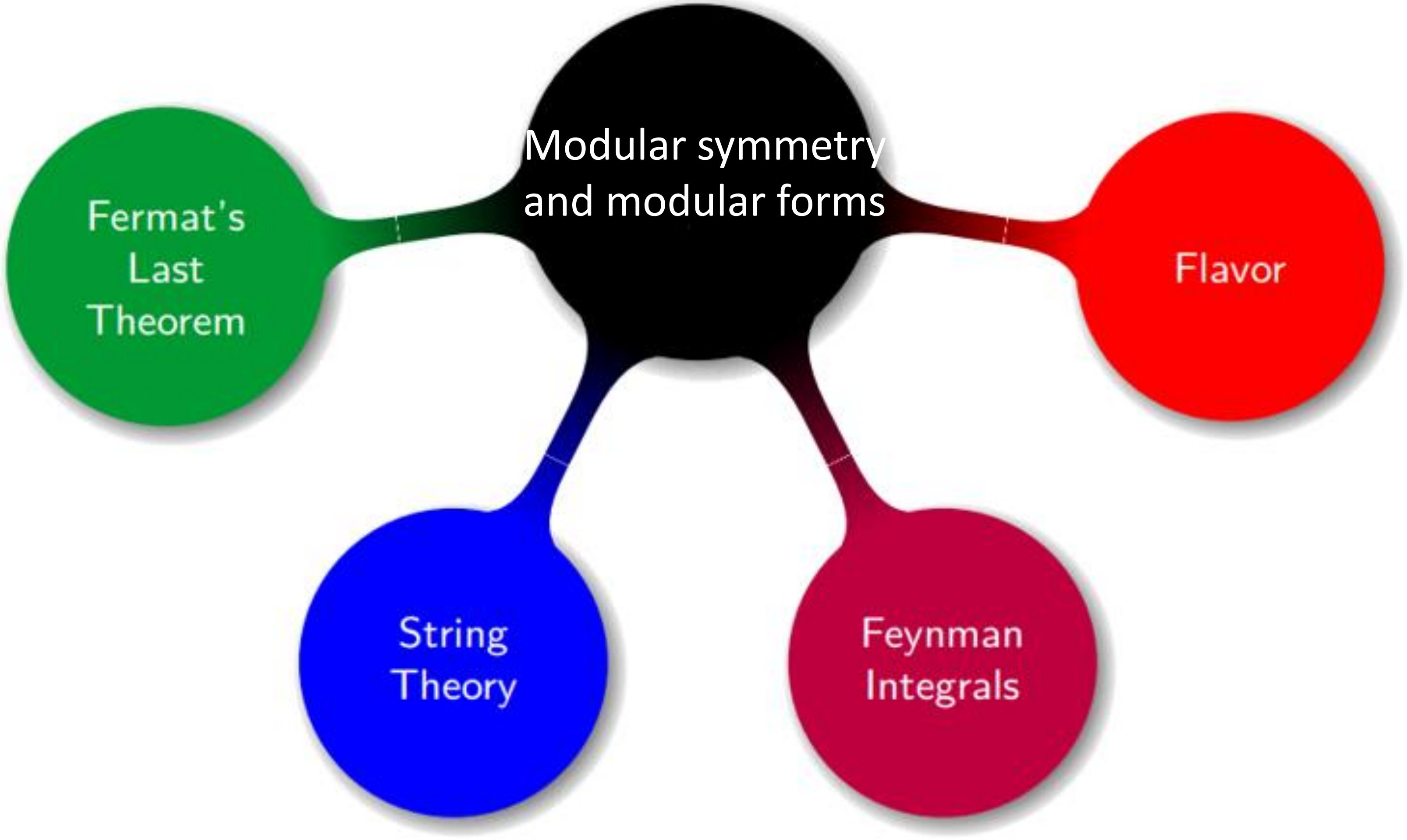


彭桓武高能基础理论研究中心（合肥）
Peng Huanwu Center for Fundamental Theory

Outline:

- Introduction to modular symmetry
- Modular inflation
- Reheating from inflaton decay
- Baryon asymmetry from non-thermal leptogenesis
- Summary

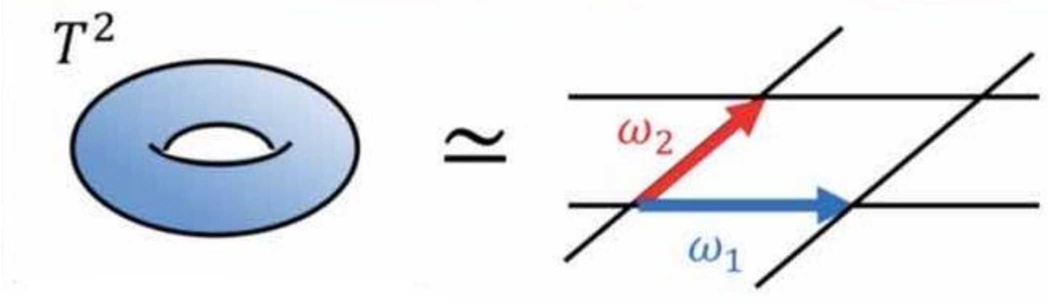
Modular symmetry and modular forms are well-known in Mathematics and some areas of physics.

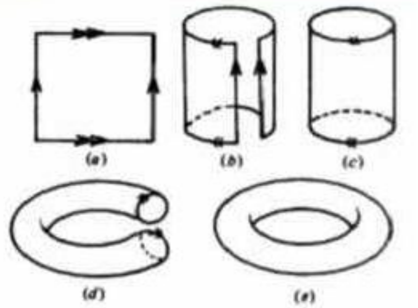


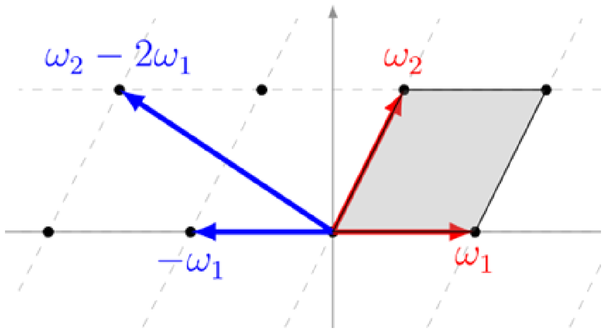
Modular symmetry

Modular symmetry $SL(2,Z)$ is the geometrical symmetry of the torus T^2

Torus $T^2 \cong \mathbb{C}/\Lambda_{(\omega_1,\omega_2)}$, lattice $\Lambda_{(\omega_1,\omega_2)} = n_1\omega_1 + n_2\omega_2$







Up to rotations and dilations, the shape of a torus is parameterized by $\tau = \omega_2/\omega_1, \mathrm{Im}\tau > 0$

The lattice (torus) is left **invariant** by $SL(2,\mathbb{Z})$ modular transformation

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \Rightarrow \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

$$SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

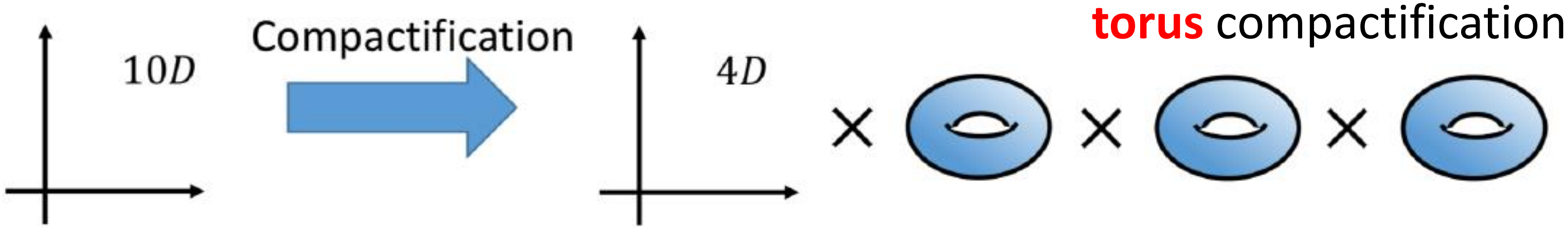
modular generators:

(**duality**) $S: \tau \to -\frac{1}{\tau}$

(**shift**) $T: \tau \to \tau + 1$

Effective 4D theories are modular invariant:

torus compactification



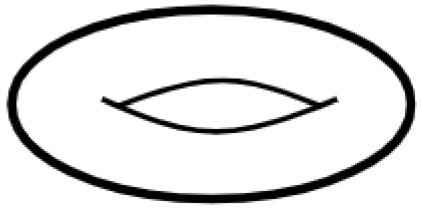
$$S = \int d^4x d^6y\ \mathcal{L}_{10D} \Rightarrow \int d^4x\ \mathcal{L}_{\rm eff}(\varphi, \tau_i)$$

Modular invariant SUSY theory

- Modular action

[Lauer, Mas, Nilles, 1989; Ferrara, Lust et al, 1989; Feruglio, 1706.08749]

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$



$SL(2,\mathbb{Z})$ on torus T^2

- Field **Non-linear** transformation

$$\psi \to (c\tau + d)^{-k} \rho(\gamma) \psi$$

weight $k \in \mathbb{Z}$ ρ is a unitary representation of Γ_N or Γ'_N

finite modular groups as G_f

- Superpotential

$$\mathcal{W} = \sum Y_{I_1 I_2 \ldots I_n}(\tau)\, \psi_{I_1} \psi_{I_2} \ldots \psi_{I_n}$$

$$G_f = \begin{cases} \Gamma_N \equiv SL(2,\mathbb{Z}) / \pm\Gamma(N) \\ \Gamma'_N \equiv SL(2,\mathbb{Z}) / \Gamma(N) \end{cases}$$

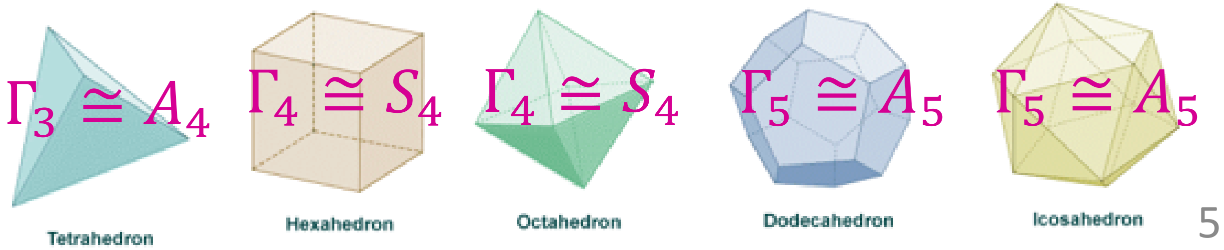
Yukawa coupling $Y_{I_1 I_2 \ldots I_n}(\tau)$ only depends on τ, and it is modular form of level N:

