How Einstein's Equation Emerges from CFT$_2$ and Realization of ER=EPR

Haitang Yang

College of Physics, Sichuan University

Some formal aspects of Field theories and Holography

USTC, April 18-20,2025

Outlines

Background

Quantum entanglement

Einstein's equation from EE in CFT$_2$

Intuitive picture

RG equation as a geometric identity

Realization of ER=EPR

Summary and discussion

Joint work with Xin Jiang(蒋昕), Houwen Wu(伍厚文) and Peng Wang(王鹏).

Based on arXiv:2406.09033 (PRD Letter, mixed state EE),
arXiv: 2501.08198 & 2501.11302 (Covariant, thermal mixed state EE),
arXiv:2410.19711 (Einstein equation from CFT$_2$), arXiv:2411.18485 (ER=EPR)

Background

Unifying gravity with gauge theory has a long history. The precursors are primarily attempts to gauging the gravity, though Einstein himself tried to geometrize gauge theories. There are insurmountable obstacles in both geometry and algebra.

AdS/CFT correspondence provides a different perspective. The first classical steps should be:

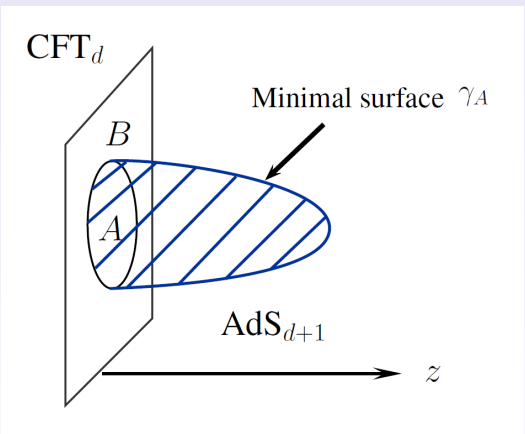
- Kinematics: How to extract the dual metric from CFT?
- Dynamics: How to get Einstein's equation from CFT? Or how to guarantee the extracted metric satisfies Einstein's equation?

The latest progress is to use the first thermodynamical law of entanglement entropy $\delta E = \delta S$, RT formula and relative entropy to show that, the geometry dual to the perturbed CFT state satisfies linearized or second-orderized Einstein's equation about pure AdS ($g = g^{(0)}_{AdS} + \epsilon g^{(1)} + \epsilon^2 g^{(2)} + \cdots$) (Faulkner etc JHEP 1312.7856 (linearized); JHEP 1705.03026 (second-orderized)).

Holography

The holographic principle states that the data of gravity are stored in its lower dimensional boundary which looks like a holographic image.

Inspired by AdS/CFT and Bekenstein-Hawking entropy, Ryu and Takayanagi (2006,PRL) proposed an equivalence between the entanglement entropy (EE) of CFT$_D$ and the minimal surface area in AdS$_{D+1}$,



$$S_A = \frac{\text{Area}\,(\gamma_A)}{4G_N^{D+1}}.$$

This RT formula was for pure state in the original proposal.

The Nature of Gravity?

What is the the nature of Gravity?

General relativity: Gravitational force is an illusion of spacetime curvature.

— Through Einstein's equation.

It from qubit: spacetime structure emerges from quantum entanglements.

— But how?

Entanglement in QM

Entanglement in quantum mechanics, for example, the triplet and singlet:

$$\begin{cases} |11\rangle = & |\uparrow\uparrow\rangle, & \text{No entangling,} \\ |1-1\rangle = & |\downarrow\downarrow\rangle, & \text{No entangling,} \\ |10\rangle = & \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big) & \text{Entangling,} \\ |00\rangle = & \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big) & \text{Entangling} \end{cases}$$

Define density matrix

$$\rho_{\rm tot} = |\Psi\rangle\langle\Psi|, \quad \rho_1 = {\rm Tr}_2\, \rho_{\rm tot}, \quad \text{and} \quad S_1 = -{\rm Tr}\rho_1 \log \rho_1, \quad S_2 = S_1.$$

