Quasi-modular boostrap and chiral algebra

Yiwen Pan

2409.01095 with Chenxi Zeng
2412.03155 with Wenbin Yan
Ongoing work with Peihe Yang

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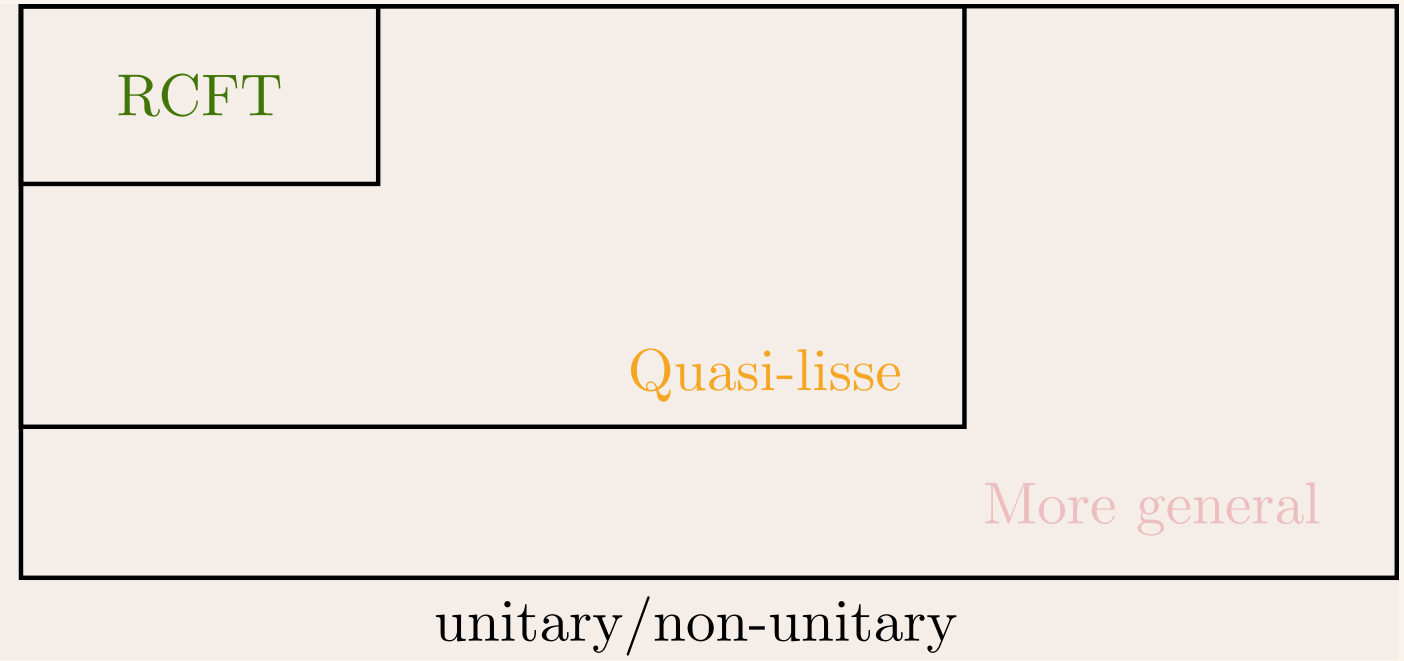
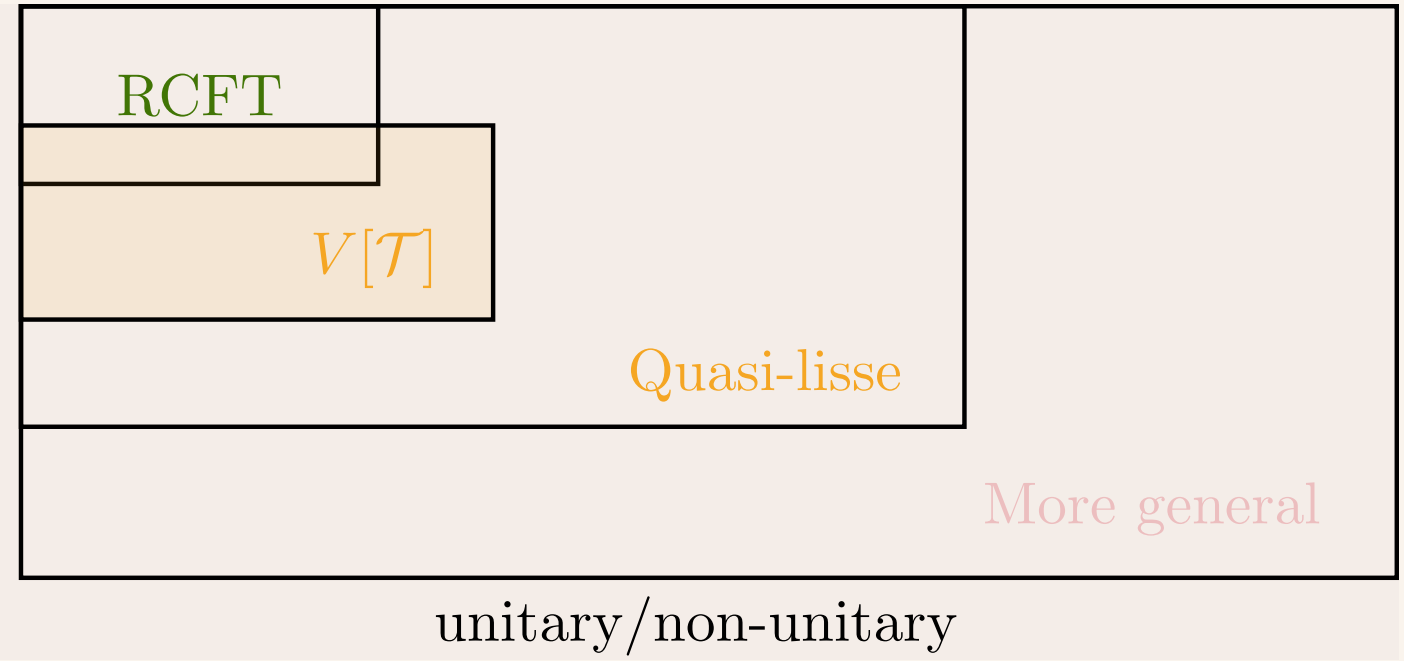
Summary



A simplified landscape of 2d chiral algebras

Summary



Infinite family of quasi-lisse chiral algebra: $\mathbb{V}[\mathcal{T}]$

Summary

- **What is the spectrum of $\mathbb{V}[\mathcal{T}]$ or in general quasi-lisse chiral algebras?**

Summary

Our goal

- Develop simple tool to explore the spectrum: (quasi-)modular bootstrap
- Uncover some interesting strucuture of quasi-lisse VOAs
- Study the spectrum of some $\mathbb{V}[\mathcal{T}]$: A_1 class-$\mathcal{S}$, 4d $\mathcal{N}=4$ $SU(N)$, $\mathcal{T}_{p,N}$ theories
- Extend "4d mirror symmetry"

Outline

- background and motivation

  > RCFT, MLDE, SCFT/VOA

- quasi-modular bootstrap
- Modularity of $\mathbb{V}[\mathcal{T}_{p,N}]$

Overview: SCFT/VOA

`[Beem, Lemos, Liendo, Peelaers, Rastelli, Rees]`

- 4d $\mathcal{N}=2$ SCFT $\mathcal{T}$
- **Schur operators**: quarter-BPS ($Q^1_{-}, \tilde{Q}_{2\dot{-}}, S^{-}_1, \tilde{S}^{2\dot{-}}$)

$$E - 2R - j_1 - j_2 = r + j_2 - j_1 = 0\,.$$

- Schur operators on plane: **chiral algebra** (VOA) $\mathbb{V}[\mathcal{T}]$
- $\mathbb{V}$: 4d $\mathcal{N}=2$ SCFT $\mathcal{T} \to$ 2d chiral algebra $\mathbb{V}[\mathcal{T}]$

