Theories with SU$(1,2)$ symmetry

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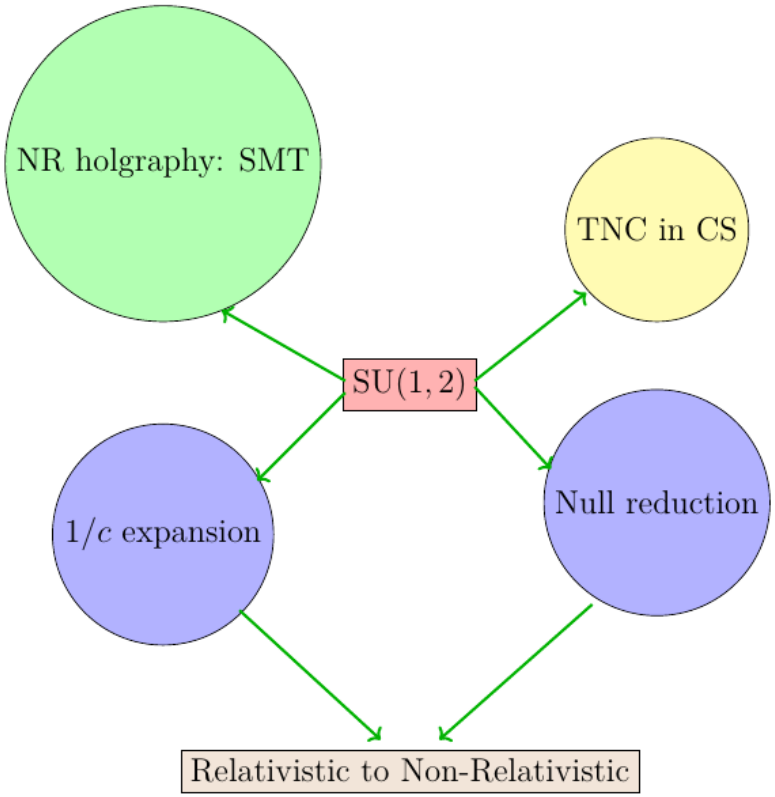
1. Introduction
2. SU$(1,2)$ gravity
   - Review SO$(2,3)$
   - SU$(1,2) \otimes U(1)$
3. Discussions

Introduction: Motivations

Related to various aspects in holography.

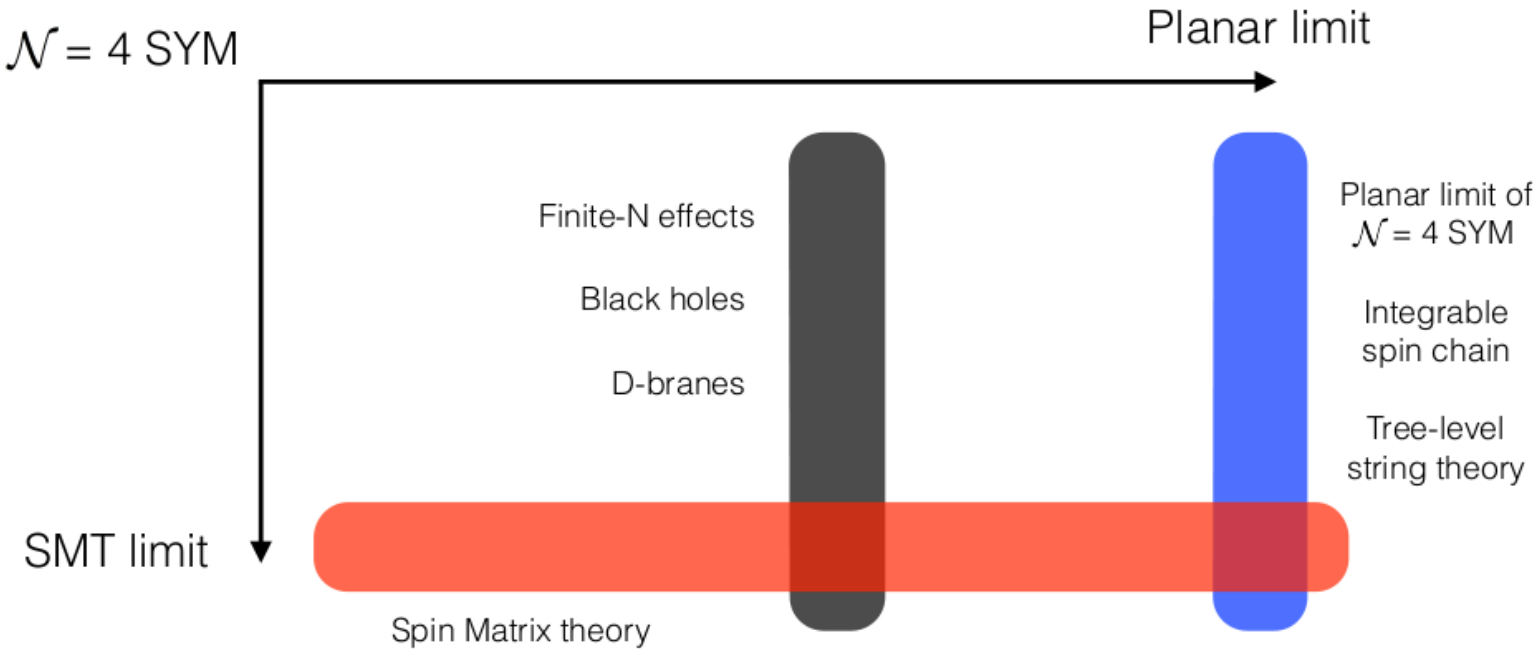
- Spin Matrix theory: PSU$(1,2|3)$ subsector
- Null reduction of SO$(2,4)$: SU$(1,2)$, non-relativistic conformal symmetry
- Classify and generate families of NC gravity from Einstein gravity
- Chern-Simons formulation of torsional Newton-Cartan gravity



