Conformal Field Theories On Magic Triangle

Kimyeong Lee
BIMSA

Some formal aspects of Field theories and Holography
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Kaiwen Sun, KL, Haowu Wang 24,23
Kaiwen Sun,KL, TBA

Jin-Beom Bae (WorldQuant)

Zhihao Duan (Queen-Mary)

Jeff Harvey (Chicago)

Sungjay Lee (KIAS)

Linfeng Li (KIAS)

Brandon Rayhaun (Stonybrook)

Matthieu Sarkis (Luxembourg)

Kaiwen Sun (USTC)

Haowu Wang (Wuhan)

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Magic Square & Triangle

Division Algebra $\mathbb{A}$

Finite dimensional division algebra over $\mathbb{R}$

(vector space, bilinear multiplication, multiplicative identity 1, inverse a^{-1} for nonzero elements a such that $aa^{-1} = a^{-1}a = 1$)

Normed division algebra over $\mathbb{R}$ is possible only for d=1,2,4,8

$\mathbb{R}$: associative and commutative

$\mathbb{C}$: complex number

$\mathbb{H}$: Hamilton's quaternions: associative and non commutative

$\mathbb{O}$: octonions (8,0), (4,4) if neither

$\mathbb{S}$: sextonions from split octonions (light cone, null), not a normed division algebra

$\mathbb{T}$: trinion from (2,2) quaternions ???

Magic Square

Projective plane $\mathbf{P}^2(\mathbf{K}, \mathbf{K}')$

Freudanthal, Tits, Vinberg

$$\dim \mathbf{P}^2 = 2 \cdot \mathrm{d}_K \cdot \mathrm{d}_{K'}$$

$$\mathbf{P}^2(\mathbb{R}, \mathbb{R}) = SU(2)/U(1) = \mathbb{P}^2$$

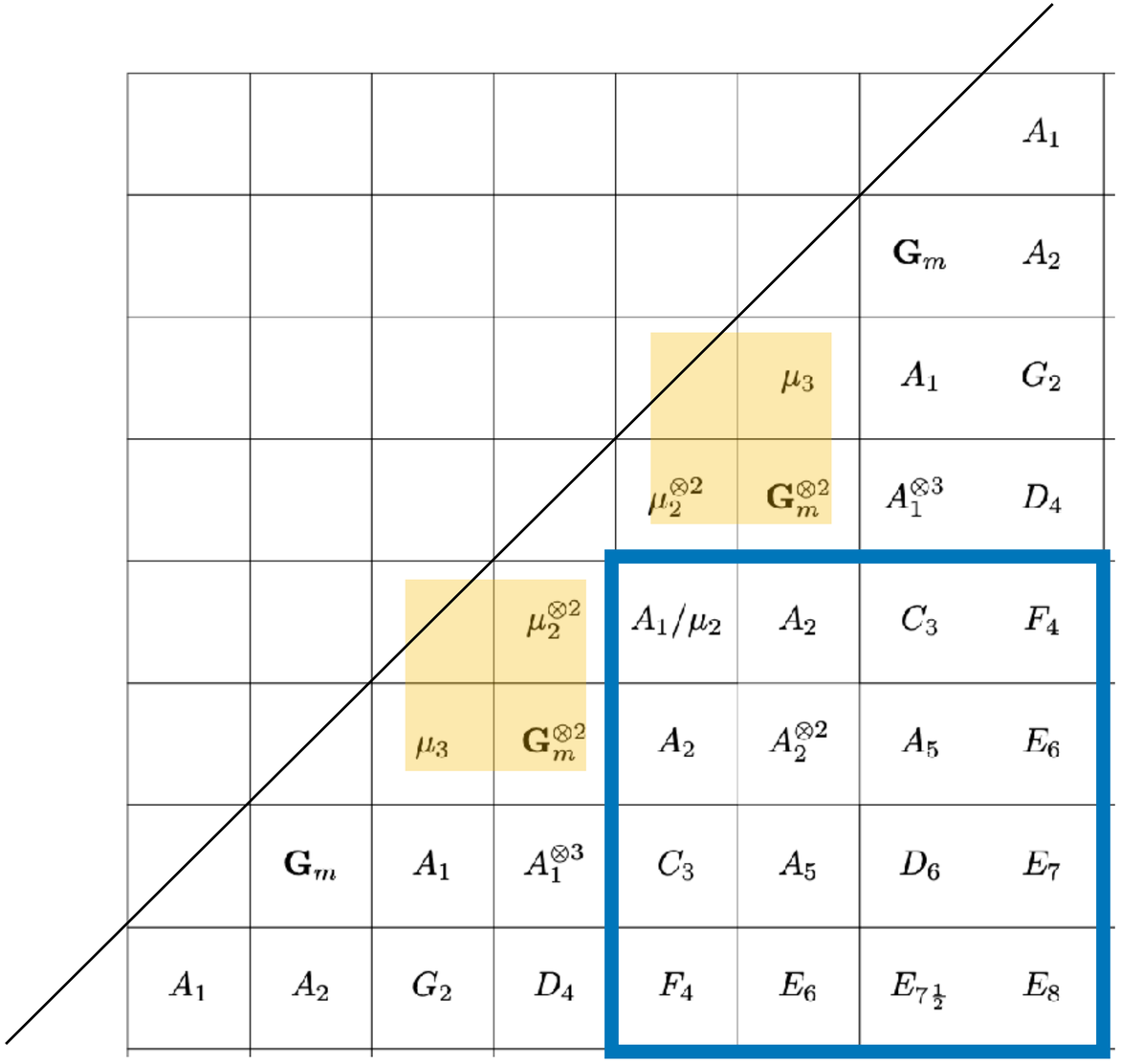
Symmetry group: $SO(3) = A_1$

$$\mathbf{P}^2(\mathbb{R}, \mathbb{Q}) = F_4/SO(9)$$

A_1	A_2	C_3	F_4	$\mathbb{R}$
A_2	$A_2 \times A_2$	A_5	E_6	$\mathbb{C}$
C_3	A_5	D_6	E_7	$\mathbb{H}$
F_4	E_6	E_7	E_8	$\mathbb{O}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	