> `[Xie, Yan, Yau]` `[Cordova, Gaiotto, Shao]` `[Lemos, Peelaers]` `[Beem, Rastelli]` `[Buican, Nishinaka]` `... many others`

- Counting of Schur operators = Schur index $\mathcal{I}_{\mathcal{T}}(q,b)$ = vacuum character of $\mathbb{V}[\mathcal{T}]$

Overview: SCFT/VOA

- Non-trivial: contains a **Virasoro** subalgebra
- **Non-unitary**: $c_{2\mathrm{d}} = -12c_{4\mathrm{d}}$
- Flavor symmetry $f \to$ affine Lie subalgebra $\widehat{f}_k$
- Crucial superconformal invariant

  > independent of exactly marginal coupling

- Representation theory of $\mathbb{V}[\mathcal{T}]$ encodes **non-local operators** in $\mathcal{T}$

  > `[Cordova, Gaiotto, Shao]` `[Bianchi, Lemos]` `[Nishinaka, Sasa, Zhu]` `[Zheng, YP, Wang]`

Overview: SCFT/VOA

- **Quasi-lisse** (generally non-rational): non-trivial Higgs branch

  > `[Arakawa, Kawasetsu]` `[Arakawa]` `[Beem, Rastelli]`
  >
  > $$\text{rational} \subset \text{lisse} \subset \textbf{quasi-lisse}$$

- More intricate representation theory
- **ordinary** modules (easy), **non-ordinary** modules (hard):

  $\mathrm{ch}(b,q) := \mathrm{tr}\, q^{L_0 - \frac{c_{2\mathrm{d}}}{24}} \prod_{i=1}^{r} b_i^{h_0^i}$

  > ordinary characters: **unflavoring** limit exists
  >
  > $$\mathrm{ch}(q, b \to 1) = \mathrm{ch}(q)$$
  >
  > non-ordinary characters: **no** unflavoring limit,
  >
  > $$\mathrm{ch}_{\text{non-ord}}(q,b) \xrightarrow{b\to 1} \infty$$
  >
  > There are also **logarithmic** ordinary/non-ordinary modules

Overview: modular bootstrap

- RCFTs: **modular bootstrap**, **unflavored modular linear differential equations (MLDE)**

$$\text{Wronskian: } \left[D_q^{(N)} + \sum_{i=0}^{N-1} \phi_r(\tau) D_q^{(N-r)}\right] \mathrm{ch}(q) = 0, \quad \text{weight} = 2N$$

> $\mathrm{ch}(q)$ denotes any (unflavored) character, ϕ_r weight-$2r$ modular forms
> $D_q^{(N)} := \partial_{(2N-2)} \circ \cdots \circ \partial_{(2)} \circ \partial_{(0)}$, $\partial_{(k)} := q\partial_q + kE_2(\tau)$
>
> `[Mathur, Mukhi, Sen]` `[Eguchi, Ooguri]` `[Chandra, Mukhi]`
> `[Gaberdiel, Hampapura, Mukhi]` `[Bae, Duan, Lee, Lee, Sarkis]`
> `[Duan, Lee, Lee, Li]` `[Das, Gowdigere, Santara]`
> `[Kaidi, Lin, Martinez]` `[Mukhi, Rayhaun]` `... many others`

- Requires **unflavoring** limit $\mathrm{ch}(q)$; **covariant** under modular group

Overview: modular bootstrap

overview of the modular bootstrap

quasi-modular bootstrap

quasi-modular bootstrap

- Modular bootstrap: study or classify **ordinary** modules of $\mathbb{V}[\mathcal{T}]$

  > `[Kaidi, et.al.]` `[Deb, Meneghelli, Rastelli]`

- **non-ordinary** modules: needs **flavor refinement**
- **Quasi-modular bootstrap**
- **Flavored** MLDE

$$\text{eq} = \sum_{\substack{k,\ell_1,\ldots,\ell_r \\ 2k+\ell_1+\ldots+\ell_r \leq 2n}} \phi_{k,\ell_1,\ldots,\ell_r}(\tau, b_i) D_q^{(k)} D_{b_1}^{\ell_1} \ldots D_{b_r}^{\ell_r} \text{ch}(q, b_i) = 0$$

> $D_{b_i} := b_i \partial_{b_i}$, ϕ are products of Eisenstein series, **quasi-Jacobi forms**

quasi-modular bootstrap

- **Modular transformation** and **spectral flow** $\sigma_{\mathbf{n}}$

$$S:(\tau,\mathfrak{b}_i)\to\left(-\frac{1}{\tau},\frac{\mathfrak{b}_i}{\tau}\right),\quad T:\tau\to\tau+1\,,\qquad \sigma_{\mathbf{n}}:\mathfrak{b}_i\to\mathfrak{b}_i+n_i\tau$$

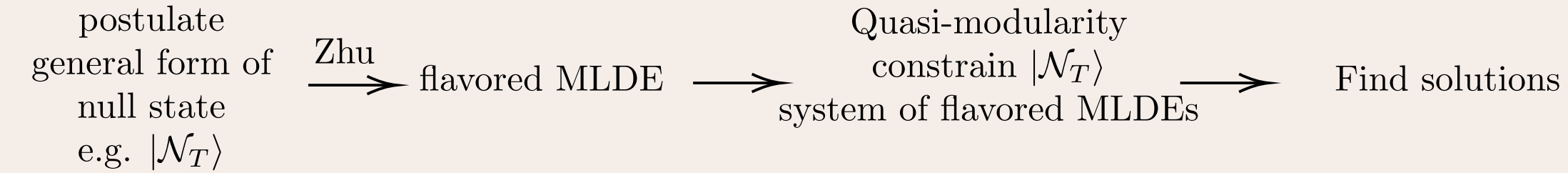
> $q=e^{2\pi i\tau}$, $b_i=e^{2\pi i\mathfrak{b}_i}$
> $(\sum_{i=1}^{r}n_i\alpha_i^\vee,\alpha)\in\frac{1}{2}\mathbb{Z}$, all $\alpha\in\Delta$

- **quasi-modularity**: eq transforms **inhomogeneously**

$$\mathrm{eq}\xrightarrow{S,T,\sigma}\tau^{\mathrm{wt}}\mathrm{eq}+\tau^{\mathrm{wt}-1}\mathfrak{b}_i\mathrm{eq}_i+\tau^{\mathrm{wt}-2}\mathfrak{b}_i\mathfrak{b}_j\mathrm{eq}_{ij}+\cdots$$

- **Quasi-modularity** (postulation): all $\mathrm{eq}_{ij\ldots}$ annhilate any flavored character $\mathrm{ch}(q,b)$

quasi-modular bootstrap



overview of the quasi modular bootstrap [YP, Zeng]

- $|\mathcal{N}_T\rangle$: "nilpotency null"

  [Arakawa, Kawasetsu] [Beem, Rastelli] [Deb, Meneghelli, Rastelli]

$$L_{-2}^{n_T}|0\rangle = |c_2\rangle + |\mathcal{N}_T\rangle\ , \qquad |c_2\rangle \in \text{span}\{a_{-h_a-1}|b\rangle\}$$

Why quasi-modular postulation?