Motivation 1: Spin Matrix theory (field theory)

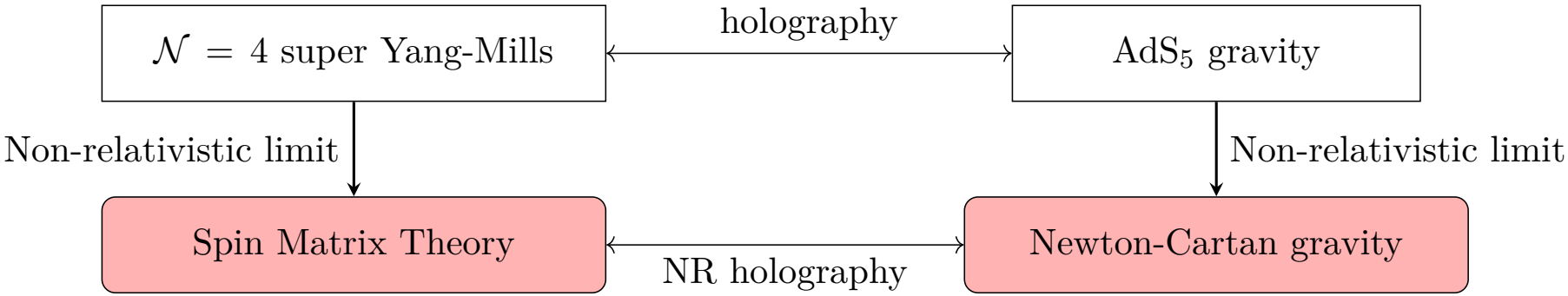
Controlled finite N effects: Spin Matrix Theory limits [Harmark, Orselli, 2014]

- Decoupling limits of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3 \Rightarrow$ the theory reduces to a subsector with only one-loop contributions of the dilatation operator [Harmark, Orselli, 2006][Harmark, Kristjansson, Orselli, 2006-07]



Non-relativistic nature of SMT

- Emergent $\mathrm{U}(1)$ global symmetry $\Rightarrow$ mass conservation
- Antiparticles decouple
- Low-energy excitations (magnons) have non-relativistic dispersion relations
- Bulk duals are non-relativistic string theories (see review [Oling, Yan, 2022])



Decoupling limits

Zoom-in close to a unitarity (BPS) bound of $\mathcal{N}=4$ SYM

$$E \geq J \equiv \sum_{i=1,2} a_i S_i + \sum_{j=1}^{3} b_j Q_j$$

(S_i rotations on S^3, Q_i Cartan generators of $\mathrm{SU}(4)$ R-symmetry)

SMT limit ($\lambda = g^2 N$)

$$\boxed{\lambda \to 0\,, \qquad H_{\text{int}} = \frac{H-J}{\lambda} \text{ finite }, \qquad N \text{ fixed}}$$

PSU$(2,2|4)$ symmetry group of $\mathcal{N}=4$ SYM gets reduced to a spin subgroup

PSU$(1,2|3)$ sector contains all the others!