No entangling state $|11\rangle$:

$$\rho_{\rm tot} = |11\rangle\langle 11|, \quad \rho_1 = {\rm Tr}_2\, \rho_{\rm tot} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad S_1 = -{\rm Tr}\rho_1 \log \rho_1 = 0.$$

Entangling singlet $|00\rangle$:

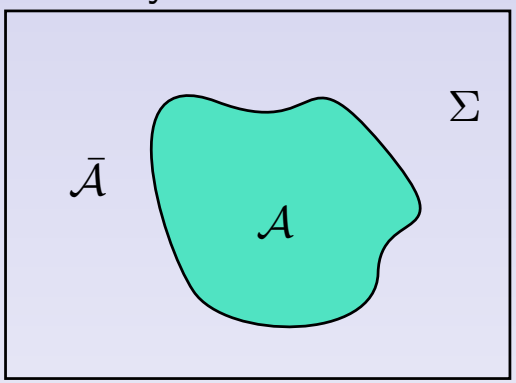
$$\rho_{\rm tot} = |00\rangle\langle 00|, \quad \rho_1 = {\rm Tr}_2\, \rho_{\rm tot} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad S_1 = -{\rm Tr}\rho_1 \log \rho_1 = \log 2.$$

Entanglement in QFT

Entanglement in QFT$_d$, on a time-slice $\Sigma = \mathcal{A} \cup \bar{\mathcal{A}}$, $\mathcal{A}$ entangles with $\bar{\mathcal{A}}$,

$d = 2$ system,

$d = 3$ system



The entanglement entropy (EE) S_A is defined as

$$\rho_{\rm tot} = \frac{1}{Z}|\Psi\rangle\langle\Psi|, \qquad \rho_A = {\rm Tr}_{\bar{A}}\rho_{\rm tot}, \qquad S_A = -{\rm Tr}\rho_A \log \rho_A,$$

where $|\Psi\rangle$ is the wave function of Σ and Z is the partition function.

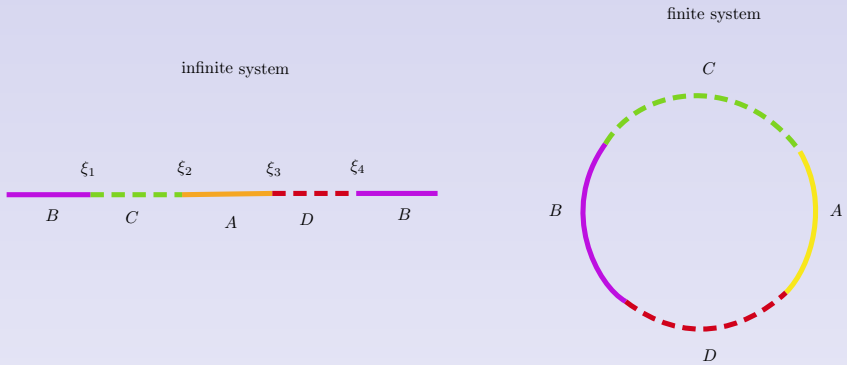
Note: $\Sigma = \mathcal{A} \cup \bar{\mathcal{A}}$ is a pure state. $\mathcal{A}$ and $\bar{\mathcal{A}}$ are complementary and jointed. So the EE is divergent! Say, EE in infinite CFT$_2$ is

$$S_A = \frac{c}{6}\log\frac{\ell_A}{\epsilon},$$

where ϵ is the UV cut-off. The asymptotic behavior makes it very difficult, if not completely impossible, to extract exact relations. So, we turn to the mixed states.

Mixed states: introduction

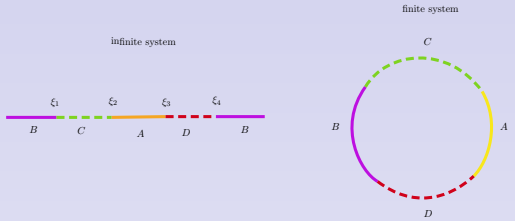
Pictorially, a bipartite mixed state is



Q: What is the entanglement between A and B, i.e. $S_{\text{vN}}(A:B)$?