Freudanthal-Tits Magic Square

Magic Triangle



							A_1
						$\mathbf{G}_m$	A_2
					μ_3	A_1	G_2
				$\mu_2^{\otimes 2}$	$\mathbf{G}_m^{\otimes 2}$	$A_1^{\otimes 3}$	D_4
			$\mu_2^{\otimes 2}$	A_1/μ_2	A_2	C_3	F_4
		μ_3	$\mathbf{G}_m^{\otimes 2}$	A_2	$A_2^{\otimes 2}$	A_5	E_6
	$\mathbf{G}_m$	A_1	$A_1^{\otimes 3}$	C_3	A_5	D_6	E_7
A_1	A_2	G_2	D_4	F_4	E_6	$E_{7\frac{1}{2}}$	E_8

Cvitanovic, Deligne-Gross

New Results

Magic Triangle & exceptional series

	$\mathbb{O}'$	$\mathbb{S}'$	$\mathbb{H}'$	$\mathbb{C}'$	$\mathbb{R}'$	0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{S}$	$\mathbb{O}$
a	$-\frac{8}{5}$	$-\frac{3}{2}$	$-\frac{4}{3}$	-1	$-\frac{2}{3}$	0	1	2	4	6	8
$\mu = 1 + a/2$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{3}{2}$	2	3	4	5

a: Landsberg-Manivel's parameter

$1 + a/2$: Deligne-Gross' parameter

										0?	1/5
									0?	$\mathrm{Vir}_{5,2}$	1/4
								0?	$\mathrm{Vir}_{5,3}$	A_1	1/3
							0?	$(U_1)_3$	$UA_{1\frac{1}{2}}$	A_2	1/2
						0?	$\mathrm{Vir}^e_{6,5}$	$(A_1)_3$	$AG_{1\frac{1}{2}}$	G_2	2/3
					0?	D_{2A}	M_2	$A_1^{\otimes 3}$	$AD_{3\frac{1}{2}}$	D_4	1
				0?	D_{2A}	$(A_1)_8^d$	$(A_2)_2$	C_3	$C_{3\frac{1}{2}}$	F_4	3/2
			0?	$\mathrm{Vir}^e_{6,5}$	M_2	$(A_2)_2$	$A_2^{\otimes 2}$	A_5	$A_{5\frac{1}{2}}$	E_6	2
		0?	$(U_1)_3$	$(A_1)_3$	$A_1^{\otimes 3}$	C_3	A_5	D_6	$D_{6\frac{1}{2}}$	E_7	3
	0?	$\mathrm{Vir}_{5,3}$	$UA_{1\frac{1}{2}}$	$AG_{1\frac{1}{2}}$	$AD_{3\frac{1}{2}}$	$C_{3\frac{1}{2}}$	$A_{5\frac{1}{2}}$	$D_{6\frac{1}{2}}$	$D_{6\frac{1}{2}\frac{1}{2}}$	$E_{7\frac{1}{2}}$	4
0?	$\mathrm{Vir}_{5,2}$	A_1	A_2	G_2	D_4	F_4	E_6	E_7	$E_{7\frac{1}{2}}$	E_8	5
1/5	1/4	1/3	1/2	2/3	1	3/2	2	3	4	5	

Wess-Zumino-Witten Chiral Algebra

Symmetry group G with Lie algebra $\mathfrak{g}$

$d_{\mathfrak{g}}$, $h^{\vee}_{\mathfrak{g}}$: dimension and dual Coxeter number

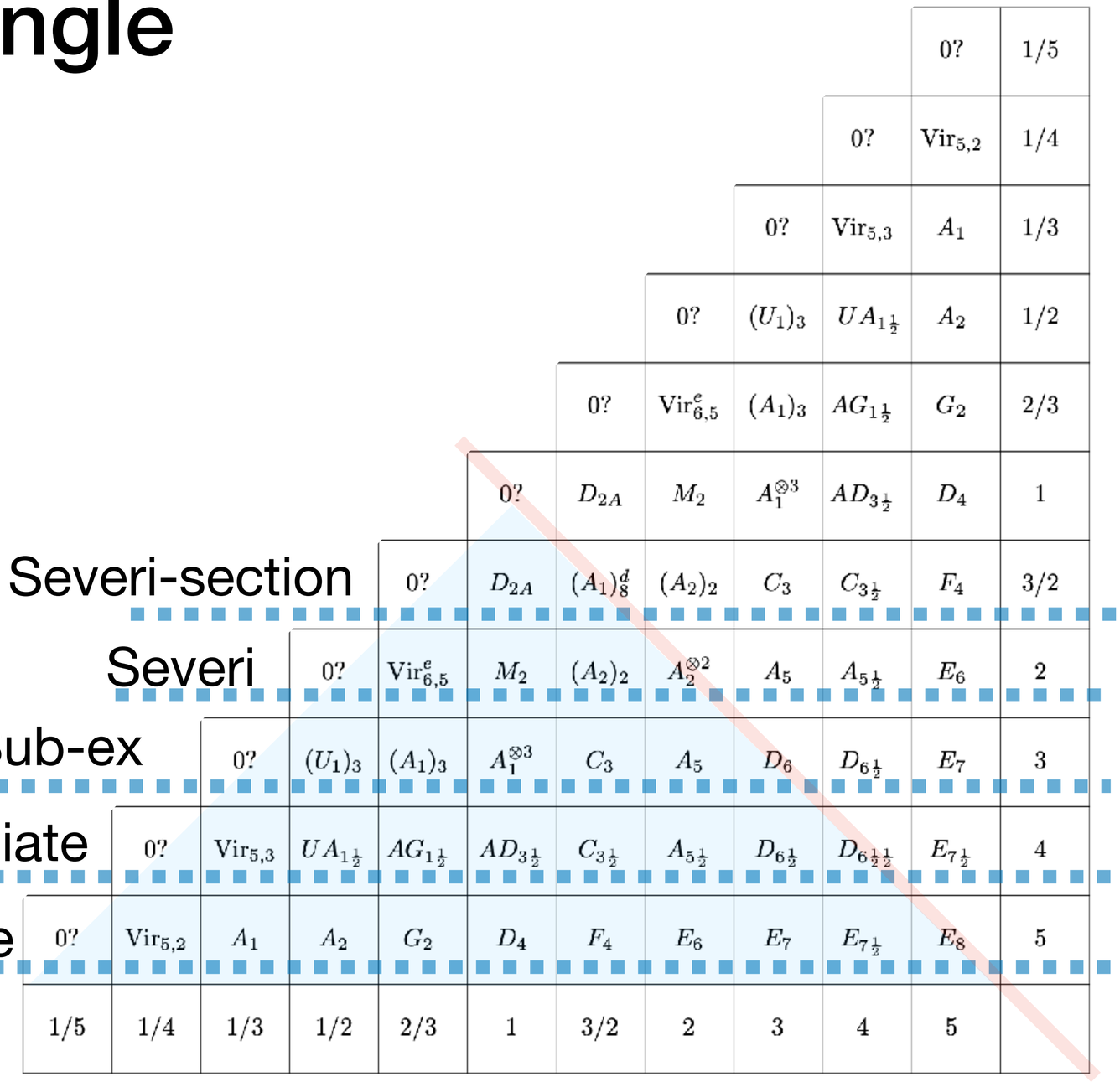
2-dim chiral algebra: RCFTs or VOAs with current algebra with level k