Useful for decoding black hole like operators [Chi-ming's talk]

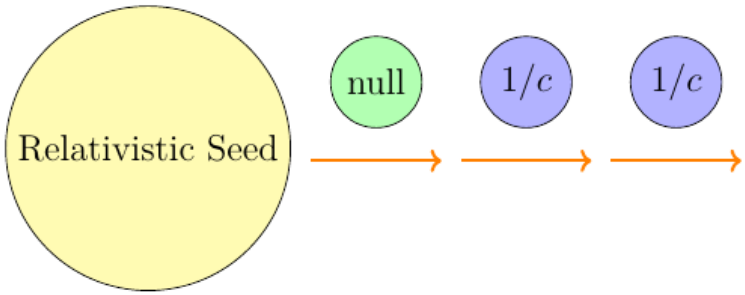
Motivations 2: Relativistic to non-relativistic

How to formulate non-relativistic gravity systematically? How to get non-relativistic limit from relativistic gravity? (like BPS)

$$E = \sqrt{m^2c^4 + p^2c^2} = mc^2 + \frac{p^2}{2m} + \mathcal{O}(c^{-1})$$

In gravitational theories or string theories, we can

- $1/c$ expansion, very technical [Dieter, 17; Hansen, Obers, Hartong, 20]
- Null reduction [Julia, Nicolai, 94; Duval et.al, 85]



Conjecture

Possibly NC gravities with high enough symmetries all come from a seed theory with applications of two algorithms. (High enough $\approx$ conformal.)

$1/c$ expansion algorithm

[Hansen, Obers, Hartong, 20]

Start with metric

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \qquad g^{\mu\nu} = -c^{-2} T^\mu T^\nu + \Pi^{\mu\nu}$$

In this case the pre-Newtonian expansion is

$$\Gamma^\rho_{\mu\nu} = c^2 \overset{(-2)}{C}{}^\rho_{\mu\nu} + C^\rho_{\mu\nu} + S^\rho_{\mu\nu} + \frac{1}{c^2} \overset{(2)}{C}{}^\rho_{\mu\nu}$$

Then the Ricci Scalar is

$$R = \frac{c^2}{4} \Pi^{\mu\nu} \Pi^{\rho\sigma} T_{\mu\rho} T_{\nu\sigma} + \Pi^{\mu\nu} \overset{(C)}{R}{}_{\mu\nu} - \frac{1}{c^2} T^\mu T^\nu \overset{(C)}{R}{}_{\mu\nu}$$

And Einstein Hilbert action is [Hartong, Oling, 22]

$$\mathcal{L} = c^6 (\overset{(-6)}{\mathcal{L}} + c^{-2} \overset{(-4)}{\mathcal{L}} + ..)$$

EOM of subleading orders include the EOM of previous orders.

Schrödinger gravity

[Hartong, YL, Obers, 16]

Gauging Schrödinger algebra in $d = 2 + 1$ Chern-Simons theory to acquire gravity needs to introduce three more generators to guarantee non-degenerate bilinear product + Jacobi identity:

$$\{H, D, K, P_a, G_a, J, N\} + \{S, Y, Z\}$$

such that the action is well-defined

$$CS[A] = \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

The Lagrangian is

$$\begin{aligned} \mathcal{L} = & 2c_1 [e^1 \wedge R(G_2) - e^2 \wedge R(G_1) + \tau \wedge \omega^1 \wedge \omega^2 - f \wedge e^1 \wedge e^2 \\ & + \zeta \wedge (d\tau - 2b \wedge \tau) + \alpha \wedge (db - f \wedge \tau) + \beta \wedge (df + 2b \wedge f) - m \wedge d\omega] \\ & - 2c_2 [b \wedge db - \tau \wedge df - 2b \wedge f \wedge \tau] + c_3 \omega \wedge d\omega. \end{aligned}$$

1. What is the seed theory of this?

Motivation 3: Missing piece in Chern-Simons theory

There are three types of Newton-Cartan gravity:

- Torsionaless $d\tau = 0$: Absolute time
- Twistless torsional $d\tau - zb \wedge \tau = 0$: not absolute time but space can still be sliced by time.
- Torsional $\tau \wedge d\tau \neq 0$

Combined with Chern-Simons,

- Torsionaless by gauging extended Bargmann [Papageorgiou, Schroers, 09']
- Twistless torsional by gauging extended Schrödinger [Hartong, YL, Obers, 16']
- Super NC gravity by gauging extended super Bargmann. [Bergshoeff, Rosseel, 16']
- Many others ...

