Topological Holography for Flavor Symmetries

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Yi-Nan Wang

Peking University

PCTS, USTC

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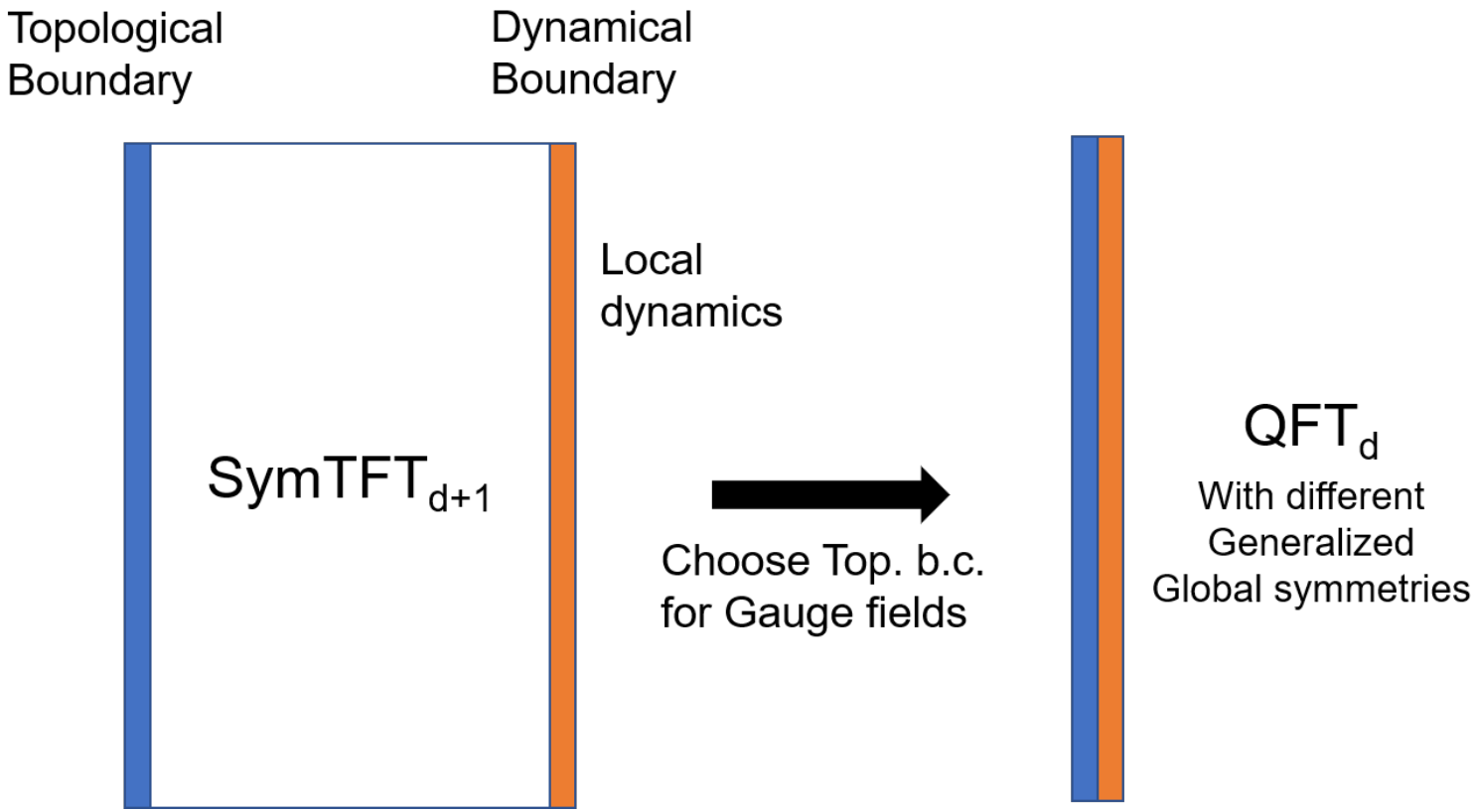
Topological Holography/SymTFT



- SymTFT also contains the anomaly polynomial information
- In AdS/CFT: SymTFT$_{d+1}$ - topological terms in the bulk (Witten 98')
- In CMT: SymTFT$_{d+1}$ - topological order (Kong, Levin, Wen etc.)