central charge: $c_k = \dfrac{k d_{\mathfrak{g}}}{k + h^{\vee}_{\mathfrak{g}}}$

CFTs on Magic Triangle

$$\mathscr{T}(\mu,\mu) = \mathscr{T}(\nu,\mu)$$

$$\mu\nu \geq 1$$



										0?	1/5
									0?	$\mathrm{Vir}_{5,2}$	1/4
								0?	$\mathrm{Vir}_{5,3}$	A_1	1/3
							0?	$(U_1)_3$	$UA_{1\frac{1}{2}}$	A_2	1/2
						0?	$\mathrm{Vir}^e_{6,5}$	$(A_1)_3$	$AG_{1\frac{1}{2}}$	G_2	2/3
					0?	D_{2A}	M_2	$A_1^{\otimes 3}$	$AD_{3\frac{1}{2}}$	D_4	1
				0?	D_{2A}	$(A_1)_8^d$	$(A_2)_2$	C_3	$C_{3\frac{1}{2}}$	F_4	3/2
			0?	$\mathrm{Vir}^e_{6,5}$	M_2	$(A_2)_2$	$A_2^{\otimes 2}$	A_5	$A_{5\frac{1}{2}}$	E_6	2
		0?	$(U_1)_3$	$(A_1)_3$	$A_1^{\otimes 3}$	C_3	A_5	D_6	$D_{6\frac{1}{2}}$	E_7	3
	0?	$\mathrm{Vir}_{5,3}$	$UA_{1\frac{1}{2}}$	$AG_{1\frac{1}{2}}$	$AD_{3\frac{1}{2}}$	$C_{3\frac{1}{2}}$	$A_{5\frac{1}{2}}$	$D_{6\frac{1}{2}}$	$D_{6\frac{1}{2}\frac{1}{2}}$	$E_{7\frac{1}{2}}$	4
0?	$\mathrm{Vir}_{5,2}$	A_1	A_2	G_2	D_4	F_4	E_6	E_7	$E_{7\frac{1}{2}}$	E_8	5
1/5	1/4	1/3	1/2	2/3	1	3/2	2	3	4	5	

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1/2 Magic Triangle

30 CFTS

				D_{2A}	$(A_1)_8^e$				
			$\mathrm{Vir}_{6,5}^e$	M_2	$(A_2)_2$	$A_2^{\otimes 2}$			
		$(U_1)_3$	$(A_1)_3$	$A_1^{\otimes 3}$	C_3	A_5	D_6		
	$\mathrm{Vir}_{5,3}$	$UA_{1\,\frac{1}{2}}$	$AG_{1\,\frac{1}{2}}$	$AD_{3\,\frac{1}{2}}$	$C_{3\,\frac{1}{2}}$	$A_{5\,\frac{1}{2}}$	$D_{6\,\frac{1}{2}}$	$D_{6\,\frac{1}{2}\,\frac{1}{2}}$	
$\mathrm{Vir}_{5,2}$	A_1	A_2	G_2	D_4	F_4	E_6	E_7	$E_{7\,\frac{1}{2}}$	E_8

$\mathcal{T}(\mu,\nu)_k$ on Magic Triangle

Uniform formula

Dimension Formula: $d_{\mathfrak{g}} = \dfrac{12(\mu\nu-1)(\mu\nu+\mu+\nu-4)}{(\mu+1)(\nu+1)}$

Dual Coxeter number: $h^{\vee}_{\mathfrak{g}} = \mu\nu+\mu+\nu-5$

For level $k=1$, central charge $c(\mu,\nu)_1 = \dfrac{12(\mu\nu-1)}{(\mu+1)(\nu+1)}$

four chiral primary operators ϕ_i of conformal weights: $h_i,\ i=0,1,2,3$

$$h_i = 0,\ \frac{\mu\nu-1}{(\mu+1)(\nu+1)},\ \frac{\mu}{\mu+1},\ \frac{\nu}{\nu+1}$$

Hilbert space $\mathscr{H}_i = \{\,|\phi_i\rangle + \text{chiral decendants}\}$

Characters $\chi_{0,1,2,3}$: torus partition functions

Characters with $\chi_i = q^{\alpha_i} \sum_{n=0}^{\infty} a_n^{(i)} q^n$. With $\alpha_i(\mu,\nu) = -c/24 + h_i$

$$\alpha_0 = -\alpha_1 = \frac{\mu\nu - 1}{2(\mu+1)(\nu+1)},\ \alpha_2 = \frac{\mu\nu + 2\mu + 1}{2(\mu+1)(\nu+1)},\ \alpha_3 = \frac{\mu\nu + 2\nu + 1}{2(\mu+1)(\nu+1)}$$

Symmetry: $\sigma : \mu \to 1/\mu,\ \nu \to 1/\nu$

central charge $c(1/\mu,1/\nu) = -c(\mu,\nu)$

$\sigma : (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \to (\alpha_1, \alpha_0, \alpha_3, \alpha_2)$

$\lambda_{1,2,3}$ are invariant and so is MLDE.

This suggests the theory on the dark side ($\mu\nu < 1$) is identical to that on light side $(\mu\nu > 1)$.