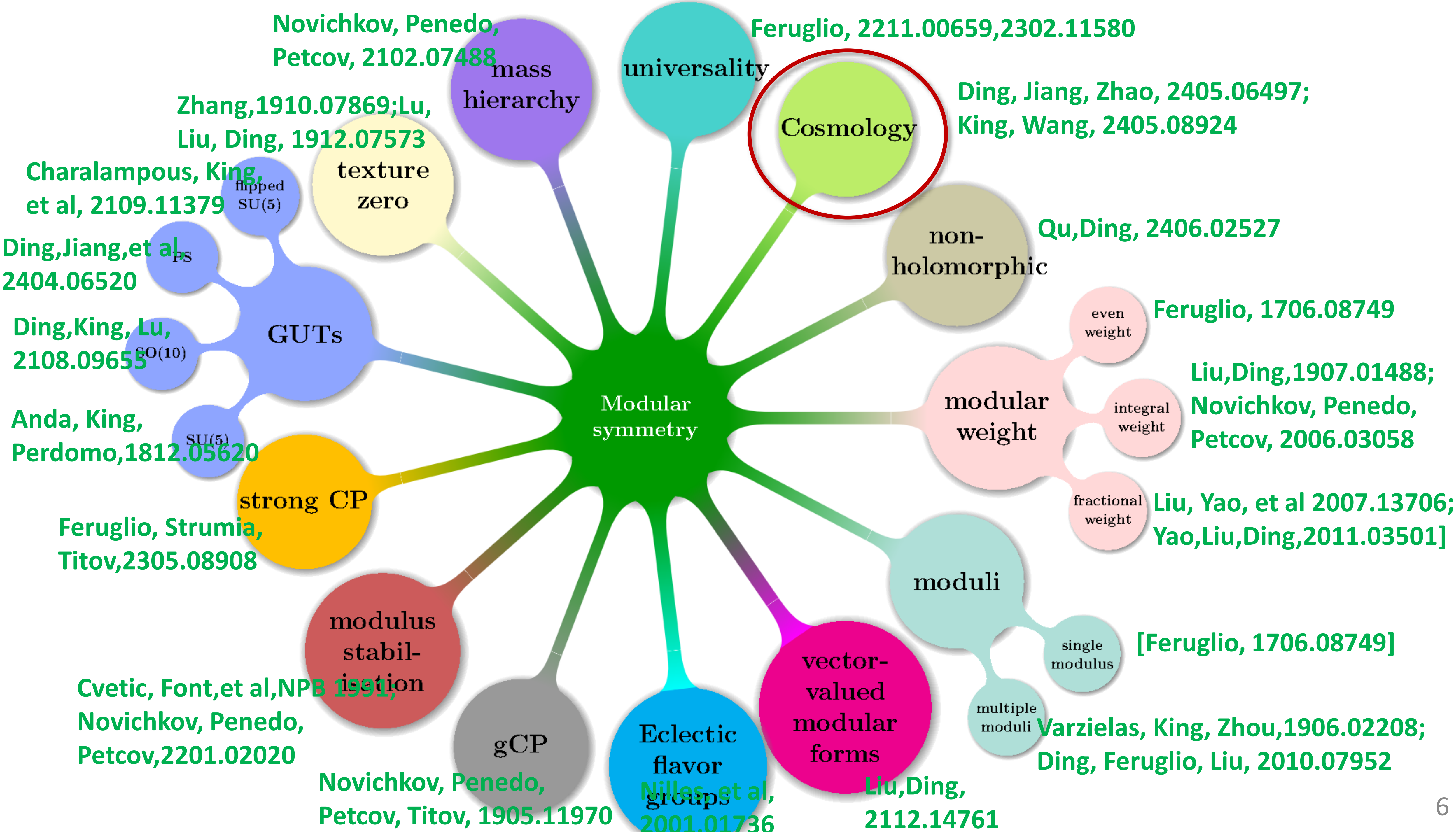
Principal congruence subgroup of level N

$$\Gamma(N) = \{\gamma \in SL(2,\mathbb{Z}) \,|\, \gamma = 1_2 \text{ mod } N\}$$

$$Y_{I_1 I_2 \ldots I_n}(\tau) \to Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$

[review: Ding, King, arXiv:2311.09282, Rept.Prog.Phys (2024)]

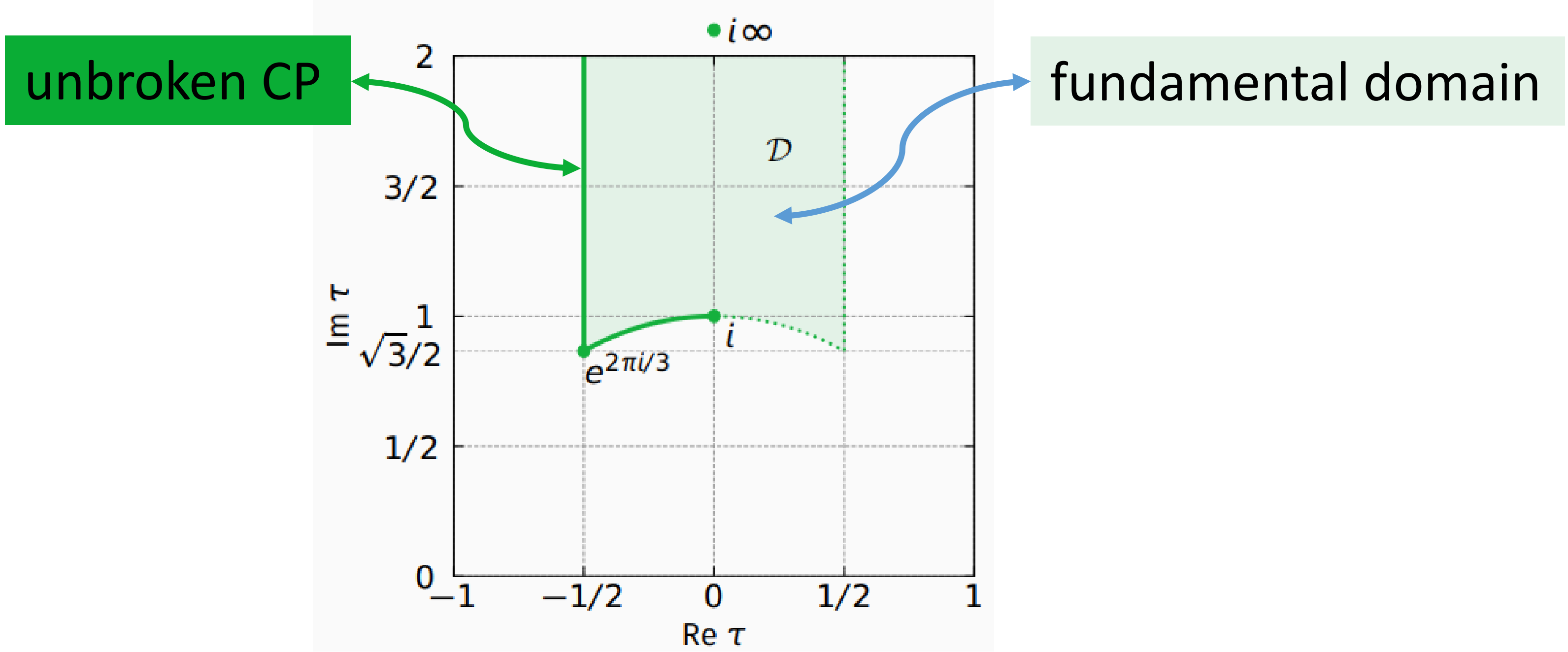




Modular symmetry
mass hierarchy
Novichkov, Penedo, Petcov, 2102.07488
universality
Feruglio, 2211.00659,2302.11580
Cosmology
Ding, Jiang, Zhao, 2405.06497; King, Wang, 2405.08924
non-holomorphic
Qu,Ding, 2406.02527
modular weight
even weight
Feruglio, 1706.08749
integral weight
Liu,Ding,1907.01488; Novichkov, Penedo, Petcov, 2006.03058
fractional weight
Liu, Yao, et al 2007.13706; Yao,Liu,Ding,2011.03501]
moduli
single modulus
[Feruglio, 1706.08749]
multiple moduli
Varzielas, King, Zhou,1906.02208; Ding, Feruglio, Liu, 2010.07952
vector-valued modular forms
Liu,Ding, 2112.14761
Eclectic flavor groups
Nilles, et al, 2001.01736
gCP
Novichkov, Penedo, Petcov, Titov, 1905.11970
modulus stabil-isation
Cvetic, Font,et al,NPB 1991; Novichkov, Penedo, Petcov,2201.02020
strong CP
Feruglio, Strumia, Titov,2305.08908
GUTs
SU(5)
Anda, King, Perdomo,1812.05620
SO(10)
Ding,King, Lu, 2108.09655
PS
Ding,Jiang,et al, 2404.06520
flipped SU(5)
Charalampous, King, et al, 2109.11379
texture zero
Zhang,1910.07869;Lu, Liu, Ding, 1912.07573

The complex modulus τ

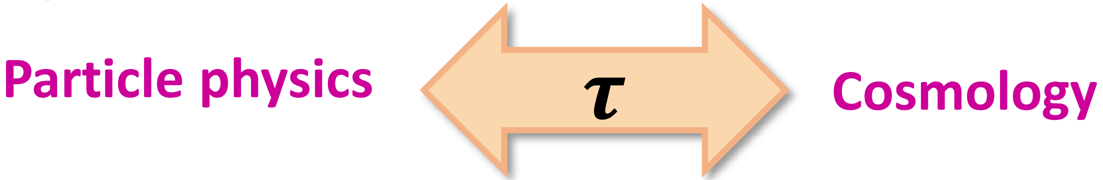
- Modular symmetry can be thought as a gauge symmetry. With a gauge choice τ can be restricted to the **fundamental region**,