- In all known examples: $\mathrm{eq}_{ij\ldots}$ annhilate ch_0
- Used to classify quasi-lisse $\widehat{\mathfrak{su}}(2)_k$ algebras: **integrable**, **boundary admissible**, general admisslbe

> [YP, Zeng]

n_T	**allowed k**
2	$\mathbf{1}$, $\mathbf{-4/3}$
3	$\mathbf{2}$, $\mathbf{-8/5}$, $-1/2$
4	$\mathbf{3}$, $\mathbf{-12/7}$
5	$\mathbf{4}$, $\mathbf{-16/9}$, $-5/4$
6	$\mathbf{5}$, $\mathbf{-20/11}$, $-7/5$, $-2/3$, $1/2$

quasi-modular bootstrap

- Observation: effective modular and spectral flow action on **null states**

$$\begin{array}{ccccccc} \mathrm{eq} & \overset{S,T,\sigma_{\mathbf{n}}}{\to} & \mathrm{eq}_i & \oplus & \mathrm{eq}_{ij} & \oplus \cdots \\ \downarrow & & \downarrow & & \downarrow & \\ |\mathcal{N}\rangle & \overset{S,T,\sigma_{\mathbf{n}}}{\to} & |\mathcal{N}_i\rangle & & |\mathcal{N}_{ij}\rangle & \cdots \end{array}$$

- explicitly

$$|\mathcal{N}\rangle \overset{S}{\to} \sum_{\ell\geq 0} \sum_{i_1,\ldots i_\ell=1}^{r} \frac{1}{\ell!} \tau^{h[\mathcal{N}]-\ell} \mathfrak{b}_{i_1} \cdots \mathfrak{b}_{i_\ell} h_1^{i_1} \ldots h_1^{i_\ell} |\mathcal{N}\rangle \ ,$$

$$|\mathcal{N}\rangle \xrightarrow{\sigma_{\mathbf{n}}} \sum_{\ell\geq 0} \sum_{i_1\leq\ldots\leq i_\ell} \frac{(-1)^\ell}{\ell!} n_{i_1} \ldots n_{i_\ell} h_1^{i_1} \cdots h_1^{i_\ell} |\mathcal{N}\rangle \ .$$

quasi-modular bootstrap

- Observation from cases

  $\mathrm{ch}_0(q,b) \xrightarrow{S,T,\sigma_{\mathbf{n}}}$ **untwisted** and **twisted**, ordinary, non-ordinary, logarithmic spectrum

  > quasi-lisse $\widehat{\mathfrak{su}}(2)_k$, non-admissible $\widehat{\mathfrak{so}}(8)_{-2}$
  > $\sigma_{\mathbf{n}}$: twisted $\leftrightarrow$ untwisted sector, ordinary $\leftrightarrow$ non-ordinary sector
  > S, T: non-log $\leftrightarrow$ log sector

- Modularity from the vacuum character is important: denotes $\mathcal{V}$ **the span** modular orbit of ch_0

$$\mathcal{V} = \mathrm{span}\left\{\mathrm{ch}_0, S\,\mathrm{ch}_0, T\,\mathrm{ch}_0, STS\,\mathrm{ch}_0, S^7T^{2025}ST^{1024}\,\mathrm{ch}_0, \dots\right\}$$

  > **Finite** dimensional
  > Sometimes need instead the span of $\Gamma^0(2)$-orbit

Modularity of $\mathcal{T}_{p,N}$ theory

Overview: Argyres-Douglas Theory

- Strongly-coupled 4d $\mathcal{N} = 2$ superconformal theory
- class-$\mathcal{S}$ construction: genus-zero Σ

  > `[Wang, Xie]` `[Xie]`

- Punctures: one **irregular puncture** $\mathrm{j}^b[k]$, (optional) one regular puncture
- Simple series: $D_p(SU(N)) = (A_{N-1}^N[p-N], F)$

  > `[Cecotti, Del Zotto]` `[Cecotti, Del Zotto, Giacomelli]`

  - $\mathrm{j} = A_{N-1}$
  - One irregular puncture $A_{N-1}^N[p-N]$
  - One full regular puncture F: $SU(N)$ **flavor symmetry**
  - p, N **coprime** $\Rightarrow$ irregular puncture carries **no** flavor symmetry

Overview: Argyres-Douglas Theory

- The Schur index of $D_p(SU(N))$

$$\mathcal{I}_{D_p(SU(N))}(q,b) = \mathrm{PE}\left[\frac{q-q^p}{(1-q)(1-q^p)}\chi^{SU(N)}_{\mathrm{adj}}(b)\right]$$
$$= \left[\frac{\eta(p\tau)}{\eta(\tau)}\right]^{\frac{1}{2}(3r-\dim \mathfrak{g})} \prod_{\substack{A,B=1 \\ A\neq B}}^{N} \frac{\vartheta_1(\mathfrak{b}_A-\mathfrak{b}_B|p\tau)}{\vartheta_1(\mathfrak{b}_A-\mathfrak{b}_B|\tau)}$$

Overview: Argyres-Douglas Theory

- SCFT/VOA: 4d $\mathcal{N}=2$ SCFT $\mathcal{T} \to \mathbb{V}[\mathcal{T}]$
- For $D_p(SU(N)) = (A_{N-1}^N[k=p-N], F)$: $\mathbb{V}$ is W-algebra

> `[Song, Xie, Yan]` `[Xie, Yan, Yau]` `[Wang, Xie]` `[Xie, Yan]`
>
> $$\mathcal{W}_{\kappa=-h^\vee+\frac{b}{b+k}}(A_{N-1},\ f=[1^N])$$
>
> f denotes a nilpotent orbit, specified by a partition of N, here $[1^N]$
>
> level $\kappa = -h^\vee + \frac{b}{b+k} = -h^\vee + \frac{N}{p}$ is **boundary admissible**

- Simplest case: $p=3$, $N=2$, $D_3(SU(2)) = (A_1, D_3) = H_1$ AD, $\mathbb{V} = \widehat{\mathfrak{su}}(2)_{-\frac{4}{3}}$

$\mathcal{T}_{p,N}$ theory

- Given several 4d $\mathcal{N}=2$ SCFTs with flavor symmetry G
- **Gauge** the diagonal G by adding dynamical G vector multilets
- Gauge may introduce non-zero β function: **non-conformal**
- Special arrangement: β function **vanishes**
- several $D_{p_i}(SU(N))$ can be treated as "matter" (like trinion theory): add vector multiplet to gauge them