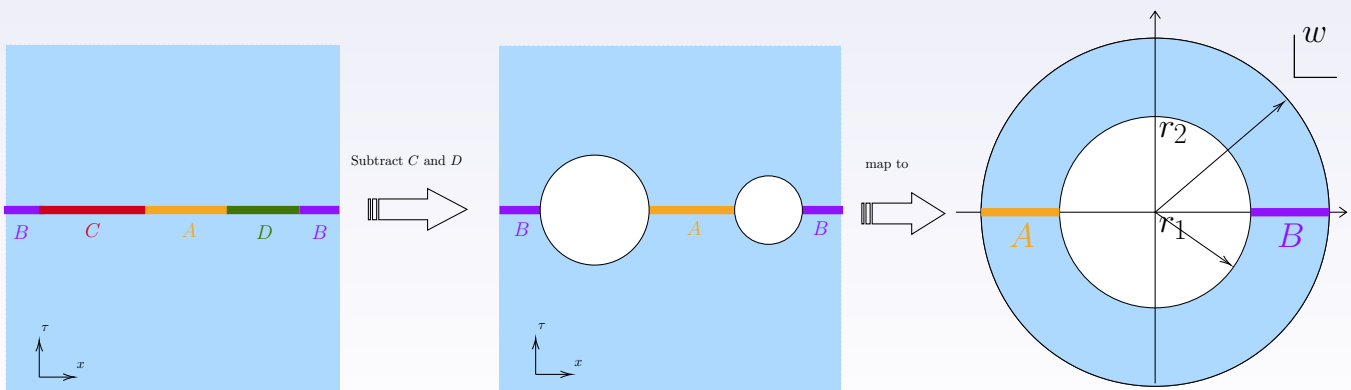
1. $\rho_{ACBD} = |\Psi_{ACBD}\rangle\langle\Psi_{ACBD}|$ is pure, so calculation of $S_{\text{vN}}(A:CBD)$, $S_{\text{vN}}(AD:CB)\cdots$ leads no difference from the pure states and they are divergent.
2. Since A and B are not complementary, $\rho_{AB} = \text{Tr}_{CD}\rho_{ACBD}$ is not pure, but mixed.
3. As $C, D \to 0$, we recover the pure state consisting of A and B.
4. In order to calculate $S_{\text{vN}}(A:B)$, we need to make A and B complementary somehow, and then construct a pure state ψ_{AB}.

Mixed states: two approaches to calculate $S(A:B)$ in CFT



There are two approaches to make A and B complementary and then calculate $S_{\text{vN}}(A:B)$ in CFT:

1. Purification (Takayanagi and Umemoto:2017 Nature Phys): Replace unknown C and D by arbitrary states $\bar{A}$ and $\bar{B}$, define $S_{\text{vN}}(A:B) := \min_{\bar{A}\bar{B}} S_{\text{vN}}(A\bar{A}:B\bar{B})$ optimizing over $\bar{A}$ and $\bar{B}$. — not workable in practice.
2. Subtraction (JWWY:2406, PRD Letter): Simply subtract unknown C and D from the system in a covariant way.



Entanglement for mixed states in CFT$_2$

$S_{\text{vN}}(A:B)$ is (JWWY2406 PRD Letter(static) JWWY2501(generic)),

$$S_{\text{vN}}(A:B) = \frac{c}{12}\log\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right) + \frac{c}{12}\log\left(\frac{1+\sqrt{\bar{\eta}}}{1-\sqrt{\bar{\eta}}}\right), \tag{1}$$

with the cross ratio

$$\eta = \frac{\xi_{21}\xi_{43}}{\xi_{31}\xi_{42}}, \quad \xi_{ij} \equiv \xi_i - \xi_j, \tag{2}$$

where $\xi_i = \tilde{x}_i + i\tilde{t}_i$ are endpoints of A and B, parameters in CFT$_2$.

As a consequence of the RG equation $\ell\frac{d}{d\ell}S = \frac{c}{6}$ and conformal symmetries, the EE eqn. (1) turns out to be the universal expression for CFT$_2$s. Different CFTs, such as a finite temperature system or a finite size system, are characterized by different cross ratios eqn. (2).

This entanglement entropy is indeed finite!

The divergent EE of pure states are various simple limits of this EE.