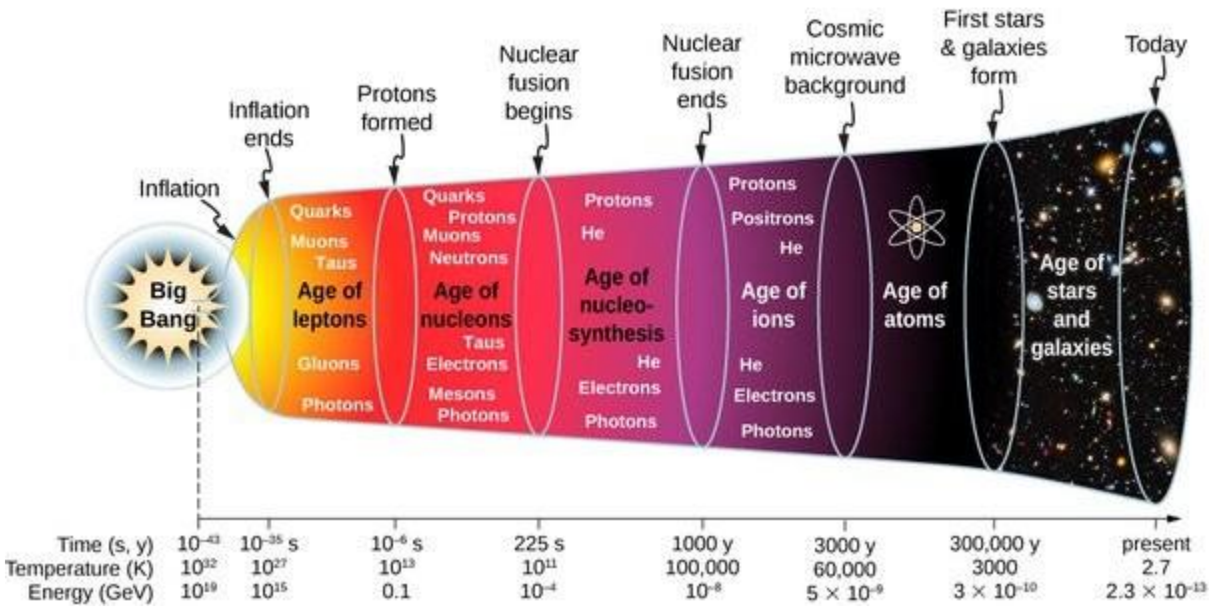
- Modular symmetry is broken by the vacuum expectation value of τ

• Bottom-up approach: $\langle\tau\rangle$ is treated as a free parameter and its value is obtained by fitting the data

• Top-down approach: modulus stabilization, $\langle\tau\rangle$ is the global minimum of the modular invariant scalar potential for τ



Modular inflation



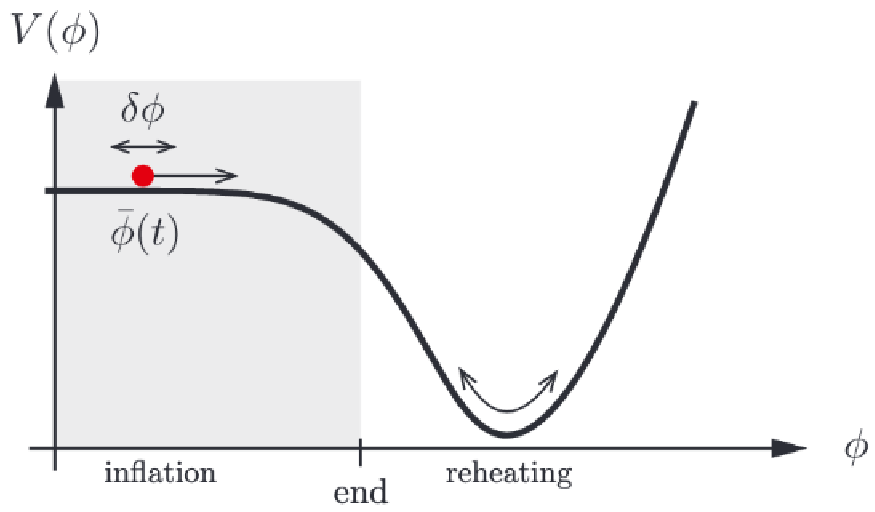
Accelerated expansion of the early universe favored by cosmic microwave background (CMB) observations

$$
\begin{aligned}
\ln\left(10^{10} A_s\right) &= 3.044 \pm 0.014\,(68\%\mathrm{CL})\,, \\
n_s &= 0.9649 \pm 0.0042\,(68\%\mathrm{CL})\,, \\
r &< 0.036\,(95\%\mathrm{CL})\,.
\end{aligned}
$$

A_s: scalar amplitude
n_s: spectral index
r: tensor-to-scalar ratio

[Planck Collaboration,1807.06211]

- Inflation can be realized by a slow-rolling scalar field (inflaton)



SM cannot explain inflation!

Modular inflation: modulus τ plays the role of inflaton, a plateau in the scalar potential is necessary

A cosmological probe of modular symmetry!

General modular invariant scalar potential for τ

- The **modular invariant** effective action in SUGRA for $\tau + S$ (dilaton) [Cvetic et al., Nucl. Phys. B 361 (1991)]

$$\mathcal{G}(\tau,\bar{\tau},S,\bar{S}) = \mathcal{K}(\tau,\bar{\tau},S,\bar{S}) + \ln|\mathcal{W}(\tau,S)|^2$$

$$\mathcal{G}(\tau,\bar{\tau},S,\bar{S}) \xrightarrow{\gamma} \mathcal{G}(\tau,\bar{\tau},S,\bar{S})$$

Kähler potential:

$$\mathcal{K}(\tau,\bar{\tau},S,\bar{S}) = K(S,\bar{S}) - 3\ln[-i(\tau-\bar{\tau})]$$

$$K(S,\bar{S}) = -\ln(S+\bar{S}) + \delta k(S,\bar{S})$$

Tree level ($-\ln(S+\bar{S})$) — **Non-perturbative effects: important to stabilize dilaton S** ($\delta k(S,\bar{S})$)

Superpotential:

$$\mathcal{W}(\tau,S) = \Lambda_W^3 \frac{\Omega(S)H(\tau)}{\eta^6(\tau)}, \quad H(\tau) = \sum_{m,n} (j(\tau)-1728)^{m/2} j(\tau)^{n/3} \mathcal{P}_{m,n}(j(\tau))$$

- $\eta(\tau)$:Dedekind eta function with modular weight 1/2; $j(\tau)$: Klein j-invariant
- $\Omega(S)$ is technically arbitrary, $\mathcal{P}_{m,n}(j)$ is an arbitrary polynomial function of $j(\tau)$

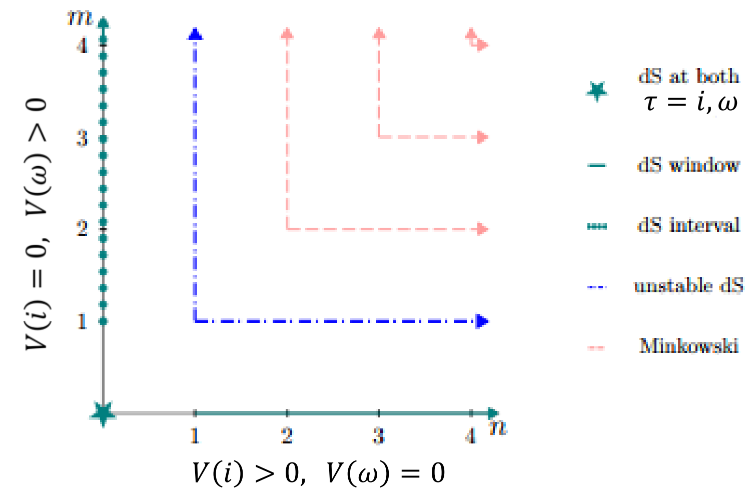
Scalar potential:

$$V(\tau,S) = \Lambda^4 e^{K(S,\bar{S})}|\Omega(S)|^2 Z(\tau,\bar{\tau})\left[(A(S,\bar{S})-3)|H(\tau)|^2 + \hat{V}(\tau,\bar{\tau})\right]$$

$$Z(\tau,\bar{\tau}) = \frac{1}{i(\tau-\bar{\tau})^3|\eta(\tau)|^{12}}, \quad A(S,\bar{S}) = \frac{K^{S\bar{S}}|\Omega_S + K_S\Omega|^2}{|\Omega|^2}, \quad \hat{V}(\tau,\bar{\tau}) = -\frac{(\tau-\bar{\tau})^2}{3}\left|H_\tau(\tau) - \frac{3i}{2\pi}H(\tau)\hat{G}_2(\tau,\bar{\tau})\right|$$

Free parameters: $A(S,\bar{S})$, (m,n), $\mathcal{P}_{m,n}(j)$

➢ Vacuum structure of modulus at fixed points $\tau = i,\ e^{2\pi i/3} \equiv \omega$



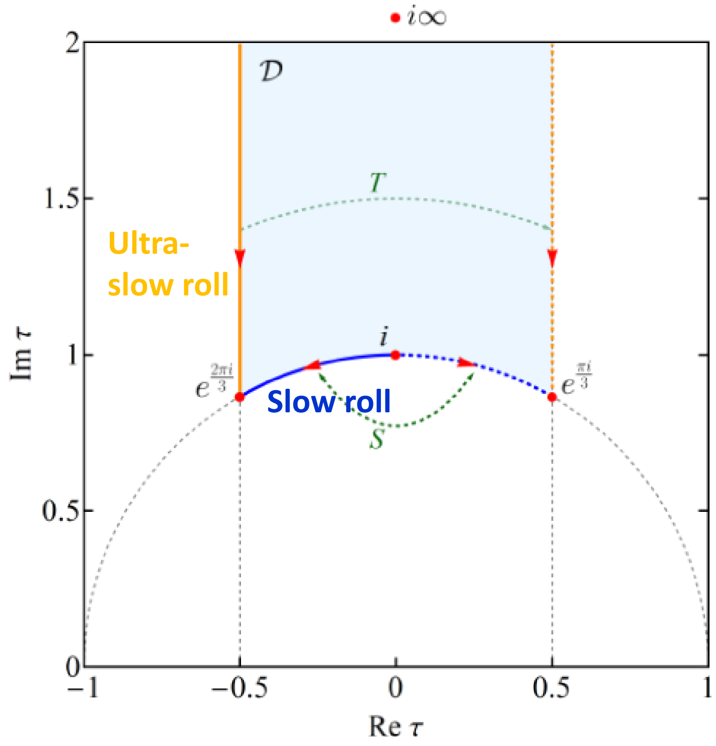
[Leedom, Righi,Westphal,2212.03876]

τ rolls toward the global minimum

➢ **Slow roll along the boundary of fundamental domain**

Boundary of fundamental domain is the **"valle"** of the scalar potential!