MLDE and Characters $\chi_{0,1,2,3}$

For k=1, $\ell = 0$ and the characters satisfies MLDE equation

$$[D^4 + \lambda_1 E_4 D^2 + \lambda_2 E_6 D + \lambda_3 E_4^2]\chi = 0$$

with parameters λ_i as

$$\begin{aligned}
\lambda_1 &= -\frac{11\mu^2\nu^2 + 11\mu^2 + 11\nu^2 + 11 - 28\mu\nu + 4\mu^2\nu + 4\mu + 4\nu + 4\mu\nu^2}{36(\mu+1)^2(\nu+1)^2}, \\
\lambda_2 &= \frac{49\mu^2\nu^2 + 49 + 8\mu^2\nu + 8\mu + 8\nu + 8\mu\nu^2 - 5\mu^2 - 5\nu^2 - 56\mu\nu}{216(\mu+1)^2(\nu+1)^2}, \\
\lambda_3 &= -\frac{(\mu\nu-1)^2(\mu\nu+2\mu+1)(\mu\nu+2\nu+1)}{16(\mu+1)^4(\nu+1)^4}
\end{aligned}$$

MLDE and Characters $\chi_{0,1,2,3}$

Parameters λ_i are invariant under $(\mu,\nu) \to (1/\mu, 1/\nu)$ and $(\mu,\nu) \to (\nu,\mu)$

For $\nu = 5$, the series is the Mathur, Mukhi, Sen or Cvitanovich-Deligne series with 2- characters of $h = 0,\ h = \mu/(\mu+1)$

MLDE: Kaneko-Zagier equation $\Delta_{Kz}\chi = 0,\ \ \Delta_{KZ} = D^2 - \dfrac{(5\mu-1)(7\mu+1)}{144(\mu+1)^2}$

The quartic equation becomes $\Delta_{KZ}(\mu \to 1/\mu) \cdot \Delta_{KZ}\chi = 0.$

$\mathscr{T}(a,b)$ level 1

$d_G, h_G^\vee$

c, h_1, h_2, h_3

				$0,-1$ $\frac{6}{5},\frac{1}{10},\frac{3}{5},\frac{1}{2}$	$3,\frac{1}{4}$ $\frac{12}{5},\frac{1}{5},(\frac{3}{5})_2$				
			$0,-1$ $\frac{4}{5},\frac{1}{15},\frac{2}{3},\frac{2}{5}$	$2,0$ $2\frac{1}{6},\frac{2}{3},\frac{1}{2}$	$8,\frac{3}{2}$ $\frac{16}{5},\frac{4}{15},\frac{2}{3},\frac{3}{5}$	$16,3$ $4\frac{1}{3},(\frac{2}{3})_2$			
		$1,0$ $1,\frac{1}{12},\frac{3}{4},\frac{2}{5}$	$3,\frac{2}{3}$ $\frac{9}{5},\frac{3}{20},\frac{3}{4},\frac{2}{5}$	$9,2$ $3\frac{1}{4},\frac{3}{4},\frac{1}{2}$	$21,4$ $\frac{21}{5},\frac{7}{20},\frac{3}{4},\frac{3}{5}$	$35,6$ $5,\frac{5}{12},\frac{3}{4},\frac{2}{3}$	$66,10$ $6\frac{1}{2},(\frac{3}{4})_2$		
	$1,\frac{2}{3}$ $\frac{3}{5},\frac{1}{20},\frac{4}{5},\frac{1}{4}$	$4,\frac{3}{2}$ $\frac{8}{5},\frac{2}{15},\frac{4}{5},\frac{1}{3}$	$8,2\frac{1}{3}$ $\frac{12}{5},\frac{1}{5},\frac{4}{5},\frac{2}{5}$	$18,4$ $\frac{18}{5},\frac{3}{10},\frac{4}{5},\frac{1}{2}$	$36,6\frac{1}{2}$ $\frac{24}{5},\frac{2}{5},\frac{4}{5},\frac{3}{5}$	$56,9$ $\frac{28}{5},\frac{7}{15},\frac{4}{5},\frac{2}{3}$	$99,14$ $\frac{33}{5},\frac{11}{20},\frac{4}{5},\frac{3}{4}$	$144,19$ $7\frac{1}{5},\frac{3}{5},(\frac{4}{5})_2$	
$1,\frac{3}{2}$ $\frac{2}{5},\frac{1}{5}$	$3,2$ $1,\frac{1}{4}$	$8,3$ $2,(\frac{1}{3})_2$	$14,4$ $2\frac{4}{5},\frac{2}{5}$	$28,6$ $4,(\frac{1}{2})_3$	$52,9$ $5\frac{1}{5},\frac{3}{5}$	$78,12$ $6,(\frac{2}{3})_2$	$133,18$ $7,\frac{3}{4}$	$190,24$ $7\frac{3}{5},\frac{4}{5}$	$248,30$ 8

Magic Coset Relation

CD series $\mathscr{T}(5,5) = (E_8)_1$: $\mathscr{T}(5,1/\nu) = (E_8)_1/\mathscr{T}(5,\nu)$

$(E_8)_1$: $\chi_0 = j(\tau)^{1/3} = q^{-1/3}(1 + 248q + \cdots)$

$\mathscr{T}(\mu,5)$: $c(\mu,5) = 2(5\mu - 1)/(\mu + 1),\ \ h = 0,\ \mu/(\mu + 1)$

$\chi_0^{(\mu,5)}, \chi_1^{(\mu,5)}$, modular invariant partition function $Z = |\chi_0|^2 + \mathscr{N}\,|\chi_1|^2$

$\mathscr{T}(1/\mu,5)$: $c(\mu,5) = 2(5 - \mu)/(\mu + 1),\ \ h = 0,\ 1/(\mu + 1)$

$\chi_0^{(1/\mu,5)}, \chi_1^{(1/\mu,5)}$, partition function $Z = |\chi_0|^2 + \mathscr{N}\,|\chi_1|^2$

Together $j^{1/3}(\tau) = \chi_0^{(\mu,5)}\chi_0^{(1/\mu,5)} + N\chi_0^{(\mu,5)}\chi_1^{(1/\mu,0)}$.