2. Why the TNC is missing in Chern-Simons theory?

Motivation 4: Null reduction of SO$(2, 2n)$

[Lambert, Mouland, Orchard, 21']

Consider $2n$-Minkowski spacetime

$$ds^2_M = -2d\hat{x}^+ d\hat{x}^- + d\hat{x}^i d\hat{x}^i$$

Under the coordinate transformation

$$\hat{x}^+ = 2R \tan\left(\frac{x^+}{2R}\right)$$

$$\hat{x}^- = x^- + \frac{x^i x^i}{4R} \tan\left(\frac{x^+}{2R}\right)$$

$$\hat{x}^i = x^i - R \tan\left(\frac{x^+}{2R}\right) \Omega_{ij} x^j$$

we get the metric

$$ds^2_M = \frac{1}{\cos^2\left(\frac{x^+}{2R}\right)} ds^2$$

where

$$ds^2 = -2dx^+ \left(dx^- + \frac{1}{2}\Omega_{ij} x^j dx^i\right) + dx^i dx^i$$

Dual AdS slicing

[Pope, Sadrzadeh, Scuro, 99']

Take the embedding coordinate and introduce the extra (y, x^+, x^-, x^i), the AdS spacetime adapted to the conformal flat metric is

$$ds^2 = -\frac{1}{4R^2}\left(dx^+ + e^y R^2\left(dx^- + \frac{1}{2}\Omega_{ij}x^j dx^i\right)\right)^2 + ds^2_{\mathbb{CP}^n}$$

where

$$ds^2_{\mathbb{CP}^n} = \frac{1}{4}dy^2 + \frac{1}{4}e^y dx^i dx^i + \frac{R^2}{4}e^{2y}\left(dx^- + \frac{1}{2}\Omega_{ij}x^j dx^i\right)^2$$

Then the conformal boundary $y \to \infty$ reduces to conformally flat Minkowski. The residue symmetry commute with translation on x^+ is $\mathrm{SU}(1,n) \oplus U(1)$

$\mathbf{SU}(1,2)$ is the first non-trivial example

$$ds^2 = -2dx^+\left(dx^- + \frac{1}{2}x_1 dx_2 - \frac{1}{2}x_2 dx_1\right) + dx_1^2 + dx_2^2$$

State-operator correspondence in SU(1, 2) CFT

[Baiguera, Harmark, YL, Yan, 24']

$$d\hat{s}^2 = 2\hat{\tau}(d\hat{u} - \hat{m}) + \hat{h}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu$$

State picture from SMT [Baiguera, Harmark, YL, 20',21',22']:

$$\tau = R^2 dx^0 - \frac{R^2}{2}(\cos\theta - 1)\, d\varphi\,, \qquad m = \frac{1}{2} dx^0 + \frac{1}{4}(\cos\theta + 1)\, d\varphi$$

$$h_{\mu\nu} dx^\mu dx^\nu = \frac{R^2}{4}(d\theta^2 + \sin^2\theta d\varphi^2)$$

Operator geometry [Lambert, Mouland, Orchard, 21'] :

$$\hat{\tau} = d\hat{t} + \hat{r}^2 d\hat{\varphi}\,, \qquad \hat{m} = \frac{1}{2} d\hat{\varphi}, \quad \hat{h}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu = d\hat{r}^2 + \hat{r}^2 d\hat{\varphi}^2$$

There exists an exact conformal map between the two geometries!

$$x^0 = -\operatorname{arccot}\left(\frac{\mathbf{t}^2 + \mathbf{r}^4 - 1}{\mathbf{t}}\right), \qquad \varphi = \hat{\varphi} - \arctan\left(\frac{\mathbf{r}^2 - 1}{\mathbf{t}}\right),$$

$$\theta = \arccos\left[1 - \frac{8\,\mathbf{r}^2}{\mathbf{t}^2 + \left(\mathbf{r}^2 + 1\right)^2}\right], \qquad u = \hat{u} + \frac{1}{2}\arctan\left(\frac{\mathbf{r}^2 - 1}{\mathbf{t}}\right)$$

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SO(2, 3) vs SU(1, 2)

Distinguish from Zhaolong's talk. Notice

$$Sp(4, R) \approx SO(2, 3)$$

Similar to gauging SO(2, 3). The theory can be put in either three dimensional manifold or four dimensional manifold, with different interpretations.