> non-class-$\mathcal{S}$ operation

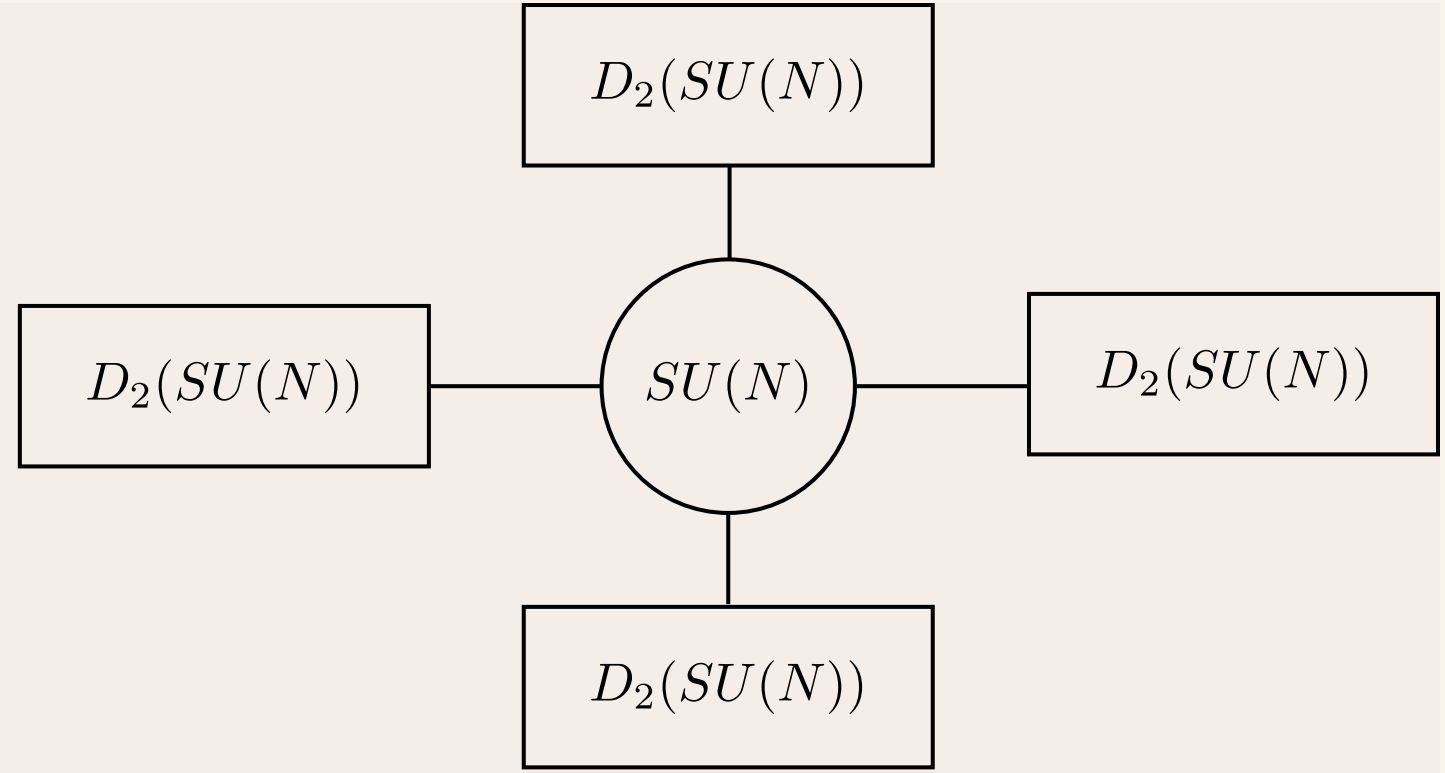
$\mathcal{T}_{p,N}$ theory

- Conformal gauging: 4 special possibilities (among many)

> `[Buican, Nishinaka]` `[Del Zotto, Vafa, Xie]` `[Kang, Lawrie, Song]`

$$\mathcal{T}_{2,N}, \qquad \mathcal{T}_{3,N}, \qquad \mathcal{T}_{4,N}, \qquad \mathcal{T}_{6,N}\ .$$