Magic Coset Relation

CD series $\mathcal{T}(5,5) = (E_8)_1$: $\mathcal{T}(5,1/\nu) = (E_8)_1/\mathcal{T}(5,\nu)$

Our discovery: $\mathcal{T}(\mu,1/\nu) = \mathcal{T}(\mu,\rho)/\mathcal{T}(\nu,\rho)$, $\mu > \nu$

$$
\begin{aligned}
\chi_0^{\mathcal{T}_{\mu,\rho}} &= x_0 \chi_0^{\mathcal{T}_{\mu,1/\nu}} \chi_0^{\mathcal{T}_{\nu,\rho}} + y_0 \chi_3^{\mathcal{T}_{\mu,1/\nu}} \chi_2^{\mathcal{T}_{\nu,\rho}}, \\
\chi_1^{\mathcal{T}_{\mu,\rho}} &= x_1 \chi_1^{\mathcal{T}_{\mu,1/\nu}} \chi_1^{\mathcal{T}_{\nu,\rho}} + y_1 \chi_2^{\mathcal{T}_{\mu,1/\nu}} \chi_3^{\mathcal{T}_{\nu,\rho}}, \\
\chi_2^{\mathcal{T}_{\mu,\rho}} &= x_2 \chi_2^{\mathcal{T}_{\mu,1/\nu}} \chi_0^{\mathcal{T}_{\nu,\rho}} + y_2 \chi_1^{\mathcal{T}_{\mu,1/\nu}} \chi_2^{\mathcal{T}_{\nu,\rho}}, \\
\chi_3^{\mathcal{T}_{\mu,\rho}} &= x_3 \chi_0^{\mathcal{T}_{\mu,1/\nu}} \chi_3^{\mathcal{T}_{\nu,\rho}} + y_3 \chi_3^{\mathcal{T}_{\mu,1/\nu}} \chi_1^{\mathcal{T}_{\nu,\rho}},
\end{aligned}
$$

Magic Coset Relation

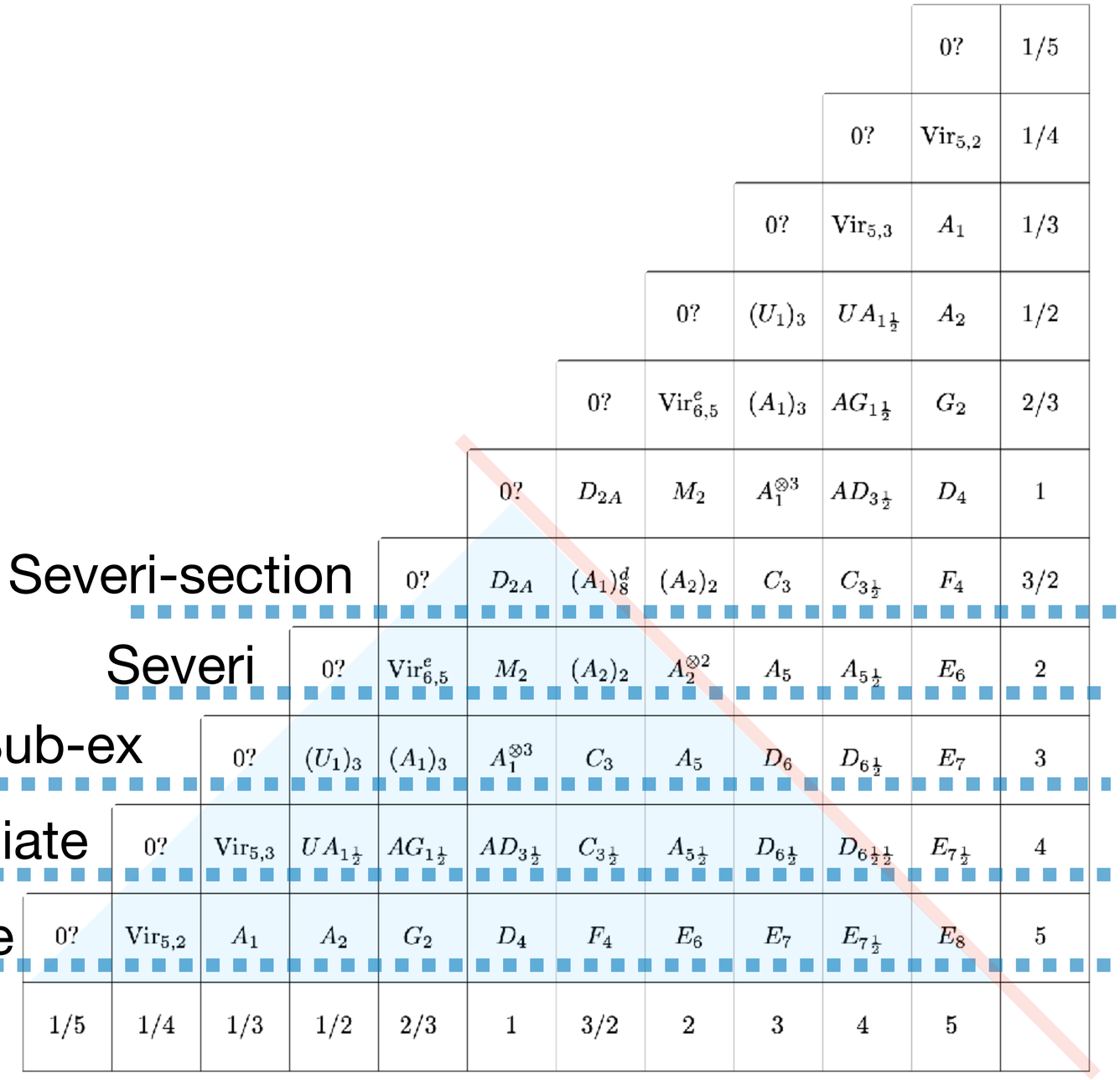
$$\chi_0 = q^{-c/24}\Big(1 + d_{\mathfrak{g}} q + \sum_{n=2}^{\infty} a_n q^n\Big),$$
$$\chi_1 = q^{-c/24+h_1}\Big(R_1 + \sum_{n=1}^{\infty} b_n q^n\Big),$$
$$\chi_2 = q^{-c/24+h_2}\Big(R_2 + \sum_{n=1}^{\infty} c_n q^n\Big),$$
$$\chi_3 = q^{-c/24+h_3}\Big(R_3 + \sum_{n=1}^{\infty} d_n q^n\Big),$$

$$d_{\mathfrak{g}}^{(\mu,\rho)} = d_{\mathfrak{g}}^{(\mu,1/\nu)} d_{\mathfrak{g}}^{(\nu,\rho)} + y_0 R_1^{(\mu,1/\nu)} R_2^{\nu,\rho},$$
$$R_1^{(\mu,\rho)} = x_1 R_1^{\mu,1/\nu} R_1^{(\nu,\rho)},$$
$$R_2^{(\mu,\rho)} = x_2 R_2^{(\mu,1/\nu)} + y_2 R_1^{\mu,1/\nu} R_2^{(\nu,\rho)},$$
$$R_3^{(\mu,\rho)} = x_3 R_3^{(\nu,\rho)} + y_3 R_3^{(\mu,1/\nu)} R_1^{(\nu,\rho)}.$$