Topological Holography/SymTFT

• In the CMT/topological order community (categorical language), typically only tackle finite symmetries, e.g. $\mathbb{Z}_N$

• What about continuous symmetries e.g. $U(1)$, $SU(2)$ which appear more frequently in high energy physics?

• In our work, realize the SymTFT of non-abelian continuous symmetry G as a non-abelian BF theory, and derived the relevant symmetry operators

$$S_{BF} = \frac{1}{2\pi} \int_{M_{d+1}} \mathrm{Tr}(B \wedge F) \,.$$

p-form symmetry

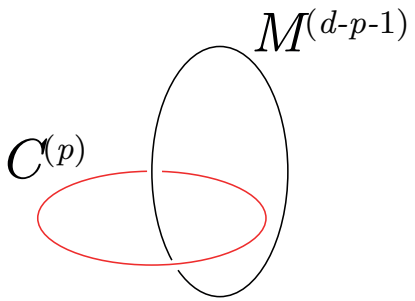
- A p-form symmetry with symmetry group G is generated by $(d-p-1)$-dimensional topological operators $U(g, M^{(d-p-1)})$ $(g \in G)$:

$$U(g_1, M^{(d-p-1)})U(g_2, M^{(d-p-1)}) = U(g_1 g_2, M^{(d-p-1)})\,.$$

and acts on p-dimensional object(operator) $V^i(\mathcal{C}^{(p)})$.

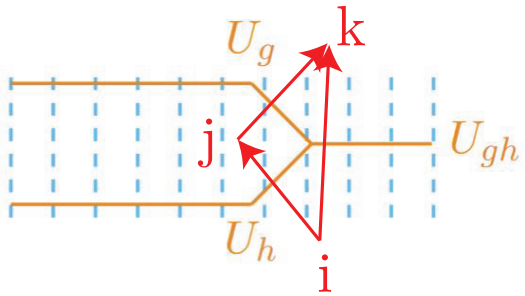
- $U(g, M^{(d-p-1)})$ has non-trivial action on $V^i(\mathcal{C}^{(p)})$ when $M^{(d-p-1)}$ and $\mathcal{C}^{(p)}$ are non-trivially linked.

$$U(g, M^{(d-p-1)})V^i(\mathcal{C}^{(p)}) = R^i{}_j(g)V^j(\mathcal{C}^{(p)})\,.$$



Gauging of p-form symmetry

- Gauging finite abelian p-form symmetry $G^{(p)}$:
- Flat connections $A_{(p+1)} \in C^{p+1}(M_d, G^{(p)})$, equivalent classes mod out gauge transformation: $[A_{(p+1)}] \in H^{p+1}(M_d, G^{(p)})$, $p = 0$ example:



$$A_{ij}A_{jk} = A_{ik}$$

Gauging: (1) coupling $\sim A \wedge J$

(2) summing over all configurations of $[A_{(p+1)}]$, i.e. the holonomies

Gauging of p-form symmetry

- Dual symmetry after the gauging $G^{(p)}$:

$$\widehat{G}^{(d-p-2)} = \mathrm{Hom}(G^{(p)}, U(1)) \sim \widehat{G}^{(d-p-2)} = \mathrm{Rep}(G^{(p)})$$

- $(d-p-2)$-form symmetry generated by the topological operator

$$W_R(C^{(p+1)}) = \exp\left(i \int_{C^{(p+1)}} A_{(p+1),R}\right) \ , \ R \in \mathrm{Rep}(G^{(p)})$$