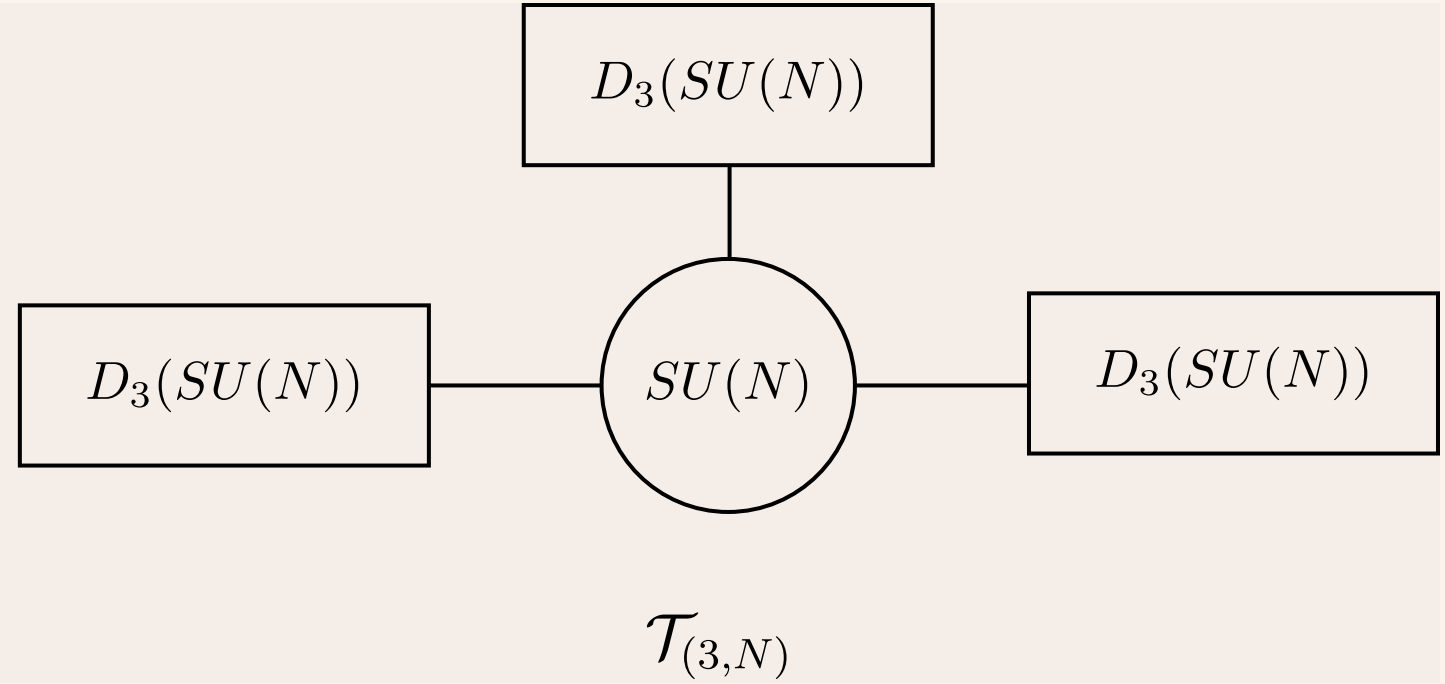
$\mathcal{T}_{p,N}$ theory



$\mathcal{T}_{(2,N)}$

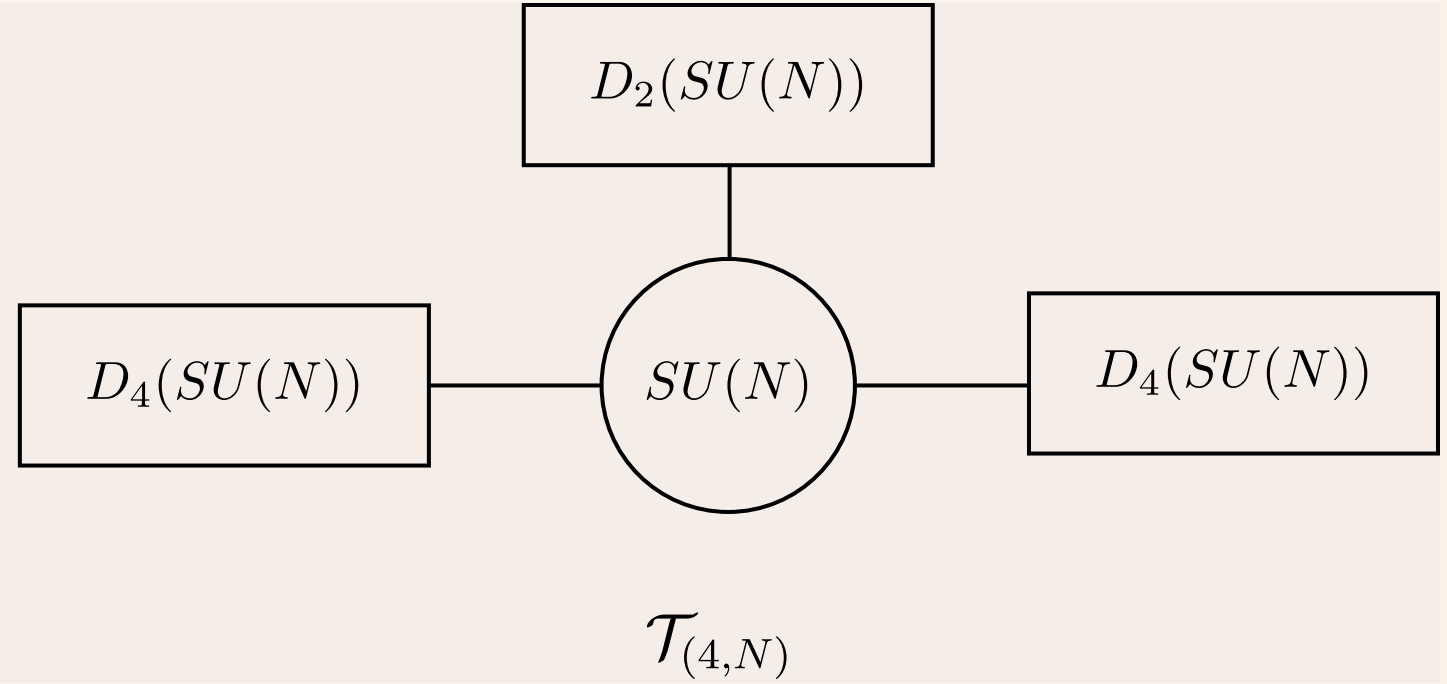
$N = \text{odd}$

$\mathcal{T}_{p,N}$ theory



$$N = 1, 2 \mod 3$$

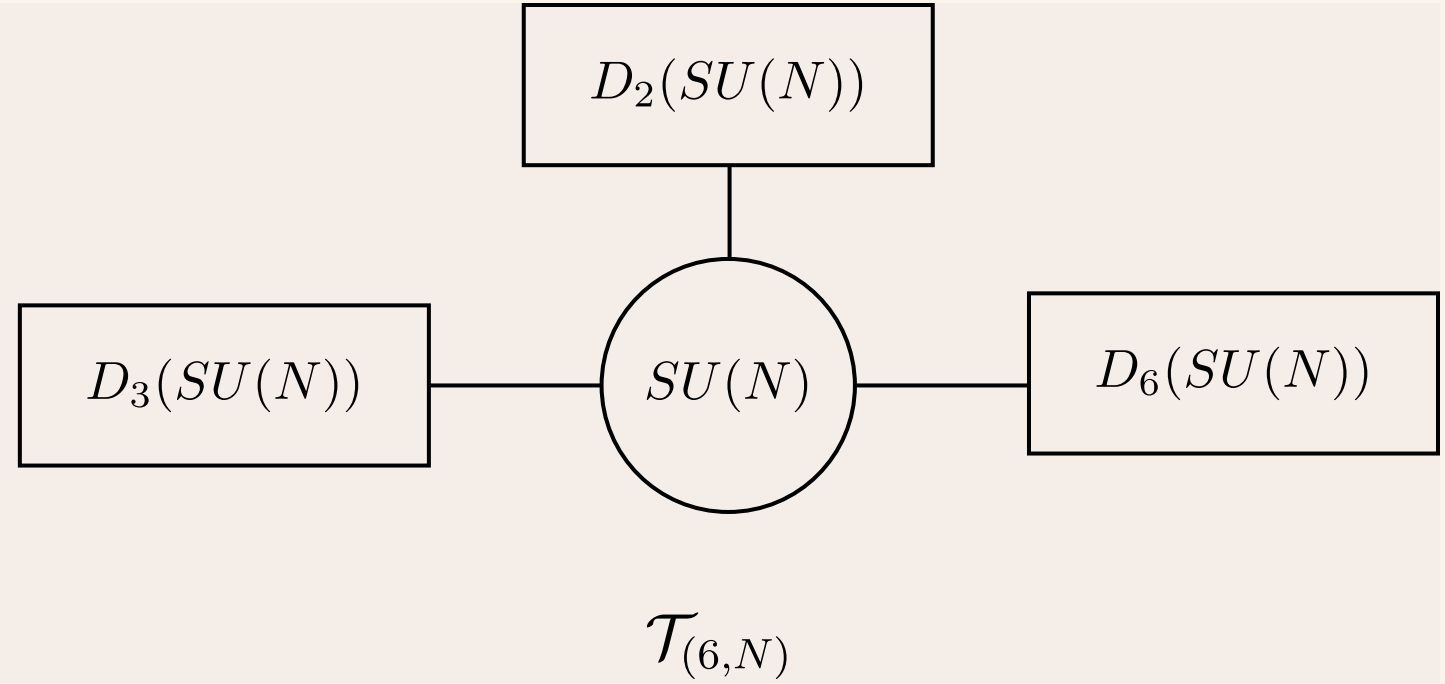
$\mathcal{T}_{p,N}$ theory



$\mathcal{T}_{(4,N)}$

$N = \text{odd}$

$\mathcal{T}_{p,N}$ theory



$$N = 1, 5 \mod 6$$

$\mathcal{T}_{p,N}$ theory

- Interesting property: $a_{4\mathrm{d}} = c_{4\mathrm{d}}$

  > all 4d $\mathcal{N} = 4$ SYMs also share the same property

- Close relation between $\mathcal{T}_{p,N}$ and 4d $\mathcal{N} = 4$ $SU(N)$ theory

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q) = q^{-\frac{c_{2\mathrm{d}}[\mathcal{T}_{p,N}] - p c_{2\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}} \mathcal{I}_{SU(N)}(b \to q^{\frac{p}{2}-1}, q^p) \,.$$

  > Schur index $\mathcal{I}_{SU(N)} = q^{-\frac{c_{2\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}}(1 + \cdots)$, while $\mathcal{I}_{\mathcal{T}_{p,N}} = q^{-\frac{c_{2\mathrm{d}}[\mathcal{T}_{p,N}]}{24}}(1 + \cdots)$

- **Vector space isomorphism** between $\mathbb{V}[\mathcal{T}_{SU(2)}]$ and $\mathbb{V}[\mathcal{T}_{3,2}]$

  > `[Buican, Nishinaka]`
  > **vacuum** module of $\mathbb{V}[\mathcal{T}_{SU(2)}] \to$ **vacuum** module of $\mathbb{V}[\mathcal{T}_{3,2}]$

Closed form of Schur index

- Closed form of $\mathcal{T}_{SU(N)} \Rightarrow$ closed form $\mathcal{I}_{\mathcal{T}_{p,N}}$
- Several way of getting closed form

> `[Bourdier, Drukker, Felix]` `[YP, Peelaers]` `[Hatsuda, Okazaki]` `[Du, Huang, Wang]`

- Direct integration using integration formula
- Fermi-gas formalism

$$\mathcal{I}_{\mathcal{T}_{SU(N)}} = \frac{1}{N!} \frac{u^{N/2} q^{-\frac{N^2}{6}} \vartheta_4(\mathfrak{b}) \eta(\tau)^{3(N-1)} b^{-\frac{2N^2}{3}}}{\vartheta_1(\mathfrak{u} + N(\mathfrak{b} + \frac{\tau}{2})) \vartheta_1(\mathfrak{u})^{N-1}} \sum_{\vec{n}} \frac{N! (-1)^{C_{\vec{n}}}}{\prod_{\ell=1}^{N} \ell^{n_\ell} n_\ell!} \prod_{\ell=1}^{N} Z_\ell^{n_\ell}$$

> $b = e^{2\pi i \mathfrak{b}}$ is the $SU(2)$ **flavor fugacity**; $\mathcal{I}_{\mathcal{T}_{SU(N)}}$ **independent** of $u = e^{2\pi i \mathfrak{u}}$

Closed form of Schur index

- Special choice: $u \to 1, \mathfrak{u} \to 0$

$$\mathcal{I}_{\mathcal{T}_{SU(N)}} = \begin{cases} \dfrac{\vartheta_4(\mathfrak{b}|\tau)}{\vartheta_4(N\mathfrak{b}|\tau)} \mathbb{E}_N(b,q), N = 2\ell + 1 \, , \\ \\ \dfrac{\vartheta_4(\mathfrak{b}|\tau)}{\vartheta_1(N\mathfrak{b}|\tau)} \mathbb{E}_N(b,q), N = 2\ell \, . \end{cases}$$

> $\mathbb{E}_N$ is a **polynomial** of Eisenstein series

Closed form of Schur index

- Overall prefactor

$$f_N(\mathfrak{b},\tau) := \begin{cases} \dfrac{\vartheta_4(\mathfrak{b}\mid\tau)}{\vartheta_4(N\mathfrak{b}\mid\tau)}, & N=2\ell+1, \\ \dfrac{\vartheta_4(\mathfrak{b}\mid\tau)}{i^{N^2-1}\vartheta_1(N\mathfrak{b}\mid\tau)}, & N=2\ell \end{cases}$$

- $\mathcal{T}_{p,N}$ limit: $\mathfrak{b}\to(\frac{p}{2}-1)\tau, \tau\to p\tau, b\to q^{\frac{p}{2}-1}, q\to q^p$

$$f_N(\mathfrak{b},\tau)\to q^{\frac{c_{2\mathrm{d}}[\mathcal{T}_{p,N}]-pc_{2\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}}$$