Magic Triangle

$$\mathscr{T}(\mu,\mu) = \mathscr{T}(\nu,\mu)$$

$$\mu\nu \geq 1$$



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Magic Coset Relation

Specific cases

$$\frac{(\mathfrak{g}_{\mathrm{CD}})_1}{(\mathfrak{g}_{\mathrm{I}})_1} = \mathrm{Vir}^{\mathrm{eff}}_{5,2}, \qquad \frac{(\mathfrak{g}_{\mathrm{CD}})_1}{(\mathfrak{g}_{\mathrm{sub}})_1} = (A_1)_1,$$

$$\frac{(\mathfrak{g}_{\mathrm{I}})_1}{(\mathfrak{g}_{\mathrm{sub}})_1} = \mathrm{Vir}^{\mathrm{eff}}_{5,3}. \qquad \frac{(\mathfrak{g}_{\mathrm{sub}})_1}{(\mathfrak{g}_{\mathrm{Severi}})_1} = (U_1)_3.$$

Atoms of Magic Triangles

All the theories on the triangle are built from the theories along the diagonal lines:

$$\mathrm{Vir}_{5,2},\ \mathrm{Vir}_{5,2},\ (U_1)_3,\ \mathrm{Vir}^{e}_{6,5},\ D_{2A}$$

	$\mathrm{Vir}_{5,2}$	$\mathrm{Vir}_{5,3}$	$(U(1))_3$	$\mathrm{Vir}^{e}_{6,5}$	D_{2A}
(μ,ν)	(1/4,5)	(1/3,4)	(2/3, 3)	(1/2, 2)	(1,3/2)
d	1	1	1	0	0
$h^\vee$	$\frac{3}{2}$	$\frac{2}{3}$	0	-1	-1
c	$\frac{2}{5}$	$\frac{3}{5}$	1	$\frac{4}{5}$	$\frac{6}{5}$
h_1	~~$\frac{1}{30}$~~	$\frac{1}{20}$	$(\frac{1}{12})_2$	$(\frac{1}{15})_2$	$(\frac{1}{10})_3$
$h_2 = \frac{\mu}{\mu+1}$	$\frac{1}{5}$	$\frac{1}{4}$	$(\frac{1}{3})_2$	$(\frac{2}{5})_2$	$(\frac{1}{2})_3$
$h_3 = \frac{\nu}{\nu+1}$	~~$\frac{5}{6}$~~	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$

Level 2

Not good as level one case. For $d = 0,\ h^\vee = -1$, there is no higher level.

One can find some progress for level 2 for sub exceptional series.

They are all supersymmetric.

The central charge is

$$c = \frac{3(2a+3)(3a+4)}{(a+2)(a+4)}.$$

The 4 NS and 5-R weights are

$$0, \frac{2a+3}{2(a+4)}, \frac{a+1}{a+2}, \frac{7a^2+14a+24}{4(a+2)(a+4)}.$$

$$\frac{(2a+3)(3a+4)}{8(a+2)(a+4)}, \frac{3(2a+3)}{8(a+2)}, \frac{(3a+4)(4a+11)}{8(a+2)(a+4)}, \frac{(2a+3)(3a+4)}{8(a+2)(a+4)} + 1$$

Level 2

No universal coset relations

$$\frac{(\mathfrak{g}_{\mathrm{CD}})_2}{(\mathfrak{g}_{\mathrm{sub}})_2 \times (A_1)_2} = \mathrm{Vir}_{n+1,n}, \qquad n = \begin{cases} 3 + \frac{8}{a}, & a = 1, 2, 4, 8 \\ -4 - \frac{8}{a}, & a = -1, -2/3. \end{cases} \qquad \frac{(G_2)_2}{(A_1)_6 \times (A_1)_2} = \mathrm{Vir}_{9,8}.$$

$$\frac{(\mathfrak{g}_{\mathrm{sub}})_2}{(\mathfrak{g}_{\mathrm{Severi}})_2 \times (U_1)_6} = \mathcal{B}\mathrm{Vir}^{N=1}_{n+2,n}, \qquad n = \begin{cases} 4 + \frac{8}{a}, & a = 1, 2, 4, 8 \\ -6 - \frac{8}{a}, & a = -1, -2/3. \end{cases} \qquad \frac{(E_7)_2}{(E_6)_2 \times (U_1)_6} = \mathcal{B}\mathrm{Vir}^{N=1}_{7,5}$$

Level k

Series of CFTs

Cvitanovich-Deligne Series

Five series of CFTs of level 1

series		4	5
CD exceptional	$\mathrm{Vir}_{5,2},\ A_1,\ A_2,\ G_2,\ D_4,\ F_4,\ E_6,\ E_7$	$E_{7.5},$	E_8
intermediate exceptional (IE)	$\mathrm{Vir}_{5,3},\ UA_{1\frac{1}{2}},\ AG_{1\frac{1}{2}},\ AD_{3\frac{1}{2}},\ C_{3\frac{1}{2}}, A_{5\frac{1}{2}},\ D_{6\frac{1}{2}}$	$D_{6\frac{1}{2}\frac{1}{2}}$	$E_{7\frac{1}{2}}$
subexceptional	$U(1)_3, (A_1)_3,\ A_1^{\oplus 3},\ C_3,\ A_5,\ D_6,$	$D_{6\frac{1}{2}}$	E_7
Severi exceptional	$\mathrm{Vir}^{\mathrm{e}}_{6,5},\ M_2,\ (A_2)_2,\ A_2^{\oplus 2},\ A_5$	$A_{5\frac{1}{2}}$	E_6
Severi-Section	$D_{2A}, (A_1)_8^e,\ (A_2)_2,\ C_3$	$C_{3\frac{1}{2}}$	F_4