	3d CS	4d
SO(2, 2)	AdS_3	——
SO(2, 3)	Conformal gravity	AdS_4/CFT_3
SU(1, 2)	This work: TNC	Holography dual of SMT in 1/16-BPS

	SO(2, 2)	SU(1, 2)	SO(2, 3)
Translation	3	3	3
Boost/Rotation	3	3	3
Special conformal	0	1	3
Dilatation	0	1	1

Gauging SO(2, 3)

[Horne, Witten 89']

The basis:

$$
\begin{aligned}
&[J, P_a] = \epsilon_{ab} P_b, \quad [J, G_a] = \epsilon_{ab} G_b, \quad [J, K_a] = \epsilon_{ab} K_b \\
&[H, D] = H, \quad [K, D] = -K, \quad [H, K] = D \\
&[H, G_a] = P_a, \quad [P_a, K] = G_a, \quad [P_a, D] = P_a, \quad [K_a, D] = -K_a \\
&[H, K_a] = G_a, \quad [P_a, G_b] = \delta_{ab} H, \quad [P_a, K_b] = J\epsilon_{ab} + D\delta_{ab} \\
&[G_a, G_b] = -J\epsilon_{ab}, \quad [K, G_a] = -K_a, \quad [K_a, G_b] = -K\delta_{ab}
\end{aligned}
$$

and the gauge field

$$
\mathcal{A} = HE^0 + P_a E^a + J\Omega + G_a \Omega^a + KF + K_a F^a + DB
$$

Gauging SO(2, 3)

The affine connection by vielbein postulates is not invariant under Λ^a_K transformation

$$\begin{aligned}\tilde{\Gamma}^\sigma_{\mu\nu} = &-E^\sigma_0(\partial_\mu - B_\mu)E^0_\nu + E^\sigma_a(\partial_\mu - B_\mu)E^a_\nu \\ &+ \Omega^a_\mu(E^\sigma_0 E^a_\nu - E^\sigma_a E^0_\nu) - \Omega_\mu(E^\sigma_a \epsilon_{ab} E^b_\nu)\end{aligned}$$

In metric like fields:

$$\tilde{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left[(\partial_\mu - 2B_\mu)g_{\rho\nu} + (\partial_\nu - 2B_\nu)g_{\mu\rho} - (\partial_\rho - 2B_\rho)g_{\mu\nu}\right]$$

Both transform as

$$\delta\tilde{\Gamma}^\sigma_{\mu\nu} = \Lambda_K\left(-E^0_\mu\delta^\sigma_\nu - \delta^\sigma_\mu E^0_\nu + g_{\mu\nu}E^\sigma_0\right) + \left(-E^a_\mu\delta^\sigma_\nu - E^a_\nu\delta^\sigma_\mu + E^\sigma_a g_{\mu\nu}\right)\Lambda^a_K$$

Similar to gauging Schrödinger algebra [Hartong, Obers 15']

CS in SO(2, 3)

The Lagrangian is

$$\frac{\mathcal{L}}{c} = -2F \wedge R(H) - 2F^a \wedge R^a(P) \\ - \Omega \wedge d\Omega + \Omega^a \wedge d\Omega^a + 2\Omega \wedge \Omega^1 \wedge \Omega^2 + B \wedge dB$$

After imposing the curvature constraint, the action is

$$S_{gCS} = \int d^3x \epsilon^{\mu\nu\rho} \Gamma^{\lambda}_{\mu\sigma} \left(\partial_\nu \Gamma^{\sigma}_{\rho\lambda} + \frac{2}{3} \Gamma^{\sigma}_{\nu\tau} \Gamma^{\tau}_{\rho\lambda} \right)$$

up to some terms essentially about the boundary dynamics.