- $m = 0, n \geq 2$, slow roll from i (saddle point) to ω (Minkowski minimum) along the unit arc [Ding, Jiang, Zhao, 2405.06497]
- $m \geq 2, n \geq 2$, ultra-slow roll from $i\infty$ to ω (Minkowski minimum) along the left boundary [Ding, Jiang, Zhao, 2405.06497]
- $m = 0, n = 0$, slow roll from i (saddle point) to ω (dS minimum) along the unit arc [King, Wang, 2405.08924]



modular symmetry⇒ $\epsilon_V = \frac{1}{2}\left(\frac{V'}{V}\right)^2 \ll 1,\ \eta_V = \frac{V''}{V} \ll 1$

Slow roll in the vicinity of τ=i

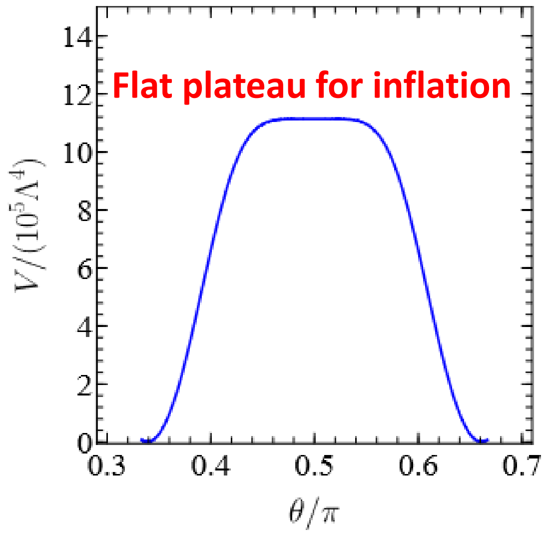
- Canonical normalization $\tau = \rho e^{i\theta}$

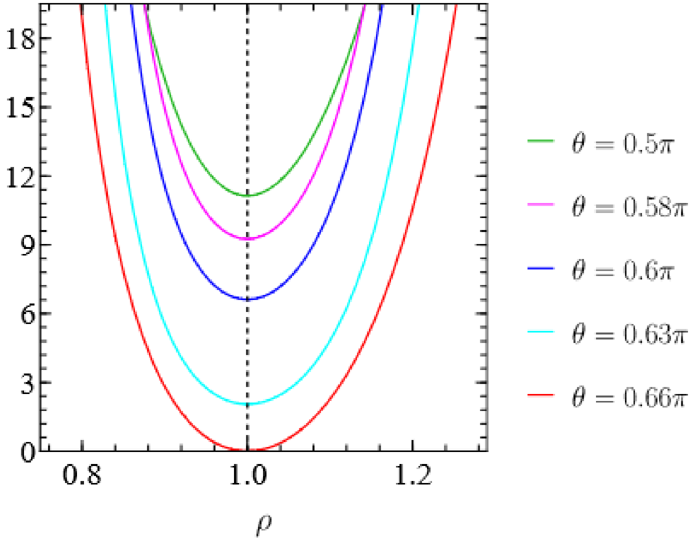
$\phi = \sqrt{3/2}\,\log(\tan(\theta/2))$

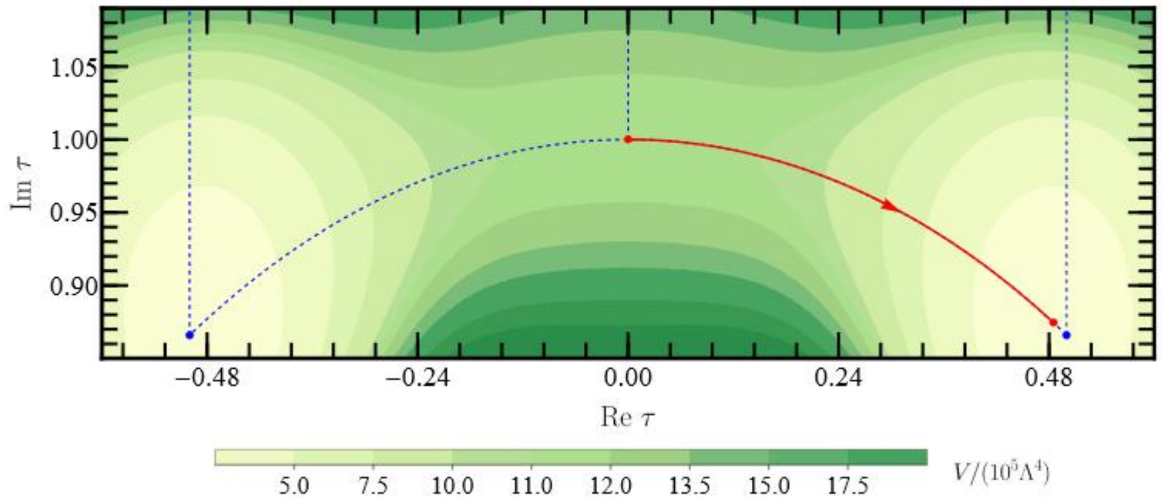
$$\mathcal{L}_{\rm kin} = \frac{3}{(-i\tau + i\,\bar\tau\,)^2}\partial_\mu\tau\partial^\mu\bar\tau = \frac{3}{4\sin^2\theta}\left(\frac{1}{\rho^2}\partial_\mu\rho\partial^\mu\rho + \partial_\mu\theta\partial^\mu\theta\right) \xrightarrow{\rho=1} \mathcal{L}_{\rm kin} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

- Hilltop-like potential: $V(\phi) = V_0[1 - C_2\phi^2 - C_4\phi^4 - C_6\phi^6 + \cdots]$

Invariant under $\phi \xrightarrow{S} -\phi$







Example: $H(\tau) = \left(j^{1/3}(\tau) - j^{1/3}(\tau_0)\right)^2 [1 + \beta(1 - j(\tau)/1728)] \Rightarrow m = 0, n = 0,1,2$

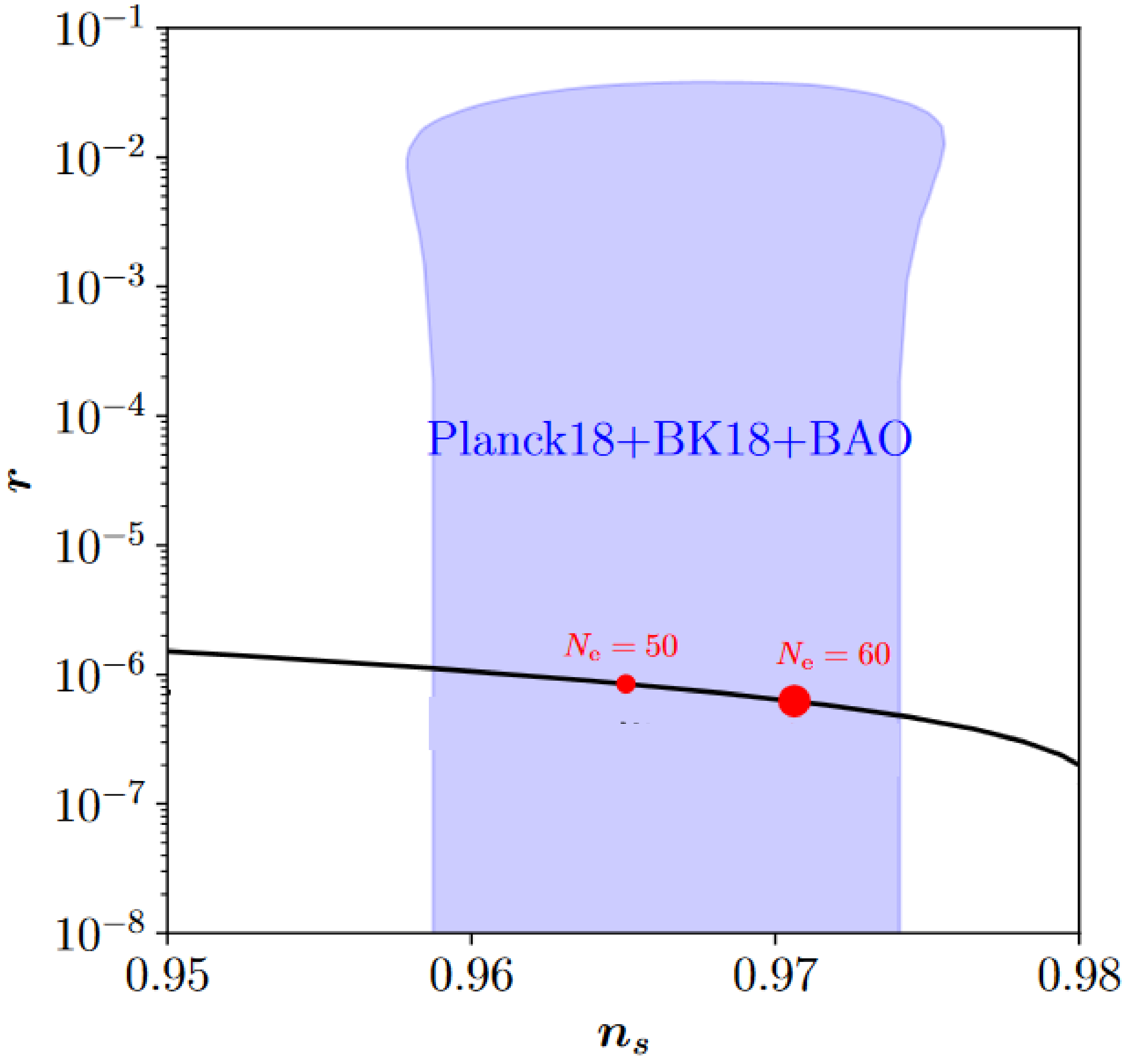
τ_0 is global minimum of the scalar potential

[Ding, Jiang, Xu, Zhao, 2411.xxxxx]

$A = 24.4895,\ \ \beta = 0.1321 \Rightarrow N_e \simeq 50.807,\ \ r \simeq 8.6\times10^{-7},\ \ n_s \simeq 0.9649,\ \ \alpha = -7.3\times10^{-4},$

inflaton mass: $m_\phi = 1.5\times10^{12}$GeV

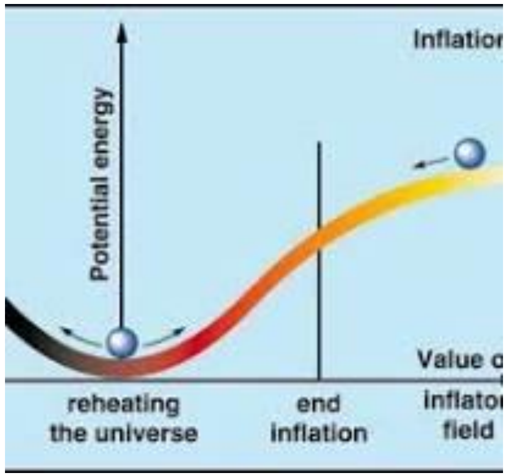
- Successful inflation can be reproduced: the tensor-to-scalar ratio $r \sim \mathcal{O}(10^{-6})$ and a negative running $\alpha \sim \mathcal{O}(-10^{-3})$ testable by future CMB measurements [Ding, Jiang, Xu, Zhao, 2411.xxxxx]



[other scenarios of modular inflation, see Gunji, Ishiwata, Yoshida, 2208.10086; Abe, Higaki, Kaneko, Kobayashi, Otsuka, 2303.02947;Kallosh,Linde, 2408.05203; Casas, Ibanez, 2409.15823; Kallosh,Linde, 2411.07552...]