CD Series

LY theory $M(5,2) = \mathrm{Vir}_{5,2} : c_{\mathrm{eff}} = 2/5, h = 0, 1/5$

	$\mathrm{Vir}_{5,2}$	A_1	A_2	G_2	D_4	F_4	E_6	E_7	$E_{7\frac{1}{2}}$	E_8
d	1	3	8	14	28	52	78	133	190	248
$h^\vee$	$\frac{3}{2}$	2	3	4	6	9	12	18	24	30
c	$\frac{2}{5}$	1	2	$2\frac{4}{5}$	4	$5\frac{1}{5}$	6	7	$7\frac{3}{5}$	8
~~h_1~~	$\frac{1}{30}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{7}{30}$	$\frac{1}{3}$	$\frac{13}{30}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{19}{30}$	$\frac{2}{3}$
h_3	$\frac{1}{5}$	$\frac{1}{4}$	$(\frac{1}{3})_2$	$\frac{2}{5}$	$(\frac{1}{2})_3$	$\frac{3}{5}$	$(\frac{2}{3})_2$	$\frac{3}{4}$	$\frac{4}{5}$	~~$\frac{5}{6}$~~
R		2	3.	7	8	26	27	56	57	

Intermediate Exceptional Series

IM $= M(5,3) = \mathrm{Vir}_{5,3}$. $c_{\mathrm{eff}} = 3/5, h = 0, 1/4, 4/5, 1/20$

$c = -3/5, h = 0, 4/3, 1/5, -1/20$, $\chi_0 = \chi^{\mathrm{eff}}_{1/20}$, $\chi_{3/4} = \chi^{\mathrm{eff}}_{4/5}$, $\chi_{1/5} = \chi^{\mathrm{eff}}_{1/4}$, $\chi_{-1/20} = \chi^{\mathrm{eff}}_{0}$

$$d_{\mathfrak{g}}^{\mathrm{IES}} = \frac{12(2a+3)(a+2)}{a+4} \qquad c_G^{IE} = \frac{24(2a+3)}{5(a+4)} \qquad h = 0, \ \frac{2(2a+3)}{5(a+4)}, \ \frac{a+2}{a+4}, \ \frac{4}{5}$$

	$-4/3$	-1	$-2/3$	0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{T}(a=8/3)$	$\mathbb{H}$	$\mathbb{S}$	$\mathbb{O}$
$\mathfrak{g}$	IM	$UA_{1.5}$	$AG_{1.5}$	$AD_{3.5}$	$C_{3.5}$	$A_{5.5}$	$A_{5.75}$	$D_{6\frac{1}{2}}$	$D_{6(\frac{1}{2}+\frac{1}{2})}$	$E_{7.5}$
$X_1 = \mathrm{d}_{\mathfrak{g}}$	1	4	8	18	36	56	70	99	144	190
$h^\vee = 5a/2+4$	2/3	3/2	7/3	4	13/2	9	32/3	14	19	24
c	$\frac{3}{5}$	$1\frac{3}{5}$	$2\frac{2}{5}$	$3\frac{3}{5}$	$4\frac{4}{5}$	$5\frac{3}{5}$	6	$6\frac{3}{5}$	$7\frac{1}{5}$	$7\frac{3}{5}$
h_1, R_1	$\frac{1}{20}, 1$	$\frac{2}{15}, 1$	$\frac{1}{5}, 2$	$\frac{3}{10}, 2$	$\frac{2}{5}, 6$	$\frac{7}{15}, 6$	$\frac{1}{2},$	$\frac{11}{20}, 12$	$\frac{3}{5}, 12$	$\frac{19}{30}, -$
h_2, R_2	$\frac{1}{4}, 1$	$\frac{1}{3}, 2$	$\frac{2}{5}, 5$	$\frac{1}{2}, 6$	$\frac{3}{5}, 20$	$\frac{2}{3}, 21$	$\frac{7}{10},$	$\frac{3}{4}, 44$	$(\frac{4}{5})_2, 45$	$\frac{5}{6}, -$
$h_3 = \frac{4}{5}, d_{R_3} = 6a+9$	1	3	5	9	15	21	25	33	45	57
Vogel $\alpha = -2$, (β, γ)		.						(6,10)		(10,16)

Subexceptional Series

	$(U(1))_3$	$(A_1)_3$	$A_1^{\otimes 3}$	C_3	A_5	D_6	$D_{6\frac{1}{2}}$	E_7
d	1	3	9	21	35	66	99	133
h	0	$\frac{2}{3}$	2	4	6	10	14	18
c	1	$1\frac{4}{5}$	3	$4\frac{1}{5}$	5	6	$6\frac{3}{5}$	7
h_1	$\frac{1}{12}$	$\frac{3}{20}$	$\frac{1}{4}$	$\frac{7}{20}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{11}{20}$	~~$\frac{7}{12}$~~
h_2	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
h_3	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	~~$\frac{5}{6}$~~
$R_2 = 12\nu - 4$	2	4	8	14	20	32	44	56

Tools

Modular Linear Differential Equation

Jin-Beom Bae, Zhihao Duan, KM, Sungjay Lee, Mathieu Sarkis '20,'21,'22

$E_{7+\frac{1}{2}}$

Mathur-Mukhi-Sen 88

Table I

l	μ	m_1	c	h	Identification
96	$\frac{11}{900}$	1	$\frac{2}{5}$	$\frac{1}{5}$	$c = -\frac{22}{5}$ minimal model ($c \leftrightarrow c - 24h$)
90	$\frac{5}{144}$	3	1	$\frac{1}{4}$	$k = 1$ SU(2) WZW model
80	$\frac{1}{12}$	8	2	$\frac{1}{3}$	$k = 1$ SU(3) WZW model
72	$\frac{119}{900}$	14	$\frac{14}{5}$	$\frac{2}{5}$	$k = 1$ G_2 WZW model
60	$\frac{2}{9}$	28	4	$\frac{1}{2}$	$k = 1$ SO(8) WZW model
48	$\frac{299}{900}$	52	$\frac{26}{5}$	$\frac{3}{5}$	$k = 1$ F_4 WZW model
40	$\frac{5}{12}$	78	6	$\frac{2}{3}$	$k = 1$ E_6 WZW model
30	$\frac{77}{144}$	133	7	$\frac{3}{4}$	$k = 1$ E_7 WZW model
24	$\frac{551}{900}$	190	$\frac{38}{5}$	$\frac{4}{5}$	?
20	$\frac{2}{3}$	248	8	$\frac{5}{6}$	$\supset k = 1$ E_8 WZW model

Two characters (Mathur-Mukhi-Sen)

- Effective Lee-Yang theory: $c = 2/5, h = 1/5,\ \chi_0' = \chi_1, \chi_1' = \chi_0$
- $\chi_0^c \chi_0^{8-c} + \mathscr{M} \chi_1^c \chi_1^{8-c} = \chi_0^{E_8} = j^{1/3}$
- Klein j-invariant $j(\tau)$: SL(2,Z) invariant, modular function.
- $E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n},\ \Delta = q \prod_{n=1}^{\infty} (1-q^n)^{14}$
- $j(\tau) = \frac{E_4^3}{\Delta} = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$
- 196883= the lowest dimension of Monster group $\mathscr{M}$