$\mathrm{SU}(1,2) \otimes U(1)$

The algebra in this basis:

$$[J, P_a] = \epsilon_{ab} P_b, \qquad [J, G_a] = \epsilon_{ab} G_b,$$
$$[H, K] = D, \qquad [D, K] = 2K, \quad [D, H] = -2H$$
$$[K, P_a] = -G_a, \quad [H, G_a] = P_a, \quad [P_a, D] = P_a, \quad [G_a, D] = -G_a$$
$$[P_a, G_b] = \delta_{ab}\left(-N - \frac{3}{2}J\right) + \frac{1}{2}\epsilon_{ab} D, \quad [P_a, P_b] = \epsilon_{ab} H, \quad [G_a, G_b] = \epsilon_{ab} K$$

The gauge field is denoted as

$$\mathcal{A} = H E^0 + P_a E^a + G_a \Omega^a + J\Omega + DB + KU + NM$$

Cuvature tensor of $\mathrm{SU}(1,2) \otimes U(1)$

$$R(H) = dE^0 - 2B \wedge E^0 + \frac{1}{2}\epsilon_{ab} E^a \wedge E^b$$
$$R^a(P) = dE^a + E^0 \wedge \Omega^a + E^a \wedge B + \epsilon_{ab} E^b \wedge \Omega$$
$$R^a(G) = d\Omega^a + E^a \wedge U - \Omega^a \wedge B + \epsilon_{ab} \Omega^b \wedge \Omega$$
$$R(J) = d\Omega - \frac{3}{2} E^a \wedge \Omega^a$$
$$R(D) = dB + E^0 \wedge U + \frac{1}{2}\epsilon_{ab} E^a \wedge \Omega^b$$
$$R(K) = dU + 2B \wedge U + \frac{1}{2}\epsilon_{ab} \Omega^a \wedge \Omega^b$$
$$R(N) = dM + E^a \wedge \Omega^a$$

The curvature constraint $R(H) = 0$ will end up in torsional Newton-Cartan gravity

Transformation

The transformation under the internal transformation:

$$\Sigma = G_a \Lambda^a + J\Lambda + D\Lambda_D + K\Lambda_K + N\sigma$$

are following

$$\begin{aligned}
\delta E^0_\mu &= 2E^0_\mu \Lambda_D \\
\delta E_\mu{}^a &= E^0_\mu \Lambda^a + E^a_\mu \Lambda_D + \epsilon_{ab} E^b_\mu \Lambda \\
\delta \Omega_\mu{}^a &= \partial_\mu \Lambda^a + E^a_\mu \Lambda_K - \Omega^a_\mu \Lambda_D + B_\mu \Lambda^a + \epsilon_{ab} \Omega^b_\mu \Lambda - \epsilon_{ab} \Omega_\mu \Lambda^b \\
\delta \Omega_\mu &= \partial_\mu \Lambda - \frac{3}{2} E^a_\mu \Lambda^a \\
\delta B_\mu &= \partial_\mu \Lambda_D + E^0_\mu \Lambda_K + \frac{1}{2} \epsilon_{ab} E^a_\mu \Lambda^b \\
\delta U_\mu &= \partial_\mu \Lambda_K + \epsilon_{ab} \Omega^a_\mu \Lambda^b - 2U_\mu \Lambda_D + 2B_\mu \Lambda_K \\
\delta M_\mu &= \partial_\mu \sigma + E^a_\mu \Lambda^a .
\end{aligned}$$

$$\tilde{\Gamma}^\rho_{\mu\nu} = -E^\rho_0 (\partial_\mu - 2B_\mu) E^0_\nu + E^\rho_a \left[(\partial_\mu - B_\mu) E_\nu{}^a - \Omega_\mu{}^a E^0_\nu - \epsilon_{ab} \Omega_\mu E^b_\nu \right]$$

is not invariant under boost, different from [Bergshoeff, Hartong, Rosseel, 14'] :

$$\delta \tilde{\Gamma}^\rho_{\mu\nu} = \Lambda^a \left(\epsilon_{ab} E^b_\mu \delta^\rho_\nu - \frac{1}{2} \epsilon_{ab} E^b_\mu E^c_\nu E^\rho_c - \frac{3}{2} \epsilon_{bc} E^a_\mu E^b_\nu E^{\rho c} \right) - \Lambda_K (E^0_\mu \delta^\rho_\nu + E^0_\nu \delta^\rho_\mu)$$