> cancelling the original coefficient
> Similar discussion in `[Jiang]`

Closed form of Schur index

- $\mathcal{I}_{\mathcal{T}_{p,N}}$ is a **polynomial** of Eisenstein series $E_k\left[\begin{smallmatrix}\pm 1\\ q^{\#}\end{smallmatrix}\right](p\tau)$

$$\mathcal{I}_{\mathcal{T}_{3,2}} = E_1\begin{bmatrix}-1\\ \sqrt{q}\end{bmatrix}(3\tau)$$

$$\mathcal{I}_{\mathcal{T}_{3,4}} = -\frac{1}{72} - \frac{1}{6}E_2(3\tau) + \frac{1}{24}E_1\begin{bmatrix}-1\\ \sqrt{q}\end{bmatrix}(3\tau)$$

Closed form of Schur index

- Reorganize $\mathcal{T}_{p,N}$

$$E_n\begin{bmatrix}\pm 1\\ b^p\end{bmatrix}(p\tau)=\frac{1}{p}\sum_{\ell=0}^{p-1}E_n\begin{bmatrix}\pm 1\\ e^{\frac{2\pi i\ell}{p}}b\end{bmatrix}(\tau)=\frac{1}{p}\sum_{\ell=0}^{p-1}E_n\begin{bmatrix}\pm 1\\ e^{\frac{2\pi i\ell}{p}}b\end{bmatrix}$$

> make modular property more manifest; omit (τ) for brevity

Modularity

- Modularity study of $\mathcal{T}_{p,N}$ initiated in `[Jiang]`
- Generalize and refine the known results
- Find relation between characters of $\mathbb{V}[\mathcal{T}_{SU(N)}]$ and $\mathbb{V}[\mathcal{T}_{p,N}]$

Modularity

- The $\mathcal{T}_{2,2\ell+1}$ theory: $p=2$, $N=2\ell+1$
- The flavor fugacity $b \to q^{\frac{p}{2}-1}=1$: just the unflavoring limit
- The Schur index examples

$$\mathcal{I}_{\mathcal{T}_{2,3}}(q) = \frac{1}{24} + \frac{1}{4}\left(E_2\begin{bmatrix}1\\1\end{bmatrix} + E_2\begin{bmatrix}1\\-1\end{bmatrix}\right).$$

$$\mathcal{I}_{\mathcal{T}_{2,5}} = \frac{1}{640}\left[3 + 20\left(E_2\begin{bmatrix}1\\-1\end{bmatrix} + E_2\begin{bmatrix}1\\1\end{bmatrix}\right) + 20\left(E_2\begin{bmatrix}1\\-1\end{bmatrix} + E_2\begin{bmatrix}1\\1\end{bmatrix}\right)^2 - 80\left(E_4\begin{bmatrix}1\\-1\end{bmatrix} + E_4\begin{bmatrix}1\\1\end{bmatrix}\right)\right].$$

Modularity

- $\mathcal{I}_{\mathcal{T}_{2,3}}$ under $SL(2,\mathbb{Z})$

$$1 \xrightarrow{SL(2,\mathbb{Z})} 1$$

$$\left(E_2\begin{bmatrix}1\\1\end{bmatrix}+E_2\begin{bmatrix}1\\-1\end{bmatrix}\right) \xrightarrow{SL(2,\mathbb{Z})} \tau^{\ell=0,1,2}\left(E_2\begin{bmatrix}+1\\+1\end{bmatrix}+E_2\begin{bmatrix}-1\\1\end{bmatrix}\right),$$
$$\tau^{\ell=0,1,2}\left(E_2\begin{bmatrix}+1\\+1\end{bmatrix}+E_2\begin{bmatrix}1\\-1\end{bmatrix}\right),$$
$$\tau^{\ell=0,1,2}\left(E_2\begin{bmatrix}+1\\+1\end{bmatrix}+E_2\begin{bmatrix}-1\\-1\end{bmatrix}\right).$$

- In total 10 linear independent objects: $\dim \mathcal{V}_{\mathcal{T}_{2,3}} = 10$

Modularity

- 12-th order MLDE in `[Jiang]`: **10** rational and **2** irrational indicial roots

$$
\begin{aligned}
0 = \Big[& D_q^{(12)} - 1510 E_4 D_q^{(10)} - 55440 E_6 D_q^{(9)} - 233400 E_4^2 D_q^{(8)} + 2364600 E_4 E_6 D_q^{(7)} \\
& + 2000 \left(31228 E_4^3 - 41013 E_6^2\right) D_q^{(6)} + 1422624000 E_4^2 E_6 D_q^{(5)} \\
& + \left(3925360000 E_4^4 + 40438916000 E_4 E_6^2\right) D_q^{(4)} \\
& + \left(420470400000 E_4^3 E_6 + 344509200000 E_6^3\right) D_q^{(3)} \\
& + \left(11688240000000 E_4^5 + 7510426000000 E_4^2 E_6^2\right) D_q^{(2)} \\
& + \left(23905224000000 E_4^4 E_6 + 31682361200000 E_4 E_6^3\right) D_q^{(1)} \Big] \mathrm{ch}_0(q)
\end{aligned}
$$

> Constant 1 is a solution

- **$10 < 12$? Did we miss two characters?**

Modularity

- **We didn't**
- A 10-th order equation, but at **modular weight 36**

$$\left[\left(\frac{55}{196}E_4^4 + E_4E_6^2\right)D_q^{(10)} - \frac{405}{11}E_4^3E_6D_q^{(9)} + \cdots\right]\mathrm{ch}_0(q) = 0$$

> polynomial coefficient $\frac{55}{196}E_4^4 + E_4E_6^2$

- span $\mathcal{V}$ of the modular orbit gives the full spectrum: 10 rational indicial roots

Modularity

- S, T matrices

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ -8 & 0 & 8 & 0 & -5 & 0 & 4 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -8 & 0 & 8 & 0 & -4 & 0 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -8 & 8 & 0 & -4 & 0 & 2 & 0 & 0 & 0 & \frac{1}{2} \\ -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Modularity

- T: 4 Jordan blocks
- Jordan block size

$$[3, 3, 3, 1]$$

Modularity

- $\mathcal{T}_{2,5}$ theory

$$\mathcal{I}_{\mathcal{T}_{2,5}} = \frac{1}{640}\left[3 + 20\left(E_2\begin{bmatrix}1\\-1\end{bmatrix} + E_2\begin{bmatrix}1\\1\end{bmatrix}\right) + 20\left(E_2\begin{bmatrix}1\\-1\end{bmatrix} + E_2\begin{bmatrix}1\\1\end{bmatrix}\right)^2 - 80\left(E_4\begin{bmatrix}1\\-1\end{bmatrix} + E_4\begin{bmatrix}1\\1\end{bmatrix}\right)\right].$$

Modularity

- Modular transformation: independent objects

$$1$$

$$\tau^{\ell=0,1,2}\left(E_2\begin{bmatrix}1\\-1\end{bmatrix}+E_2\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right)\Big|_{\substack{(\alpha,\beta)=(1,-1),\\(-1,1),(-1,-1)}}$$

$$\tau^{\ell=0,1,2,3,4}\left[\left(E_2\begin{bmatrix}1\\+1\end{bmatrix}+E_2\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right)^2-4\left(E_4\begin{bmatrix}1\\+1\end{bmatrix}+E_4\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right)\right]\Big|_{\substack{(\alpha,\beta)=(1,-1),\\(-1,1),(-1,-1)}}$$

- Dimension of $SL(2,\mathbb{Z})$

$$\dim \mathcal{V}_{\mathcal{T}_{2,5}} = 1+3\times 3+3\times 5=25$$

Modularity

- **MLDE?**
- first MLDE: order 37: **25** rational and **12** irrational indicial roots
- $37 (> 25)$; **miss?**
- **No we didn't**: at **modular weight 254**, found a 25-th order MLDE
- $\mathcal{V}$ is the full spectrum: 25 rational indicial roots
- S, T matrix
- T-matrix: 7 Jordan blocks, block sizes $[1, 3, 3, 3, 5, 5, 5]$