Hecke Transformation

Zhihao Duan, KM, Kaiwen Sun '22, '22

Hecke Transformation II

Harvey-Wu ('18)

On modular forms $f(\tau) = \sum_n a(n) q^n,\ a(n) \in \mathbb{Z}$,

$$T_m f(\tau) = m^{k-1} \sum_{\substack{ad=m \\ a,d>0}} \frac{1}{d^k} \sum_{b \pmod d} f\left(\frac{a\tau + b}{d}\right)$$

For level N vvmf $\chi_i(\tau)$, the Hecke operator T_p for prime p coprime to N,

$$(T_p f)_i(\tau) = \sum_j \rho_{ij}(\sigma_p) f_j(p\tau) + \sum_{b=0}^{p-1} f_i\left(\frac{\tau + bN}{p}\right) \text{ where } \rho(\sigma_p) = G_p = \rho(T^p S^{-1} T^{\bar{p}} S T^p S)$$

where $\bar{p}$ is the inverse of p mod N and $f_i(\gamma\tau) = \sum_j \rho(\gamma)_{ij} f_j(\tau)$.

Example

Lee-Yang Model: $c_{eff} = 2/5,\ h = 1/5$

characters: $\chi_0 = q^{-1/60} G(q),\ \chi_1 = q^{11/60} H(q)$ (Roger-Ramanujan its G,H)

$G_p = \mathbf{1}_2$ for p=1,19,41,59 mod 60, $G_p = -\mathbf{1}_2$ for p=11,29,31,49 mod 60

$G_p = -i\sigma_2$ for p=7,13,47,53 mod 60, $G_p = i\sigma_2$ for p=17,23,37,43 mod 60

The exponents: $\alpha_0^{(p)} = -\frac{p}{60},\ \alpha_1^{(p)} = \frac{(11p)}{60}$ mod 1: central charge $c_{\text{eff}} = \frac{2p}{5}$

p	7	11	13	17	19	23	29	31	37	41	43	47	49	53	59	61	67	71
$\ell^{YL}(p)$	0	2	0	2	0	2	2	0	0	2	0	2	0	2	2	6	6	8

Classification

Zhiaho Duan, Lee,Sungjay Lee, Linfeng Li '22

RCFT

- Under $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,
  - $\chi_i(-1/\tau) = S_{ij}\chi_j(\tau)$, $\chi_j(\tau+1) = e^{2\pi I h_j}\chi_j(\tau)$
  - Charge Conjugation: $S^2 = (ST)^3 = C$
- Mathur-Sen (89),...
- Thm: Every component of VVMF of integer Fourier coefficient is invariant under a principal congruence subgroup $\Gamma(N) \quad \{\gamma = \mathrm{diag}(1,1) \bmod N\}$ for some N. (Conductor)
  - The conformal characters are d-dim representation $\rho_{ij}(\gamma)$ of $SL(2,\mathbb{Z}_N) = SL(2,\mathbb{Z})/\Gamma(N)$.

RCFT Classification

- J. Kaidi, Y.H. Lin, J. Parra-Matinez 2107.
  - characters: weak holomorphy, integrality, vacuum, positivity
  - use the representation theory of $PSL(2,\mathbb{Z}_N)$
  - $SL(2,Z_N) \sim \prod SL(2,Z_{p_i^{\lambda_i}})$
- For fermionic system $\Gamma_\theta(2,Z_N) \sim \prod \Gamma_\theta(2,Z_{p_i^{\lambda_i}})$
- use the representation theory and characters to find $\theta_i = e^{2\pi i(-c/24+h_i)}$

Vogel

Vogel 1707.02072

$\alpha, \beta, \gamma, \quad t = h^\vee = \alpha + \beta + \gamma. \quad : \quad S^2(\mathfrak{g}) = Y_2 + Y_2' + Y_2''$ (symmetric product)

$$\dim_{\mathfrak{g}} = -\frac{(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha\beta\gamma}, \quad \dim_{Y_2} = \frac{t(2t-3\alpha)(2t-\beta)(2t-\gamma)(t+\beta)(t+\gamma)}{\alpha^2\beta\gamma(\alpha-\beta)(\alpha-\gamma)}.$$ + permutations of α, β, γ

$\mathfrak{g}$	$d_{\mathfrak{g}}$	$h^\vee$	α	β	γ
$\mathfrak{a}_r$	$r(r+2)$	$r+1$	-2	2	$r+1$
$\mathfrak{b}_r$	$r(2r+1)$	$2r-1$	-2	4	$2r-3$
$\mathfrak{c}_r$	$r(2r+1)$	$r+1$	-2	1	$r+2$
$\mathfrak{d}_r$	$r(2r-1)$	$2r-2$	-2	4	$2r-4$
$\mathfrak{e}_6$	78	12	-2	6	8
$\mathfrak{e}_7$	133	18	-2	8	12
$\mathfrak{e}_8$	248	30	-2	12	20
$\mathfrak{f}_4$	52	9	-2	5	6
$\mathfrak{g}_2$	14	4	-2	$10/3$	$8/3$
$\mathfrak{so}_n$	$n(n-1)/2$	$n-2$	-2	4	$n-4$

Cvitanovic-Delign Series and $E_{7\frac{1}{2}}$

Haowu Wang, Kaiwen Sun '23, '24

$E_{7\frac{1}{2}}$

Mathur-Mukhi-Sen 88, Deligne, Cohen-de Man 96

Intermediate Lie algebra

dimension: 190=133+56+1, fundamental: 57=56+1

dual Coxeter number 24

middle between E_7 $(133,18)$ and E_8 $(248,30)$

Deligne-Cvitanovic exceptional series

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_{7+\frac{1}{2}} \subset E_8$$

Intermediate Exotic Series

Haowu Wang, Kaiwen Sun '23, '24

Concluding Remarks

We identified all the theories in the Magic Triangle

We worked on Rank 1 and Rank 2 Characters

We find the Magic Coset and 5 atoms

Subexceptional series at level 2 has $N = 1$ supersymmetry

We have a uniform coset construction for some level 2 theories

Some of our theories play important role in 4,3,2d relations