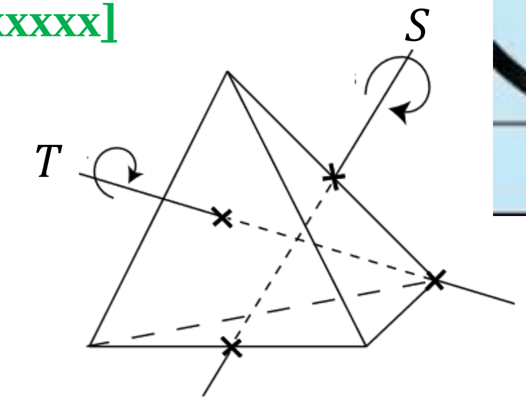
Post inflation: reheating

The Universe is reheated through the inflaton decays into SM particles. When modulus τ is the inflaton, its couplings with SM fields are determined by modular symmetry. Generally it has the largest coupling with RH neutrinos.



- modular model based on $\Gamma_3 \cong A_4$ [Ding, Jiang, Xu, Zhao, 2411.xxxxx]

	L	$\{e^c, \mu^c, \tau^c\}$	N^c	H_u	H_d
$SU(2)_L \times U(1)_Y$	$(2, -1/2)$	$(1,1)$	$(1,0)$	$(2,1/2)$	$(2,-1/2)$
A_4	$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}''$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$
k_I	1	$\{1,5,5\}$	1	0	0



- Modular invariant interactions with leptons

Charged leptons:

$$\mathcal{W}_\ell = y_1 e^c \left(LY_3^{(2)}(\tau)\right)_{1''} H_d + y_2 \mu^c \left(LY_{3I}^{(6)}(\tau)\right)_{1''} H_d + y_3 \mu^c \left(LY_{3II}^{(6)}(\tau)\right)_{1''} H_d + y_4 \tau^c \left(LY_{3I}^{(6)}(\tau)\right)_{1'} H_d + y_5 \tau^c \left(LY_{3II}^{(6)}(\tau)\right)_{1'} H_d$$

Neutrinos:

$$\mathcal{W}_\nu = g_1 \left((N^c L)_{3_S} Y_3^{(2)}(\tau)\right)_1 H_u + g_2 \left((N^c L)_{3_A} Y_3^{(2)}(\tau)\right)_1 H_u + \Lambda \left((N^c N^c)_{3_S} Y_3^{(2)}(\tau)\right)_1$$

The interactions with RH neutrinos are strong constrained by modular symmetry!

$$\mathcal{Y}_D = \begin{pmatrix} 2g_1 Y_{3,1}^{(2)}(\tau) & -g_1 Y_{\mathbf{3,3}}^{(2)}(\tau) - g_2 Y_{3,3}^{(2)}(\tau) & -g_1 Y_{3,2}^{(2)}(\tau) + g_2 Y_{3,2}^{(2)}(\tau) \\ -g_1 Y_{3,3}^{(2)}(\tau) + g_2 Y_{3,3}^{(2)}(\tau) & 2g_1 Y_{3,2}^{(2)}(\tau) & -g_1 Y_{3,1}^{(2)}(\tau) - g_2 Y_{3,1}^{(2)}(\tau) \\ -g_1 Y_{\mathbf{3},2}^{(2)}(\tau) - g_2 Y_{3,2}^{(2)}(\tau) & -g_1 Y_{3,1}^{(2)}(\tau) + g_2 Y_{3,1}^{(2)}(\tau) & 2g_1 Y_{3,3}^{(2)}(\tau) \end{pmatrix}$$

$$\mathcal{Y}_N = \begin{pmatrix} 2Y_{3,1}^{(2)}(\tau) & -Y_{\mathbf{3,3}}^{(2)}(\tau) & -Y_{3,2}^{(2)}(\tau) \\ -Y_{3,3}^{(2)}(\tau) & 2Y_{3,2}^{(2)}(\tau) & -Y_{3,1}^{(2)}(\tau) \\ -Y_{\mathbf{3},2}^{(2)}(\tau) & -Y_{3,1}^{(2)}(\tau) & 2Y_{3,3}^{(2)}(\tau) \end{pmatrix}$$

- At the global minimum $\tau = \tau_0 = 0.485 + 0.875i$ ⇒lepton masses and mixing

$\left.\begin{array}{c} \sin^2\theta_{12} = 0.307,\ \ \sin^2\theta_{13} = 0.02224,\ \ \sin^2\theta_{23} = 0.454, \delta_{CP} = 1.145\pi,\ \alpha_{21} = 1.062\pi, \\ \alpha_{31} = 1.729\pi, m_1 = 25.725\ \text{meV}, m_2 = 27.127\ \text{meV}\ , m_3 = 56.274\ \text{meV}\ , m_{\beta\beta} = 9.615\ \text{meV} \end{array}\right\}$ within 1σ

↪Quasi-degenerate heavy neutrino masses: $(M_1, M_2, M_3) = (1.37, 1.45, 2.82)\Lambda$, seesaw scale Λ is free

↪ CP violation source: complex couplings y_3, y_5, g_2

- Quantum fluctuations around global minimum $\tau = \tau_0 + \delta\tau \Rightarrow$ reheating

Expand the modular invariant interactions around τ_0:

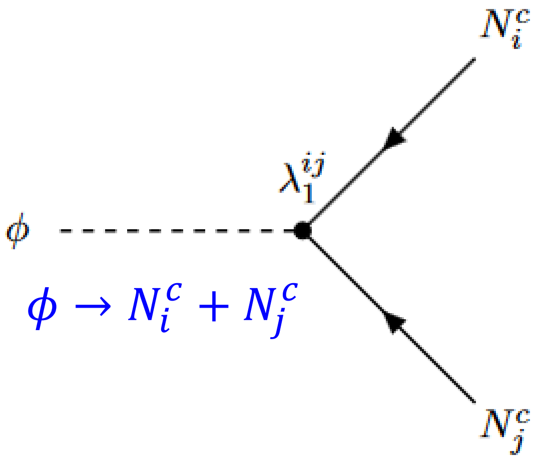
$$\lambda_1^{ij} = \frac{d\mathcal{Y}_N^{ij}(\tau)}{d\tau}\frac{d\tau}{d\phi}\bigg|_{\phi=\phi_0}$$

$$\lambda_2^{ij} = \frac{d\left(\mathcal{Y}_N^T(\tau)\times\mathcal{Y}_N^*(\tau)\right)^{ij}}{d\tau}\frac{d\tau}{d\phi}\bigg|_{\phi=\phi_0}$$

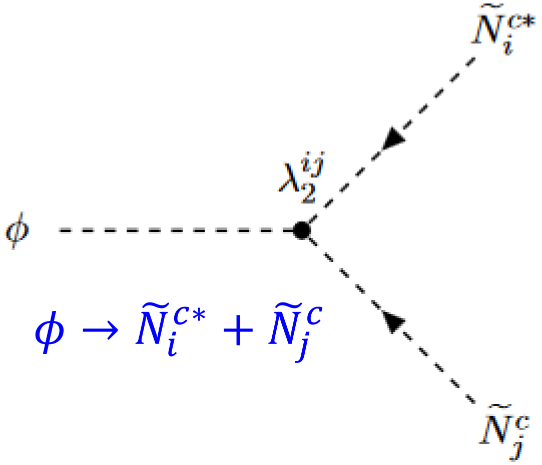
Inflaton-(s)neutrino-(s)neutrino interaction:

$$\mathcal{L} = \frac{\Lambda}{2M_{\rm pl}}\lambda_1^{ij}\phi N_i^c N_j^c + \frac{\Lambda^2}{2M_{\rm pl}}\lambda_2^{ij}\tilde{N}_i^{c*}\tilde{N}_j^c + \text{h.c.}$$

2-body decays:



$\phi \to N_i^c + N_j^c$



$\phi \to \tilde{N}_i^{c*} + \tilde{N}_j^c$

$$\Gamma(\phi \to N_i^c N_j^c) = \frac{m_\phi \Lambda^2}{8(1+\delta_{ij})\pi M_{\rm pl}^2}\left[|\lambda_1^{ij}|^2\left(1-\frac{M_i^2+M_j^2}{m_\phi^2}\right) - 2{\rm Re}[(\lambda_1^{ij})^2]\frac{M_i M_j}{m_\phi^2}\right] \times \sqrt{\left(1-\frac{(M_i-M_j)^2}{m_\phi^2}\right)\left(1-\frac{(M_i+M_j)^2}{m_\phi^2}\right)}$$

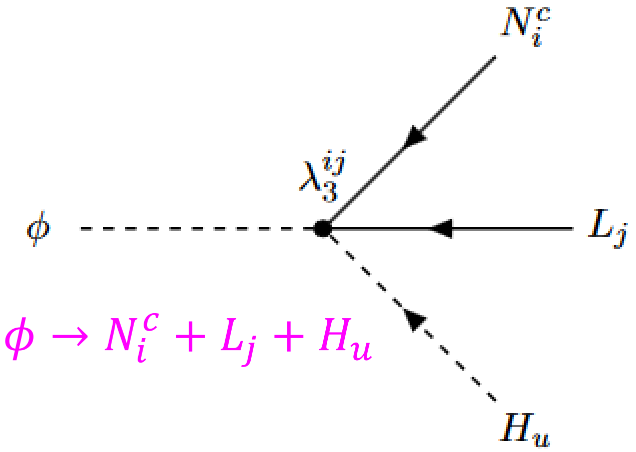
$$\Gamma(\phi \to \tilde{N}_i^c \tilde{N}_j^{c*}) = \left|\frac{\Lambda^2\lambda_2^{ij}}{M_{\rm pl}}\right|^2 \frac{1}{16\pi m_\phi}\sqrt{\left(1-\frac{(M_i-M_j)^2}{m_\phi^2}\right)\left(1-\frac{(M_i+M_j)^2}{m_\phi^2}\right)}$$

Inflaton-lepton-neutrino-Higgs interactions:

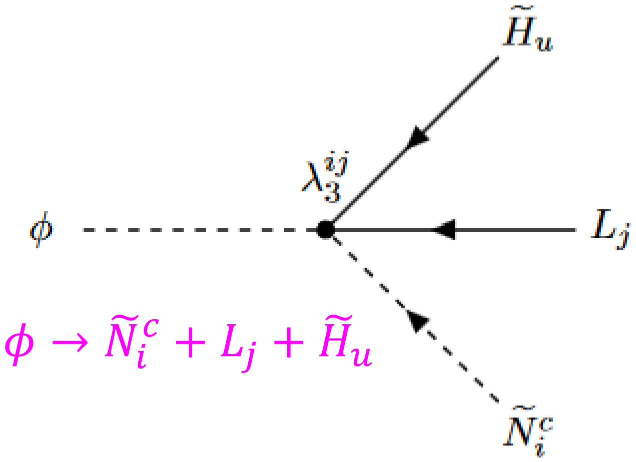
$$\lambda_3^{ij} = \frac{d\mathcal{Y}_D^{ij}(\tau)}{d\tau}\frac{d\tau}{d\phi}\bigg|_{\phi=\phi_0} \qquad \lambda_4^{ij} = \frac{d\left(\mathcal{Y}_N^{\dagger}(\tau)\times\mathcal{Y}_D(\tau)\right)^{ij}}{d\tau}\frac{d\tau}{d\phi}\bigg|_{\phi=\phi_0}$$

$$\mathcal{L} = \frac{1}{M_{\rm pl}}\lambda_3^{ij}\phi N_i^c\left(L_j\cdot H_u\right) + \frac{1}{M_{\rm pl}}\lambda_3^{ij}\phi\,\widetilde{N}_i^c\left(L_j\cdot\widetilde{H}_u\right) + \frac{1}{M_{\rm pl}}\lambda_3^{ij}\phi\,N_i^c\left(\widetilde{L}_j\cdot\widetilde{H}_u\right) + \frac{\Lambda}{M_{\rm pl}}\lambda_4^{ij}\phi\,\widetilde{N}_i^{c*}\left(\widetilde{L}_j\cdot H_u\right) + \text{h.c.}$$

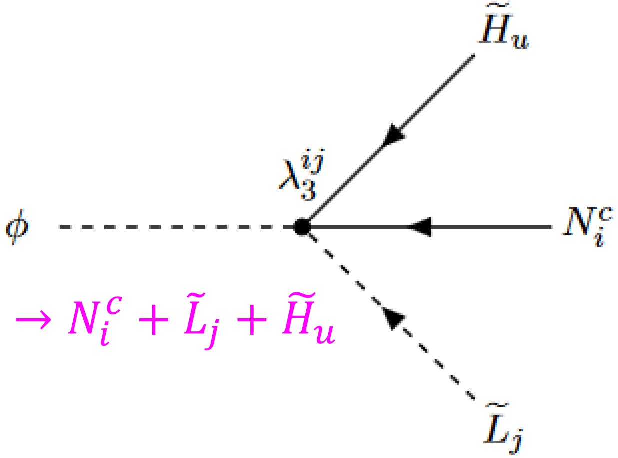
3-body decays:



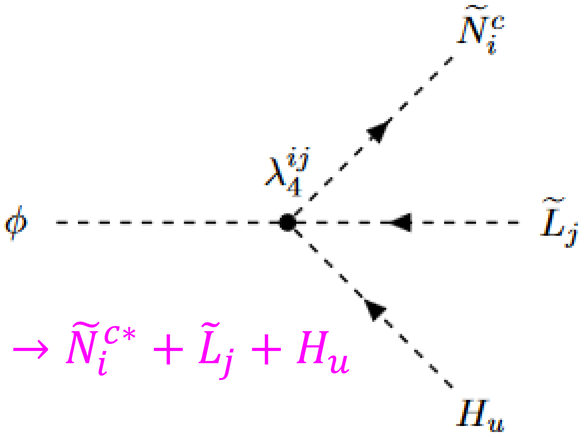
$\phi \to N_i^c + L_j + H_u$



$\phi \to \widetilde{N}_i^c + L_j + \widetilde{H}_u$



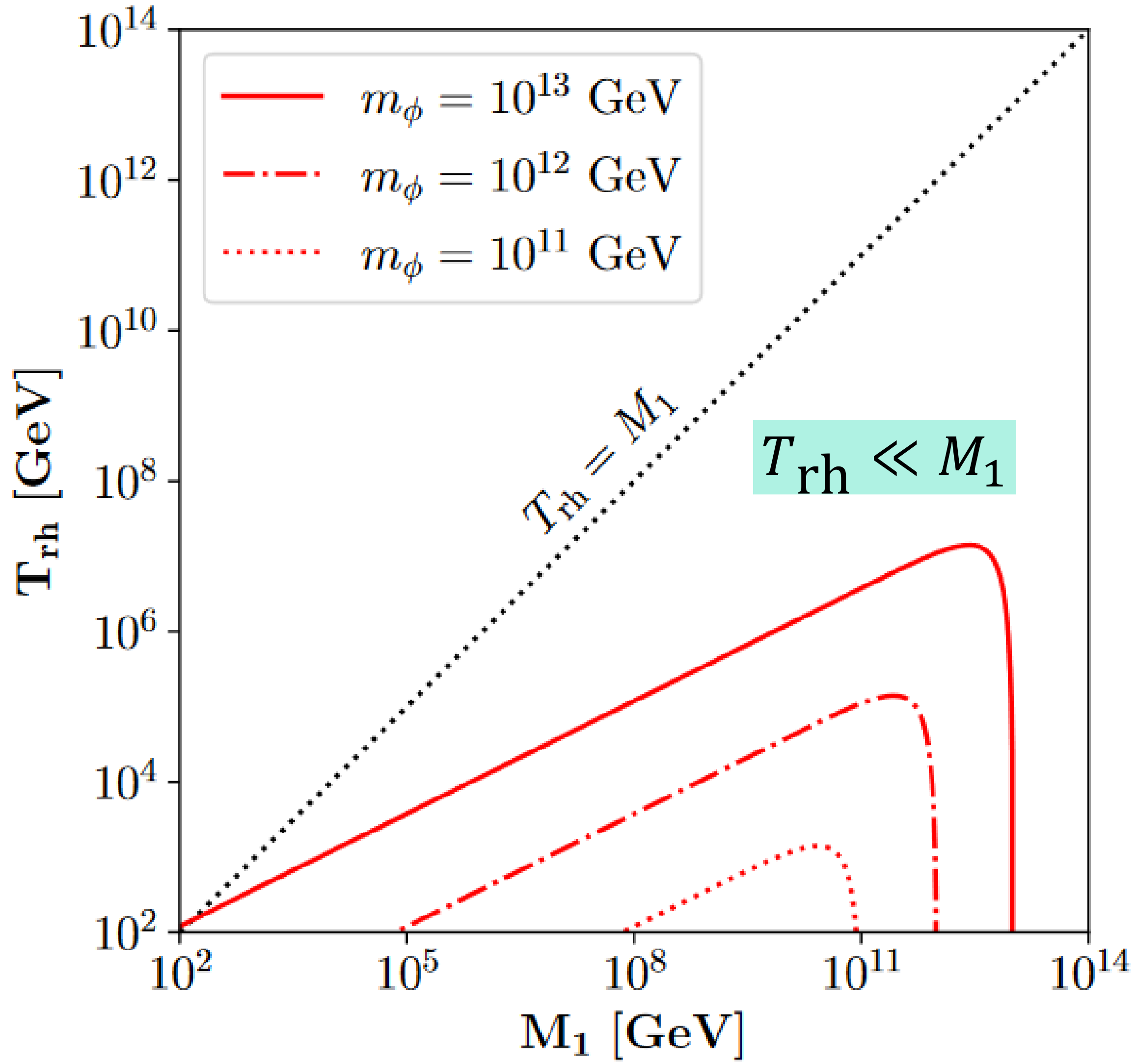
$\phi \to N_i^c + \widetilde{L}_j + \widetilde{H}_u$



$\phi \to \widetilde{N}_i^{c*} + \widetilde{L}_j + H_u$

- Reheating temperature

$$T_{\rm rh} = \sqrt{\frac{2}{\pi}\left(\frac{10}{g_*}\right)^{1/4}} \sqrt{M_{\rm pl}\Gamma_\phi}$$



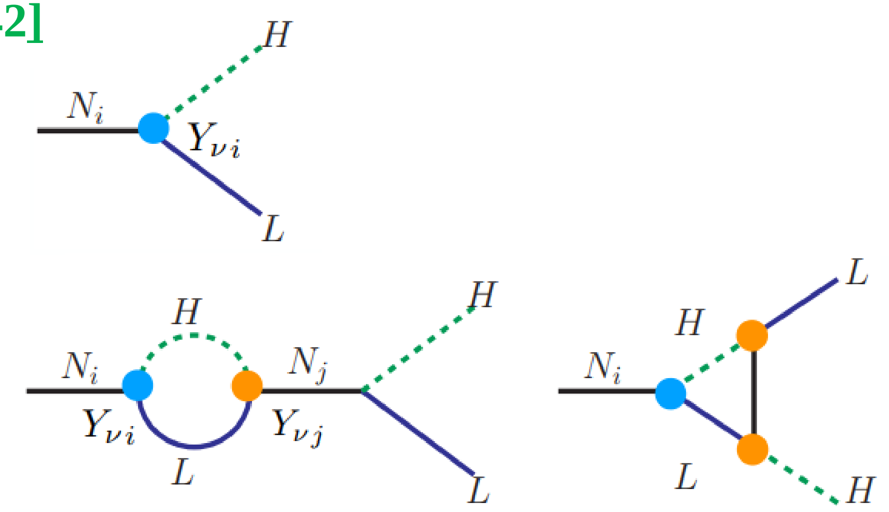
The thermal production of right-handed neutrinos is Boltzmann suppressed.