Modularity

- For all $\mathcal{T}_{2,2\ell+1}$
- Dimension of $SL(2,\mathbb{Z})$-orbit

$$1+3\times 3+3\times 5+3\times 7+\cdots+3\times(2\ell+1)=1+3\lfloor\ell\rfloor(2+\lfloor\ell\rfloor)$$

- **Conjecture**: minimal order of MLDE: $1+3\lfloor\ell\rfloor(2+\lfloor\ell\rfloor)$
- T-matrix Jordan block numbers: $1+3\ell$, block sizes

$$[1,3,3,3,5,5,5,\cdots,2\ell+1,2\ell+1,2\ell+1]$$

Modularity

- $\mathcal{T}_{3,2}$ theory

$$\mathrm{ch}_0 = q^{-1/3}\frac{i\vartheta_4(\frac{\tau}{2}|3\tau)}{\vartheta_1(\tau|3\tau)}E_1\begin{bmatrix}-1\\ \sqrt{q}\end{bmatrix}(3\tau) = \frac{1}{3}\sum_{\ell=0}^{2}E_1\begin{bmatrix}-1\\ e^{\frac{2\pi i\ell}{3}}\sqrt{q}\end{bmatrix}(\tau)$$

- Modular orbit: complete basis of $\mathcal{V}_{\mathcal{T}_{3,2}}$

$$\mathrm{ch}_0, \quad \mathrm{ch}_1 = S\,\mathrm{ch}_0$$
$$\mathrm{ch}_2 = T\,\mathrm{ch}_1, \quad \mathrm{ch}_3 = T\,\mathrm{ch}_2, \quad \mathrm{ch}_4 = T\,\mathrm{ch}_3 - \mathrm{ch}_1$$

> Different basis in `[Jiang]`

Modularity

- 5-th order unflavored MLDE

  > `[Jiang]`

$$[D_q^{(5)} - 140E_4D_q^{(3)} - 700E_6D_q^{(2)} - 2000E_4^2D_q^{(1)}]\,\mathrm{ch}_0 = 0$$

- $\mathcal{V}_{\mathcal{T}_{3,2}}$ is the full spectrum
- **Constant** $\mathrm{ch} = 1$ is a solution

Modularity

- S, T matrices

  > see also `[Jiang]`

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 3\sqrt[3]{-1} & -3 & 3-3\sqrt[3]{-1} & 2\sqrt[3]{-1} \end{pmatrix}$$

- 3 Jordan blocks, sizes $[2, 2, 1]$

Modularity

- $\mathcal{T}_{4,3}$
- basis of length 13

$$
\begin{aligned}
&\mathrm{ch}_0\\
&\mathrm{ch}_1 = S\,\mathrm{ch}_0, \quad \mathrm{ch}_2 = T\,\mathrm{ch}_1, \quad \mathrm{ch}_3 = T\,\mathrm{ch}_2 - \mathrm{ch}_1, \quad \mathrm{ch}_4 = T\,\mathrm{ch}_3, \quad \mathrm{ch}_5 = T\,\mathrm{ch}_4,\\
&\mathrm{ch}_6 = i\,\mathrm{ch}_3 - \mathrm{ch}_4 - i\,\mathrm{ch}_5 + T\,\mathrm{ch}_5, \quad \mathrm{ch}_7 = T\,\mathrm{ch}_6, \quad \mathrm{ch}_8 = T\,\mathrm{ch}_7,\\
&\mathrm{ch}_9 = S\,\mathrm{ch}_8, \quad \mathrm{ch}_{10} = T\,\mathrm{ch}_9,\\
&\mathrm{ch}_{11} = T\,\mathrm{ch}_{10}, \quad \mathrm{ch}_{12} = T\,\mathrm{ch}_{11},
\end{aligned}
$$

Modularity

- 17-th order unflavored MLDE in `[Jiang]`: **13** rational indicial roots and **4** irrational roots
- $17 > 13$
- **Guess: 13-th order unflavored MLDE?**
- **Yes**: 13-th order unflavored MLDE at modular weight 72
- $\mathcal{V}$ is the full spectrum: 13 rational indicial roots
- Compute S, T matrix
- T has 6 Jordan blocks, of sizes

$$3, 3, 2, 2, 2, 1$$

$\mathbb{V}[\mathcal{T}_{SU(N)}]$ and $\mathbb{V}[\mathcal{T}_{p,N}]$ modules

$$\mathcal{I}_{\mathcal{T}_{p,N}} = q^{-\frac{c_{2\mathrm{d}}[\mathcal{T}_{p,N}]-pc_{2\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}} \mathcal{I}_{SU(N)}(q^{\frac{p}{2}-1}, q^p)$$

- vacuum character of $\mathbb{V}[\mathcal{T}_{SU(N)}] \to$ vacuum character of $\mathbb{V}[\mathcal{T}_{p,N}]$
- **Other modules of $\mathbb{V}$?**

$\mathbb{V}[\mathcal{T}_{SU(N)}]$ and $\mathbb{V}[\mathcal{T}_{p,N}]$ modules

- 4d $\mathcal{N}=4$ $SU(2)$ has two irreducible modules + one "logarithmic" module

  > `[Adamovich]`

- Characters

$$\mathrm{ch}_0(q,b)=\frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}E_1\begin{bmatrix}-1\\ b\end{bmatrix},\quad \mathrm{ch}_1=\frac{\mathrm{ch}_0(bq,q)}{b^3q^{\frac{3}{2}}}+\mathrm{ch}_0(q,b)=\frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}$$

$$\mathrm{ch}_{\log}(q,b)=S\,\mathrm{ch}_0(q,b)$$

- For $\mathcal{T}_{3,2}$, $p=3$, $N=2$

$$q^{-\frac{\mathrm{ch}_{2\mathrm{d}}[\mathcal{T}_{3,2}]-3\,\mathrm{ch}_{2\mathrm{d}}[\mathcal{T}_{SU(2)}]}{}}\,\mathrm{ch}_1(q^{\frac{p}{2}-1},q^p)=1$$

A solution to the 5-th order MLDE

$\mathbb{V}[\mathcal{T}_{SU(N)}]$ and $\mathbb{V}[\mathcal{T}_{p,N}]$ modules

- **Higher N?** Non-vacuum modules are **not** known explicitly
- Guess from physics, quasi-modular bootstrap: non-local operators, spectral flow
- ch_1 of $SU(2)$ case:
  - From Wilson line index

$$\langle W_j \rangle = \delta_{j\in\mathbb{Z}}\,\mathrm{ch}_0 - \frac{1}{2}\frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}\sum_{\substack{m=-j\\ m\neq 0}}^{j}\frac{b^m - b^{-m}}{q^{m/2}-q^{-m/2}}\,.$$

  - From spectral-flow

$$\mathrm{ch}_1 = \frac{\mathrm{ch}_0(bq,q)}{b^3 q^{\frac{3}{2}}} + \mathrm{ch}_0(q,b) = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}$$

quasi-modularity and spectral flow

- Line defect index: linear combinations non-vacuum module characters

  > `[Cordova, Gaiotto, Shao]`

- Example: (A_1, A_2) theory

$$\langle L \rangle_{(A_1,A_2)} = q^{-\frac{1}{2}} \operatorname{ch}_0(q) - q^{-\frac{1}{2}} \operatorname{ch}_1(q)$$

  > $\mathbb{V}[(A_1, A_2)]$ is Virasoro minimal model $V_{2,5}$

- Example: (A_1, D_3) theory

$$\langle L \rangle = -q^{-\frac{1}{2}} b^{-1} \operatorname{ch}_0(q, b) + q^{-\frac{1}{2}} b^{-1} \operatorname{ch}_2(q, b)$$

quasi-modularity and spectral flow

- Line defect index: linear combinations non-vacuum module characters, coefficients are **rational** function of b, q
- 4d $\mathcal{N} = 4$ $SU(N)$ theory: Wilson line index