Einstein's equation from EE in CFT$_2$

The endpoints ξ_i, $i = 1, 2, 3, 4$ uniquely fix A and B, which amounts to eight free real parameters. However, two of these are gauge freedoms. We regroup the six independent parameters as

$$\begin{aligned} x + it &= \xi_1 + \frac{|\xi_{13}||\xi_{14}|}{|\xi_{12}||\xi_{13}| + |\xi_{34}||\xi_{24}|}\xi_{21}, \\ x' + it' &= \xi_2 + \frac{|\xi_{23}||\xi_{24}|}{|\xi_{13}||\xi_{34}| + |\xi_{12}||\xi_{24}|}\xi_{21}, \\ z &= \frac{|\xi_{14}|\sqrt{|\xi_{12}||\xi_{13}||\xi_{34}||\xi_{24}|}}{|\xi_{12}||\xi_{13}| + |\xi_{34}||\xi_{24}|}, \\ z' &= \frac{|\xi_{23}|\sqrt{|\xi_{12}||\xi_{13}||\xi_{34}||\xi_{24}|}}{|\xi_{13}||\xi_{34}| + |\xi_{12}||\xi_{24}|}. \end{aligned} \tag{3}$$

The cross ratio becomes a real number,

$$\eta = \left|\frac{\xi_{21}\xi_{43}}{\xi_{31}\xi_{42}}\right| = \frac{\Delta z^2 + \Delta x^2 - \Delta t^2}{(z+z')^2 + \Delta x^2 - \Delta t^2}. \tag{4}$$

In terms of these new coordinates (t, x, z) and (t', x', z'), the entanglement entropy $S_{\text{vN}}(A)$ can be written as

$$S_{\text{vN}}(A) = \frac{c}{6}\cosh^{-1}\left(1 + \frac{\Delta z^2 + \Delta x^2 - \Delta t^2}{2zz'}\right). \tag{5}$$

We define a metric

$$\chi(x,x') := \frac{1}{2} S^2_{\mathrm{vN}}(A), \quad g_{\mu\nu} = -[\chi_{\mu\nu'}] := -\lim_{x'\to x} \partial_{\nu'}\partial_\mu \chi, \tag{6}$$

where the bracket $[\cdots]$ denotes the coincidence limit $x \to x'$. Indices in brackets are derivatives with respect to x^μ or x'^μ hereinafter. Introduce

$$R_{\mu\nu} := g^{\rho\sigma}[\nabla_\nu \nabla_\sigma \nabla_\mu \partial_\rho \chi], \quad R := g^{\mu\nu} R_{\mu\nu}, \tag{7}$$

where the covariant derivatives $\nabla_{(\cdot)}$ are defined by $g_{\mu\nu}$. Then, plugging in everything, it is straightforward to verify

$$R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R = 0. \tag{8}$$

This is precisely the three dimensional Einstein's equation with cosmological constant,

$$\Lambda_c = \frac{R}{6} = \frac{1}{6}\Big[\chi^{\mu\nu'}\chi^{\rho\sigma'}\chi_{\rho\mu\sigma\nu}\Big]. \tag{9}$$

So, the cosmological constant is not really a free parameter, but determined by the CFT entanglement entropy. From equation (6), extracting the metric from the entanglement entropy is straightforward.

Einstein's equation from EE in CFT_2

CFT_2s on other topologies possess the same EE expression with different cross ratios which are related by conformal transformations. Since all equations are tensorial, we immediately conclude that the parameter regrouping is *once-for-all* applicable to all CFT_2s and the derivation applies to all CFT_2s.

We therefore derived Einstein's equation from the quantum entanglement of CFT_2!