Baryon asymmetry from non-thermal leptogenesis

- The first two RHNs are quasi-degenerate $M_1 : M_2 : M_3 = 1 : 1.05 : 2.05$, the CP asymmetry of RHNs decays is **enhanced** [Pilaftsis,Underwood,hep-ph/0309342]

$$\epsilon_i = \frac{\Gamma(N_i \to L + H_u) - \Gamma(N_i \to \bar{L} + \bar{H}_u)}{\Gamma(N_i \to L + H_u) - \Gamma(N_i \to \bar{L} + \bar{H}_u)}$$

$$= \frac{\mathrm{Im}\left\{\left(\mathcal{Y}_D \mathcal{Y}_D^\dagger\right)_{ij}^2\right\}}{\left(\mathcal{Y}_D \mathcal{Y}_D^\dagger\right)_{ii}\left(\mathcal{Y}_D \mathcal{Y}_D^\dagger\right)_{jj}} \frac{\left(M_i^2 - M_j^2\right) M_i \Gamma_j}{\left(M_i^2 - M_j^2\right)^2 + M_i^2 \Gamma_j^2}$$

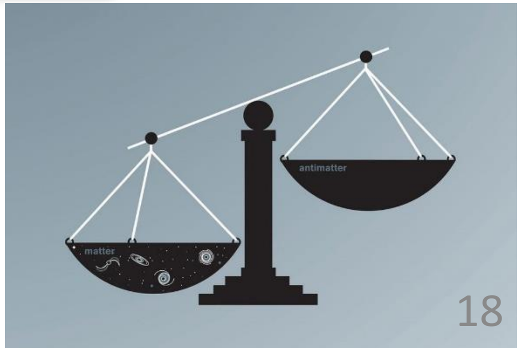


- The lepton asymmetry is converted to the Baryon asymmetry via SM sphalerons

$$Y_B \equiv \frac{n_B}{s} \simeq c_{\rm sph} \frac{3}{4} \frac{T_{\rm rh}}{m_\phi} \sum_i \epsilon_i \times \left[2\mathrm{Br}\left(\phi \to \overset{(\sim)}{N}_i + \overset{(\sim)}{N}_i\right) + \mathrm{Br}\left(\phi \to \overset{(\sim)}{N}_i + \mathrm{others}\right) \right]$$

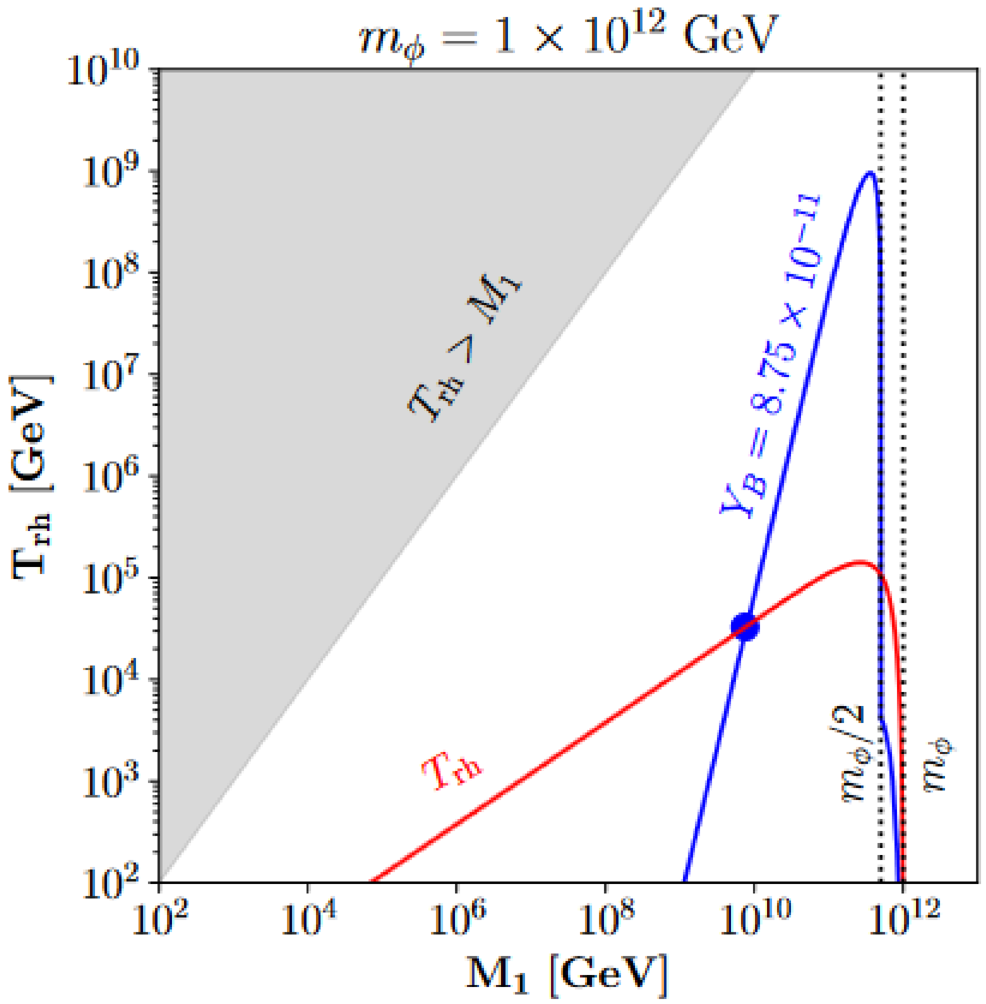
$c_{\rm sph} = -8/23$ in minimal SUSY standard model

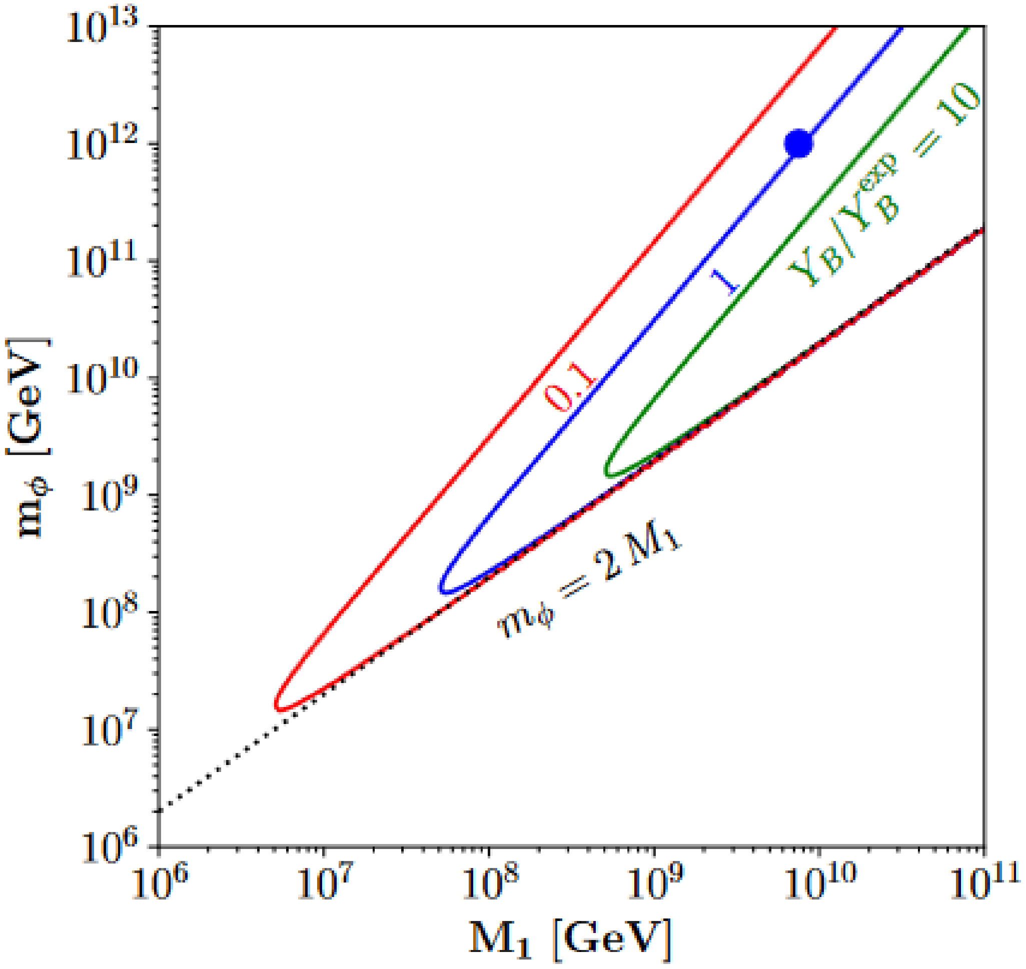
[Asaka, Hamaguchi,Kawasaki,Yanagida, hep-ph/9906366]



➢ Numerical results: Baryon asymmetry $Y_B \equiv \frac{n_B - n_{\bar{B}}}{s}$

[Ding, Jiang, Xu, Zhao, 2411.xxxxx]

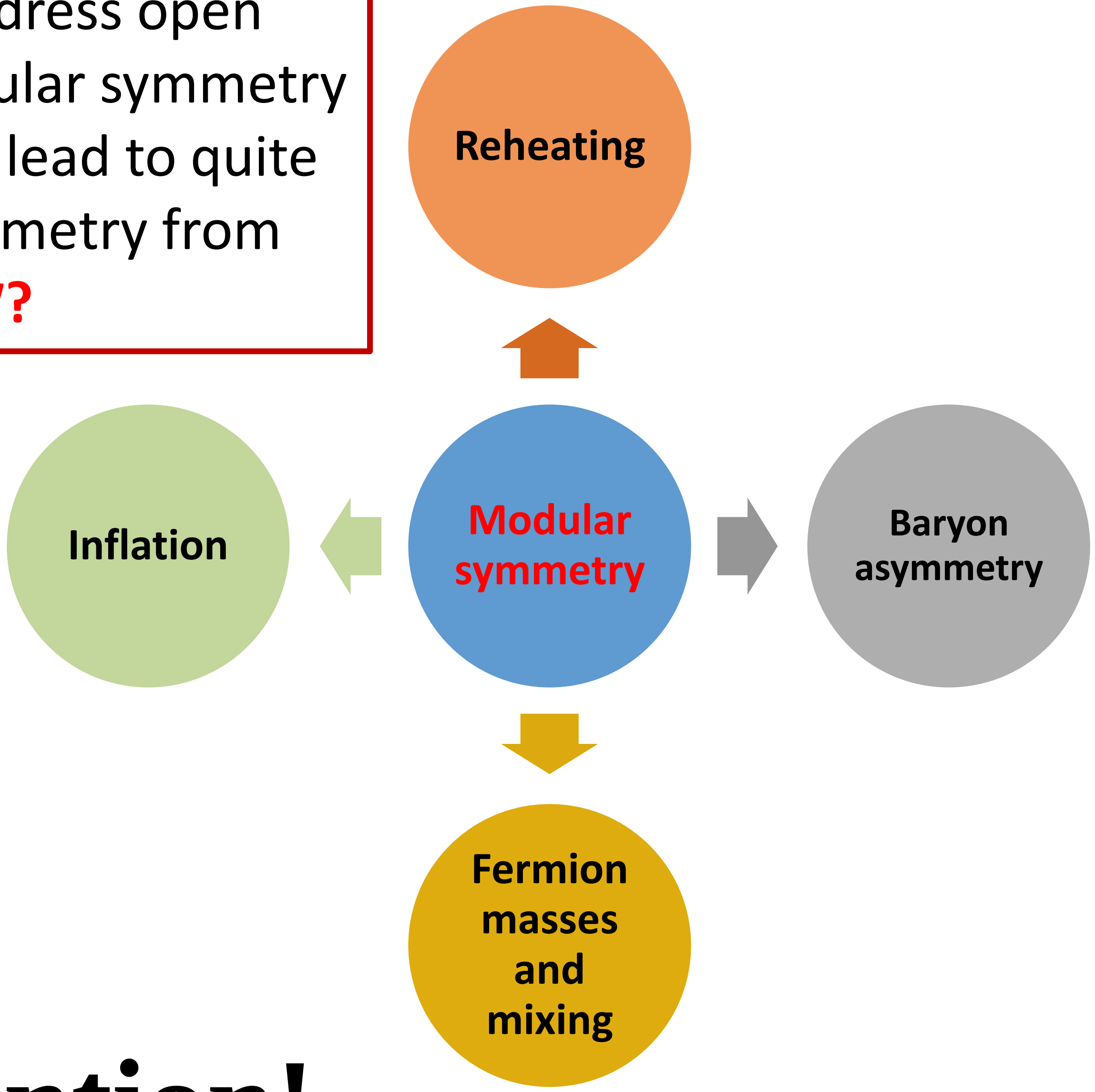




The observed baryon asymmetry can be produced the RH neutrino mass $M_1 \simeq 10^{10}$GeV