> `[Hatsuda, Okazaki]` `[Guo, Li, YP, Wang]`

$$\langle W_Y \rangle = \text{combinations of spectral flow of } \mathrm{ch}_0(q, b)$$

quasi-modularity and spectral flow

- **spectral flow of ch_0: candidate of module characters?**
- Too hard
- **Question: mapped to module characters of $\mathbb{V}[\mathcal{T}_{p,N}]$?**
- Answer seems to be **"yes"**

quasi-modularity and spectral flow

- $\mathcal{T}_{SU(2)} \to \mathcal{T}_{3,2}$
- "spectral-flow"

$$\sigma(\mathrm{ch}_0(q,b)) := \frac{\mathrm{ch}_0(bq,q)}{b^3 q^{3/2}} - \mathrm{ch}_0(b,q) = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}$$

- Trivial limit at $p=3$: $b \to q^{\frac{1}{2}}$, $q \to q^3$

$$q^{-\frac{c_{2\mathrm{d}}[\mathcal{T}_{3,2}]-3c_{2\mathrm{d}}[\mathcal{T}_{SU(2)}]}{24}} \left[\sigma(\mathrm{ch}_0(q,b))\right]_{b\to q^{\frac{1}{2}},q\to q^3} = 1$$

The simplest solution to the 5-th order MLDE

quasi-modularity and spectral flow

- $\mathcal{T}_{SU(3)} \to \mathcal{T}_{4,3}$
- "spectral-flow"

$$\sigma(\mathrm{ch}_0(q,b)) := \frac{\mathrm{ch}_0(bq,q)}{b^8 q^4} - \mathrm{ch}_0(b,q) = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}\left(-\frac{3}{2} + E_1\begin{bmatrix}-1\\ b\end{bmatrix} + E_1\begin{bmatrix}1\\ b^2\end{bmatrix}\right)$$

- Simple verification for $p=4$, $N=3$:

$$q^{-\frac{c_{2\mathrm{d}}[\mathcal{T}_{3,4}]-pc_{2\mathrm{d}}[\mathcal{T}_{SU(N)}]}{24}}\Big[\sigma(\mathrm{ch}_0(q,b))\Big]_{b\to q^{\frac{p}{2}-1},q\to q^p}$$

  solves the 13-order MLDE

  > equivalenetly: linear combination of $\mathrm{ch}_{0,1,\ldots 12}$

quasi-modularity and spectral flow

- 4d $\mathcal{N}=4$ $SU(5) \to \mathcal{T}_{p=2,N=5}$
- "Spectral flow"

$$\sigma(\mathrm{ch}_0(q,b)) := \frac{\mathrm{ch}_0(bq,q)}{b^{24}q^{12}} - \mathrm{ch}_0(b,q)$$

- Example:

$$\sigma^3(\mathrm{ch}_0(q,b))|_{b\to q^{\frac{p}{2}-1},q\to q^p}$$

  solves both the 37-order and 25-th order MLDE

quasi-modularity and spectral flow

- 4d $\mathcal{N}=4$ $SU(7) \to \mathcal{T}_{p=2,N=7}$
- "Spectral flow"

$$\sigma(\mathrm{ch}_0(q,b)) := \frac{\mathrm{ch}_0(bq,q)}{b^{48}q^{24}} - \mathrm{ch}_0(b,q)$$

- Example:

$$\sigma^4(\mathrm{ch}_0(q,b))|_{b\to 1} = -\frac{4067294\,\mathrm{ch}_{36}}{3} + \frac{1925\,\mathrm{ch}_{41}}{74} - \frac{9675\,\mathrm{ch}_{42}}{74} + \frac{75186889\,\mathrm{ch}_{45}}{888}$$

$$\sigma^5(\mathrm{ch}_0(b,q))|_{b\to 1} = -\frac{4705960}{3}\mathrm{ch}_{36} + \frac{588245}{6}\mathrm{ch}_5 \ , \qquad \cdots$$

quasi-modularity and spectral flow

- Summary:

"characters" of $\mathbb{V}[\mathcal{T}_{SU(N)}]$ $\sim$ "spectral flow" of $\mathrm{ch}_0 \to$ characters of $\mathbb{V}[\mathcal{T}_{p,N}]$

4d Mirror symmetry

- 4d mirror symmetry conjecture:

  irreducible module of $\mathbb{V}[\mathcal{T}] \leftrightarrow$ Hitchin moduli space fixed manifold

  > `[Gaiotto, Moore, Neitzke]` `[Fredrickson, Neitzke]` `[Fredrickson, Pei, Yan, Ke]` `[Shan, Xie, Yan]` `[YP, Yan]`

- Relevant correspondence

  Jordan blocks of $T \leftrightarrow$ simple modules $\leftrightarrow$ fixed varieties

  Jordan block sizes $\leftrightarrow$ dimension of fixed varieties

4d Mirror symmetry

- For some of the $\mathcal{T}_{p,N}$: **number** of fixed points can be computed relatively easily

  > `[Kang, Laurie, Lee, Sacchi, Song]` `[Jiang]`

  $$\mathcal{T}_{3,2} = ((D_4)^6[3], \mathfrak{p}), \quad \mathcal{T}_{4,3} = ((E_6)^{12}[4], \mathfrak{p}), \quad \mathcal{T}_{6,5} = ((E_8)^{30}[6], \mathfrak{p})$$

- "affine Springer-fibre technique" for $(\mathfrak{j}^b[k], f)$

  > `[Shan, Xie, Yan]`

- For simplest cases: counting some special elements in the extended Weyl group

4d Mirror symmetry

- $\mathcal{T}_{3,2}$: $\mathrm{j} = D_4$, $b = 6$, $k = 3$, $\nu = \frac{3}{2}$
- A special set of affine roots

$$S_\nu = \{\alpha + \ell\delta \in \hat{\Delta} \mid \nu(\alpha, \rho) + \ell = \nu\}$$

- Look for translations $t_{\sum_i^4 m_i \alpha_i^\vee}$ such that

$$t_{\sum_i^4 m_i \alpha_i^\vee}(S_\nu) \in \{\alpha + \ell\delta \mid \ell > 0\}$$

> and also need to mod out certain Weyl group action

- For this case: 3 such translations

4d Mirror symmetry

- $\mathcal{T}_{3,2}$:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 3\sqrt[3]{-1} & -3 & 3-3\sqrt[3]{-1} & 2\sqrt[3]{-1} \end{pmatrix}$$

- 3 Jordan blocks

4d Mirror symmetry

- $\mathcal{T}_{4,3}$: $\mathrm{j} = E_6$, $b = 12$, $k = 4$, $\nu = \frac{4}{3}$
- Find 6 translations
- T-matrix: 13×13, 6 Jordan blocks

4d Mirror symmetry

- $\mathcal{T}_{6,5}$
- Modular orbit: WIP with Peihe Yang
- Affine Springer fibre: number of fixed varieties $= 15$, should predict 15 Jordan blocks in T-matrix

Summary

- Develop simple tool to explore the spectrum: (quasi-)modular bootstrap
- Uncover some interesting strucuture of quasi-lisse VOAs
- Study the spectrum of some $\mathbb{V}[\mathcal{T}]$: A_1 class-$\mathcal{S}$, 4d $\mathcal{N}=4$ $SU(N)$, $\mathcal{T}_{p,N}$ theories
- Extend "4d mirror symmetry"

Outlook

- Compute the dimension of the fixed manifolds for $\mathcal{T}_{p,N}$ theories, compare with the Jordan block sizes
- 4d mirror symmetry for $\mathcal{N}=4$: find the full modular and spectral flow orbit of ch_0
- Gukov-Witten defects in 4d $\mathcal{N}=4$: relation to spectral flow orbit of ch_0
- Gukov-Witten, line defects in $\mathcal{T}_{p,N}$: relation to the spectrum of $\mathbb{V}[\mathcal{T}_{p,N}]$
- 4d mirror symmetry and its 3d reduction?