Intuitive picture of the derivation: Metric $\Leftrightarrow$ Geodesic

Given a metric, we know how to compute the geodesic distance between two points. On the other hand, the metric can also be extracted from geodesics. Consider a geodesic $x(\tau), \tau \in [0, t]$ connecting two points $x(\tau = 0)$ and $x'(\tau = t)$. $L(x, x')$ is the geodesic length. Recall geodesics are defined as curves whose tangent vector is parallel transported

$$\nabla_\tau \left(g_{ij} \dot{X}^i \dot{X}^j \right) = \frac{d}{d\tau} \left(g_{ij} \dot{X}^i \dot{X}^j \right) = 0.$$

The Synge's world function is a biscalar of x and x', defined as,

$$\begin{aligned} \sigma(x, x') \quad &:= \quad \frac{1}{2} L^2(x, x') = \frac{1}{2} \left[\int_0^t d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \right]^2 \\ &= \quad \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \Big|_{\text{any point}} t^2 \\ &\xrightarrow{t \to 0} \quad \frac{1}{2} g_{\mu\nu} \Delta X^\mu \Delta X^\nu \end{aligned}$$

So, the metric is

$$g_{\mu\nu} = -[\sigma_{\mu\nu'}] := -\lim_{x' \to x} \partial_{x'^\nu} \partial_{x^\mu} \sigma(x, x'),$$

or

$$g_{\mu\nu} = [\sigma_{\mu\nu}] := \lim_{x' \to x} \nabla_{x^\mu} \partial_{x^\nu} \sigma(x, x').$$

Metric $\Leftrightarrow$ Geodesic

Note raising or lowering indices and taking the limit commute. Some identities for the Synge's world function $\sigma(x,x') = \frac{1}{2}L^2(x,x')$

- $$\sigma_\mu \sigma^\mu = 2\sigma.$$
- $$g_{\mu\nu} = [\sigma_{\mu\nu}] = -[\sigma_{\mu\nu'}]$$

.

- $$[\sigma_{\alpha\mu\beta\nu}] = \frac{1}{3}\Big(R_{\alpha\nu\beta\mu} + R_{\alpha\beta\nu\mu}\Big).$$
- $$\Sigma_{\mu\nu} := [\sigma^{\alpha\beta}\,\sigma_{\alpha\mu\beta\nu}] = g^{\alpha\beta}[\sigma_{\alpha\mu\beta\nu}] = \frac{1}{3}R_{\mu\nu}.$$
- $$\Sigma := [\sigma^{\mu\nu}\sigma^{\alpha\beta}\,\sigma_{\alpha\mu\beta\nu}] = g^{\mu\nu}\Sigma_{\mu\nu} = \frac{1}{3}R.$$

The $[...]$ denotes the coincidence limit $[\Omega] := \lim_{x'\to x}\Omega(x,x')$ for any bitensor $\Omega(x,x')$. Note the indices of σ are (covariant) derivatives.

Geodesic version Einstein's equation

With respect to metric, we have Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0, \tag{10}$$

So, the geodesic version Einstein's equation is

$$\Sigma_{\mu\nu} - \frac{1}{2}[\sigma_{\mu\nu}]\,\Sigma + \frac{\Lambda}{3}[\sigma_{\mu\nu}] = 0,$$

taking the identical form as the metric Einstein equation. Note $g_{\mu\nu} = [\sigma_{\mu\nu}]$.

Geodesics in the bulk of AdS$_3$

Now considering AdS$_3$ geometry, on the conformal boundary, two segments A and B are disjoint. γ_1, γ_2 and L_{AB} are all geodesics. The picture also demonstrates explicitly how an extra dimension arises from the boundary theory: four points on the boundary fix two points in the bulk. Again, there exist two gauge freedoms. The remaining six independent parameters exactly match two points in the 3-dimensional bulk.