Chern-Simons theory

[YL, Zhang, 25']

The action is

$$\begin{aligned} & \mathrm{tr}\left(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) \\ = \quad & c_S\left[\Omega^2\wedge R^1(P)-\Omega^1\wedge R^2(P)+U\wedge R(H)-E^0\wedge\Omega^1\wedge\Omega^2-B\wedge dB+\frac{1}{3}\Omega\wedge d\Omega\right] \\ & +c_U\left(M\wedge d\Omega+\frac{3}{4}M\wedge dM+\frac{1}{3}\Omega\wedge d\Omega\right) \end{aligned}$$

reducing to Schrödinger gravity by

$$\begin{aligned} & E^0=c\tau+\beta,\quad U=\frac{1}{c}f+\frac{1}{c^2}\zeta,\quad M=\frac{1}{c}m+\mathcal{O}(c^{-2}) \\ & B=b-\frac{1}{2c}\alpha,\quad \Omega=\omega+\mathcal{O}(c^{-1}) \\ & E^i=e^i+\mathcal{O}(c^{-1}),\quad \Omega^i=\frac{1}{c}\omega^i+\mathcal{O}(c^{-2}), \end{aligned}$$

with

$$c_1=\frac{c_S}{c},\quad c_2=c_S,\quad c_3=\frac{2c_S}{3}+2c_U$$

Metric like fields

A special case $c_S = -c_U$ is called fully coupled

$$\begin{aligned}&\text{tr}\left(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\\ =&c_S\Big[-E^0\wedge\Omega^1\wedge\Omega^2+E^a\wedge\Omega^a\wedge\Omega-B\wedge dB+\frac{3}{4}M\wedge dM\Big]\end{aligned}$$

The action by metric like fields of the black part is [Hartong, YL, Obers, 16']

$$\mathcal{L}=e\Big[(h^{\alpha\nu}h^{\beta\mu}-h^{\alpha\mu}h^{\beta\nu})K_{\alpha\mu}K_{\beta\nu}+2\hat{E}_0^\mu B_\mu h^{\nu\rho}K_{\nu\rho}-2(\hat{E}_0^\mu B_\mu)^2-\tilde{\Phi}\tilde{R}\Big]$$

where $\tilde{\Phi}$ is called Newton potential

$$\tilde{\Phi}=-E_0^\mu M_\mu+\frac{1}{2}h^{\mu\nu}M_\mu M_\nu$$

Vacuum solution

An interesting solution:

$$\mathcal{A} = H\left(\frac{dt}{r^2} - \frac{1}{2}d\theta\right) + P_1\frac{dx_1}{r} + P_2\frac{dx_2}{r} - \frac{dr}{r}D$$

where $x_1 = r\cos\theta\,, x_2 = r\sin\theta$.

- In the null reduced ansatz,

$$ds^2 = -2E^0_\mu(du - M_\mu dx^\mu) + h_{\mu\nu}dx^\mu dx^\nu$$

the four dimensional metric is conformal to Omega deformed background:

$$ds^2 = \frac{1}{r^2}ds^2_4 = \frac{1}{r^2}\left[2du\left(dt - \frac{r^2}{2}d\theta\right) + dx_1^2 + dx_2^2\right]$$

- After $1/c$ expansion and decompactify θ, the solution is $z = 2$ Lifshitz [Hartong, YL, Obers, 16'; Grumiller et.al, 14']

$$\mathcal{A} = H\frac{dt}{r^2} + (P_1 - D)\frac{dr}{r} + (P_2 - Z)\frac{dx}{r}$$