Summary

Modular symmetry is a promising approach to address open problems in particle physics and cosmology. Modular symmetry strongly constrains the interactions so that it can lead to quite predictive scenarios. One can probe modular symmetry from multiple perspectives. **Modular symmetry in GW?**

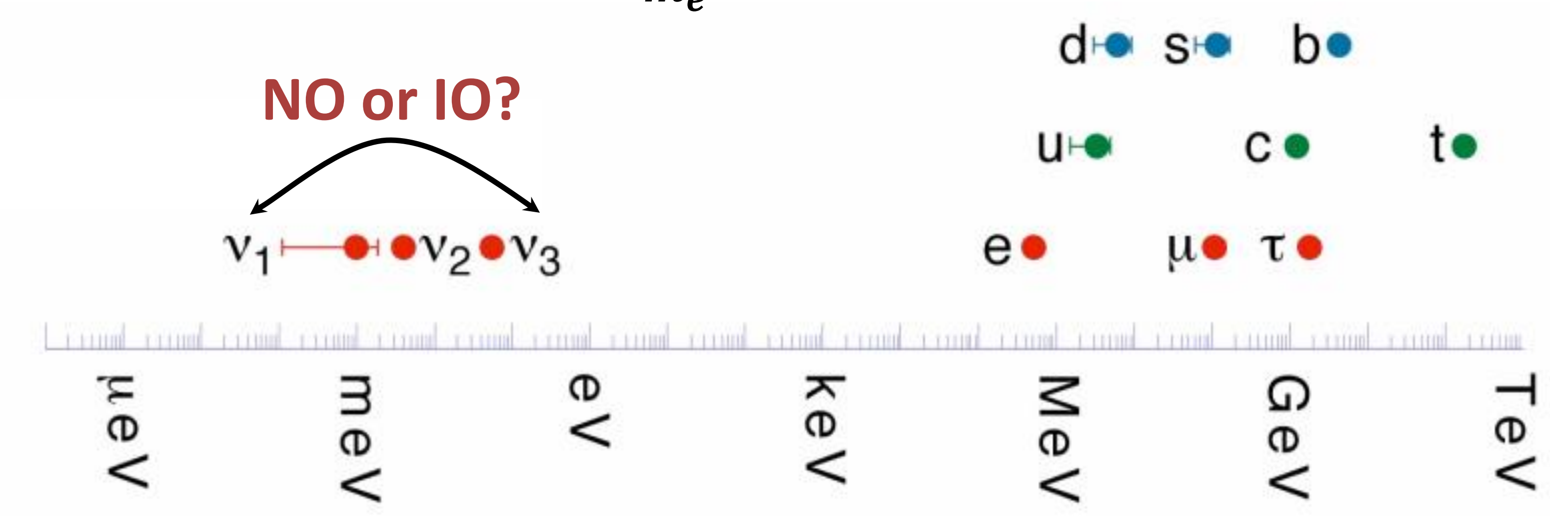


Thank you for your attention!

Backup

Flavor problems of SM

➢ **Hierarchical masses, e.g.** $\frac{m_t}{m_e} = \mathcal{O}(10^5)$



➢ **Quark mixing vs lepton mixing**

Quark mixings are small

$$|V_{CKM}| \approx \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.042 \\ 0.009 & 0.041 & 0.999 \end{pmatrix}$$

PDG(2024)

Lepton mixings are large

$$|U_{PMNS}| \approx \begin{pmatrix} 0.821 & 0.549 & 0.148 \\ 0.376 & 0.588 & 0.704 \\ 0.397 & 0.579 & 0.690 \end{pmatrix}$$

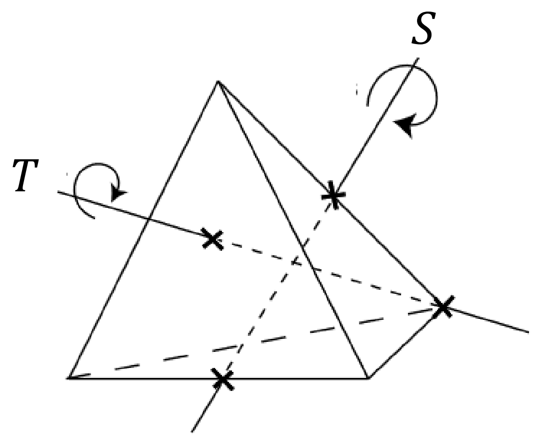
NuFIT5.3(2024)

Quark and lepton mixing matrices have distinctive structures!

A_4 modular symmetry

- $\boldsymbol{A_4 \cong \Gamma_3}$ is the symmetry group of a tetrahedron, it is the smallest non-abelian finite with 3-dim irreducible representation.

$$A_4: S^2 = T^3 = (ST)^3 = 1$$



(promising for 3 generations!)

A_4 has only 4 irreducible inequivalent representations: **1, 1', 1'',3**

singlets $\begin{cases} \mathbf{1}: & S=1, \quad T=1 \\ \mathbf{1}': & S=1, \quad T=\omega \\ \mathbf{1}'': & S=1, \quad T=\omega^2 \end{cases}$

triplets $\mathbf{3}: \quad S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

- Tensor product:

For two triplets $(\alpha_1, \alpha_2, \alpha_3) \sim 3$, $(\beta_1, \beta_2, \beta_3) \sim 3$

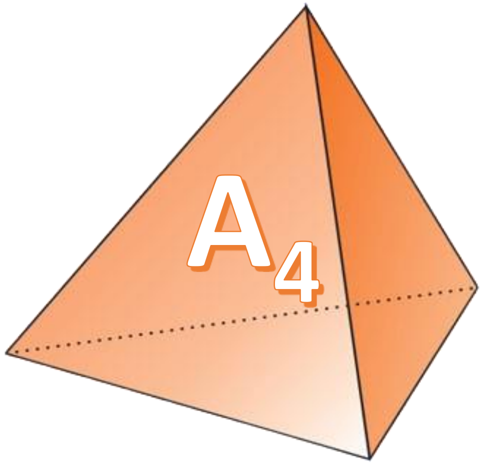
$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{\mathbf{3}} = (\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)_{\mathbf{1}} \oplus (\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1)_{\mathbf{1}'} \oplus (\alpha_2\beta_2 + \alpha_1\beta_3 + \alpha_3\beta_1)_{\mathbf{1}''}$$

$$\oplus \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \end{pmatrix}_{\mathbf{3}_S} \oplus \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}_{\mathbf{3}_A} .$$

Modular forms of level 3

- Three weight 2 and level 3 modular forms transforming as a triplet 3 of A_4

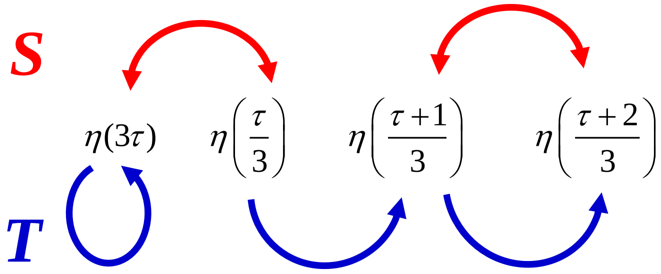
$$Y(\tau) = \left(Y_1(\tau),\quad Y_2(\tau),\quad Y_3(\tau)\right)^T$$ **A_4 triplet** **[Feruglio, 1706.08749]**



$$Y_1(\tau) = \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right]$$

$$Y_2(\tau) = \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right]$$

$$Y_3(\tau) = \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right]$$



Dedekind eta function: $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,,\quad q \equiv e^{2\pi i\tau}$

$$Y_1(\tau) = 1 + 12q + 36q^2 + 12q^3 + 84q^4 + \ldots,$$
$$Y_2(\tau) = -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + 14q^4 + \ldots),$$
$$Y_3(\tau) = -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + 8q^4 + \ldots)$$

Tensor products of $Y_{1,2,3}$ generate higher weight modular forms

- Klein j-invariant

The Klein j-invariant function is a modular form of weight zero, defined in terms of Dedekind eta function and Eisenstein series as follows:

$$j(\tau) \equiv \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\eta^{24}(\tau)} = \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\Delta(\tau)}, \quad \Delta(\tau) \equiv \eta^{24}(\tau)$$

The q-expansion of j-function is given by

$$\begin{aligned} j(\tau) = 744 &+ \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 \\ &+ 333202640600q^5 + 4252023300096q^6 + 44656994071935q^7 + \mathcal{O}(q^8) \end{aligned}$$

- The best fit values of coupling constants

$$\langle\tau\rangle = 0.485 + 0.875i \equiv \tau_0,\ \frac{y_2}{y_1} = 1582.5,\ \frac{y_3}{y_1} = 554.4e^{3.712i},\ \frac{y_4}{y_1} = 60644.8,\ \frac{y_5}{y_1} = 38717.9e^{3.195i},$$

$$\frac{g_2}{g_1} = 0.246,\ \ y_1 v_d = 0.250\ \text{MeV},\ \ \frac{(g_1 v_u)^2}{\Lambda} = 19.967\ \text{meV}$$

- The couplings between inflaton and RH neutrinos

$$\lambda_1^{ij} = \begin{pmatrix} -2.078-0.980i & 1.420i & 1.017 \\ 1.420i & 0.987-1.033i & 1.079i \\ 1.017 & 1.080 & -1.091-2.013i \end{pmatrix}$$

$$\lambda_2^{ij} = \begin{pmatrix} -5.701 & -0.002+0.106i & -0.184-4.258i \\ -0.002-0.106i & 2.856 & 1.478-0.097i \\ -0.184+4.258i & 1.478+0.097i & -6.149 \end{pmatrix}$$

$$\lambda_3^{ij} = \begin{pmatrix} -0.980+2.078i & 1.557+0.248i & -0.006+0.753i \\ 1.282-0.248i & -1.033-0.988i & -0.831-0.001i \\ -0.006-1.282i & 1.329+0.001i & 2.013-1.090i \end{pmatrix} \qquad \lambda_4^{ij} = \begin{pmatrix} 5.120i & 0.802 & 3.917i \\ 0.240 & -2.362i & 1.830 \\ -5.375i & 1.832 & -6.156i \end{pmatrix}$$