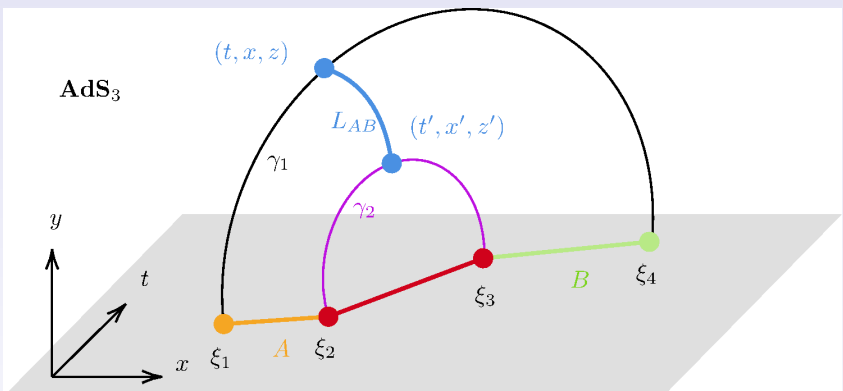
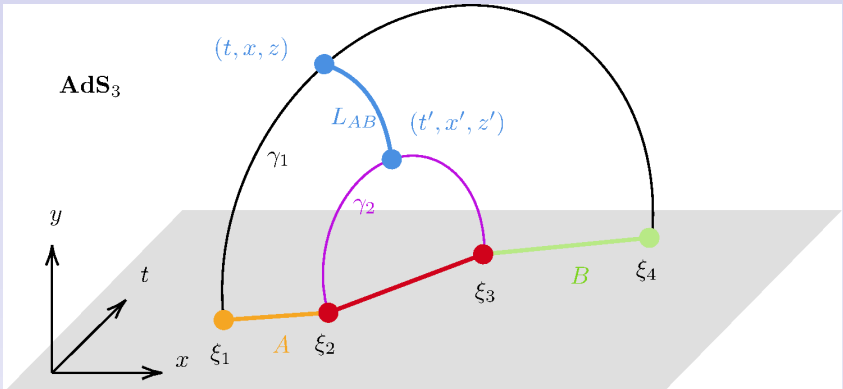


Figure: In AdS$_3$, geodesic γ_1 is completely fixed by ξ_1 and ξ_4, geodesic γ_2 is completely fixed by ξ_2 and ξ_3. L_{AB} is a geodesic and unique.

Intuitive picture



Since L_{AB} has finite length, it must satisfy Einstein's equation:

$$\Sigma_{\mu\nu} - \frac{1}{2}[\sigma_{\mu\nu}]\,\Sigma + \frac{\Lambda}{3}[\sigma_{\mu\nu}] = 0.$$

It is easy to verify exactly

$$L_{AB} = S_{\mathsf{vN}}(A).$$

This is actually the RT formula. Therefore, we reached a holographic description of the CFT Einstein's equation.

We further identified $\Lambda = 3\Lambda_c = \frac{1}{2}\left[\chi^{\mu\nu'}\chi^{\rho\sigma'}\chi_{\rho\mu\sigma\nu}\right]$.

Intuitive picture

We emphasize that this holographic description is only for illustrative purposes. It is not a necessary ingredient in our derivation.

However, we can easily deduce, once RT formula holds — which is widely believed — our derivation must hold. We only need to figure out the mixed EE for specific QFT$_2$.

RG equation as a geometric identity

For a classically scale-invariant theory living on a d-dimensional manifold $\mathcal{M}$ with the metric $ds^2 = \gamma_{ab}\, dx^a\, dx^b$, the Callan-Symanzik RG equation is:

$$\left[\ell \frac{\partial}{\partial \ell} - 2 \int_{\mathcal{M}} \gamma^{ab} \frac{\delta}{\delta \gamma^{ab}}\right] \log Z_{\mathsf{CFT}} = 0,$$

where ℓ is the length scale. Using the replica trick and Rényi entropy

$$S_{\mathsf{vN}}^{(n)} = \frac{1}{1-n} \log \left[\frac{Z_{\mathcal{M}_n}}{(Z_{\mathcal{M}})^n}\right],$$

where $\mathcal{M}_n$ is the replicated manifold, one can show the RG equation for the EE is

$$\ell \frac{\partial}{\partial \ell} S_{\mathsf{vN}} = \frac{c}{6}.$$

The constant $\frac{c}{6}$ on the right hand side is for $d = 3$.

RG equation as a geometric identity

In any geometry, not limited to gravity, there is an identity for the world function,

$$g^{\mu\nu}\sigma_\mu\sigma_\nu = 2\sigma \quad \text{for} \quad \sigma(x,x') = \frac{1}{2}L^2(x,x'). \tag{11}$$

In AdS$_3$, except the vertical ones, any other geodesic is a segment of a semi-circle,

$$(x-x_0)^2 + z^2 = r^2.$$

We can map a semi-circle to a vertical line by the isometries of AdS$_3$.