Infinite symmetry

Polyakov–Bershadsky algebra with affine u(1) extension

$$[L_m, L_n] = (m-n)L_{m+n} + c_S\pi(n^3-n)\delta_{n+m,0}, \qquad [L_m, N_n] = 0$$

$$[L_m, Y^a_r] = \left(\frac{m}{2} - r\right) Y^a_{m+r}\,, \quad [L_m, J_n] = -n\left(J_{n+m} - \frac{2}{3}N_{n+m}\right)$$

$$[Y^a_r, Y^a_s] = -(r-s)\left(\frac{3}{2}J_{r+s} - N_{r+s}\right)$$

$$[Y^1_r, Y^2_s] = L_{r+s} + \frac{2}{3\pi}\left(\frac{3}{2}J_{r+s-k} - N_{r+s-k}\right)\left(\frac{3}{2}J_k - N_k\right) - 2\pi c_S(r^2 - \frac{1}{4})\delta_{r+s,0}$$

$$[J_n, Y^a_r] = \epsilon_{ab}Y^b_{n+r}, \quad [J_m, J_n] = \frac{4\pi}{3}n(c_U + c_S)\delta_{n+m,0},$$

$$[J_m, N_n] = 2\pi n c_U\delta_{n+m,0}, \qquad [N_m, N_n] = 3c_U\pi n\delta_{n+m,0}$$

Infinite symmetry after $1/c$ expansion

[Chernyavsky, Sorokin, 19']

- Bosonic version of $\mathcal{N} = 2$ BMS without any non-linear term.
- $L_n \to M_n$ to generate BMS algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_L}{12}(m^3 - m)\delta_{m+n,0}$$
$$[L_m, M_n] = (m-n)M_{m+n} + \frac{c_M}{12}(m^3 - m)\delta_{m+n,0}$$

- $Y^a_{\pm\frac{1}{2}}$ are related to supercharges like [Hartong, YL, unpublished]

$$Y^1_r = Q^1_r\epsilon_1 + Q^2_r\epsilon_2, \quad Y^2_r = -i\epsilon_1 Q^1_r + i\epsilon_2 Q^2_r$$

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Conclusion

- We construct the $\mathrm{SU}(1,2) \otimes U(1)$ Chern-Simons as the seed theory of Schrödinger gravity.
- The infinite extended symmetry is $\mathcal{W}_3^{(2)}$ with $\mathrm{u}(1)$ affine. This is like diagonal embedding of $\mathrm{SL}(2)$ to $\mathrm{SL}(3)$.

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} \\
[L_n, W_m] &= (2n-m)W_{n+m} \\
[W_n, W_m] &= \frac{\sigma}{3}(n-m)(2m^2+2n^2-mn-8)
\end{aligned}
$$

We explained the diagonal embedding higher spin theory, which was pathological [YL, Ross, 15'] .

- We understand the origin of the $z=2$ Lifshitz solution and relations to four dimensional Omega deformed background.

$\mathrm{SU}(1,3) \otimes U(1)$

Arise from null reduction of 6d $\mathcal{N} = (2,0)$ SCFT or M5 brane dynamics [Lambert, Orchard, 20'] . This describes five dimensional manifold. Notice $SU(1,3) \approx SO(2,4)$.

- Time translation H.
- Spatial translation P_i, $i = 1,2,3,4$.
- Dilatation D
- Boost G_i
- Rotation triplet SU(2)=SO(3), J^α
- Rotation singlet $\mathcal{J}$
- Special conformal transformation K

The commutators $[P_1, P_3] = [P_2, P_4] = H$ result in

$$R(H) = d\tau - 2b \wedge \tau + e^1 \wedge e^3 + e^2 \wedge e^4 = 0$$

Affine connection is again not boost invariant

$$[P_i, G_j] = \delta_{ij}(\mathcal{J} - N) - \frac{1}{2}\Omega_{ij} D + \beta_{ij}^\alpha J^\alpha$$

Relation to 4d theory

Evidences of relations to 4d

- The SU$(1,2)$ is already non-relativistic, not likely to acquired by $1/c$ expansion.
- $$\int_{M_4} \mathrm{Tr}(F \wedge F) = \int_{M_4} d\mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$
- The vacuum solution is null reduction of Ω-deformed background

$$ds^2 = \frac{1}{r^2} ds_4^2 = \frac{1}{r^2}\left[2du\left(dt - \frac{r^2}{2} d\theta\right) + dx_1^2 + dx_2^2\right]$$

- su$(1,2) \oplus u(1)$ emerges from null reduction of so$(2,4)$.

Other works

- Galilean conformal algebra in 3d, other conformal gravity? [Jin, YL, WIP]
- Other vacuum? The basis is analogous to diagonal embedding. What about the principle embedding?
- Holography
- Carrollian $c \to 0$ limit
- Black hole like objects in non-relativistic gravity: holonomy condition