$$(x,z) \to (\tilde{x},\tilde{z}) := \left(\frac{x-x_0+r}{(x-x_0+r)^2+z^2} - \frac{1}{2r}, \frac{z}{(x-x_0+r)^2+z^2}\right).$$

In terms of $(\tilde{x},\tilde{z})$, the semi-circle is a vertical line $\sigma = \sigma(0,\tilde{z}_1;0,\tilde{z}_2)$. Now, apply the geometric identity (11),

$$z^2\left[(\partial_x\sigma)^2 + (\partial_z\sigma)^2\right] = \tilde{z}^2(\partial_{\tilde{z}}\sigma)^2 = 2\sigma.$$

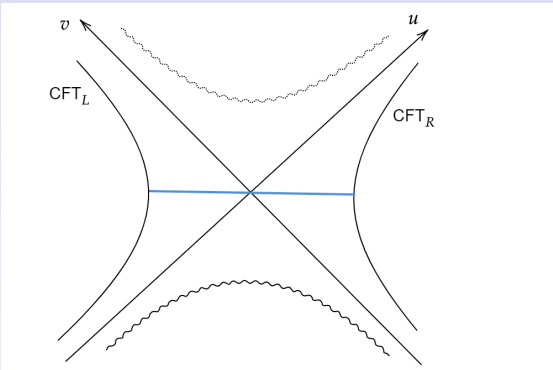
Obviously, $\tilde{z}$ is the renormalization scale ℓ. Including the coupling $c = 3R_{\text{AdS}}/2G^{(3)}$, substituting $\sigma = \frac{1}{2}L^2 = \frac{1}{2}S_{\text{vN}}^2$, we thus have in both bulk geometry and boundary CFT

$$\ell\frac{\partial}{\partial\ell}S_{\text{vN}} = \frac{c}{6},$$

precisely the RG equation.

Realization of ER=EPR

We now give a concrete realization of ER=EPR (JWWY arXiv:2411.18485). Heuristically, two uncorrelated but entangled CFTs, namely the thermofield double state, are conjectured to be dual to an Einstein-Rosen (ER) bridge (Maldacena: 2001). This leads to the ER=EPR conjecture (Maldacena, Susskind: 2013).



The Lorentzian black hole in Kruskal coordinates. The singularity is at $uv = 1$ and the boundary of AdS is at $uv = -1$.

Briefly, what we do is to calculate some specific mixed state EE between two CFTs, then show the derive metric is indeed the Einstein-Rosen bridge by using

$$g_{\mu\nu} = -[\chi_{\mu\nu'}] = -\left[\left(\frac{1}{2}S^2_{\mathrm{vN}}\right)_{\mu\nu'}\right].$$

Summary and Discussions

- We derived Einstein's equation from quantum entanglement in CFT$_2$.
- Our results make "gravity as entropic force" straightforward.
- Derive $(d+1)$-dimensional Einstein's equation from CFT$_{d>2}$? — to appear very soon.
- Note the local/global duality in gravity/QFT as shown by $g_{\mu\nu} = -[\frac{1}{2}(S^2_{\rm vN})_{\mu\nu'}]$.
- As far as we know, the mixed state EE is the unique physical object which exactly equals the bulk dual quantity, the geodesic here.

Discussions

- We showed that the metric $g_{ij} = -[\frac{1}{2}(S^2_{\rm vN})_{ij'}]$ is a derived but not a fundamental quantity.

  Perhaps this is why gravity is non-renormalizable and we should not try to quantize the metric itself. (say Klein-Gordon scalar ϕ is renormalizable, but $\Psi \equiv \partial^2\phi$ is not renormalizable.)
- Moreover, EE $S_{\rm vN}$ is already a quantum quantity, why bother perform further quantization?
- $g_{ij} = -[\frac{1}{2}(S^2_{\rm vN})_{ij'}]$ indicates that spacetime singularities are phase transitions.
- So, it seems we should not quantize $g_{\mu\nu}$ itself, but use QFT quantization through EE.
- In physics, energy dominates and links everything. But now, we tend to believe the fundamental object in physics is entropy.

Thank you!

AUTHOR: Haitang Yang
ADDRESS: College of physics, Sichuan University
Chengdu, 610064, China
EMAIL: hyanga@scu.edu.cn