- Example: $G^{(p)} = \mathbb{Z}_N$, $\widehat{G}^{(d-p-2)} = \mathbb{Z}_N$,

$$W_{\hat{g}}(C^{(p+1)}) = \hat{g}^{\int_{C^{(p+1)}} A_{(p+1)}} \ , \ \hat{g} = e^{\frac{2\pi i \hat{n}}{N}} \ (\hat{n} \in \{0, 1, \ldots, N-1\})$$

Gauging of p-form symmetry

• Partition function of the theory after gauging w/ new background gauge field $[\widehat{A}^{(d-p-1)}] \in H^{d-p-1}(M_d, \widehat{G}^{(d-p-2)})$

$$\widehat{Z}(\widehat{A}^{(d-p-1)}) = \frac{1}{|H^{p+1}(M_d, G^{(p)})|} \sum_{[A_{(p+1)}]} Z(A^{(p+1)}) e^{i \int_{M_d} \langle \widehat{A}^{(d-p-1)}, \cup A^{(p+1)} \rangle}$$

• Gauging $\equiv$ Fourier transform, between sets of functions, i.e. moduli space of flat connections $H^{p+1}(M_d, G^{(p)})$ and $H^{d-p-1}(M_d, \widehat{G}^{(d-p-2)})$!

Gauging with SymTFT

- SymTFT of $G^{(p)} = \mathbb{Z}_N$, without anomaly has a form of BF-theory

$$S_{sym} = \frac{2\pi}{N} \int_{M_{d+1}} a^{(p+1)} \cup \delta \hat{a}^{(d-p-1)}$$

- $a, \hat{a} \in \{0, 1, \ldots, N-1\}$
- Gauge invariant topological operators

$$U_g(\Sigma_{d-p-1}) = g^{\int_{\Sigma_{d-p-1}} \hat{a}^{(d-p-1)}}, \ g = e^{\frac{2\pi i n}{N}}$$

$$W_{\hat{g}}(C_{p+1}) = \hat{g}^{\int_{C_{p+1}} a^{(p+1)}}, \ \hat{g} = e^{\frac{2\pi i \hat{n}}{N}}$$

- Linking correlation function:

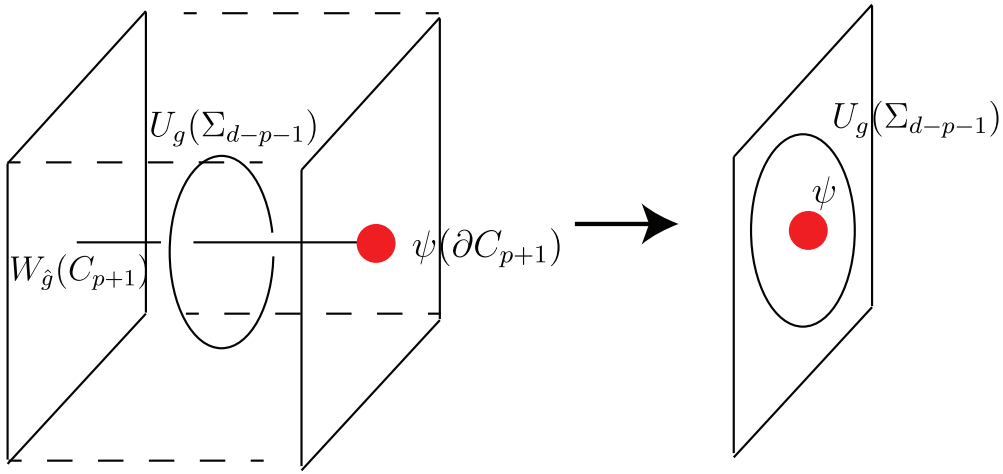
$$\langle U_g(\Sigma_{d-p-1}) W_{\hat{g}}(C_{p+1}) \rangle = \exp\left(\frac{2\pi i n \hat{n}}{N} \langle \Sigma_{d-p-1}, C_{p+1} \rangle\right)$$

Topological boundary conditions

(1) Neumann b.c. for U_g, $\hat{a}^{(d-p-1)}$, Dirichlet b.c. for $W_{\hat{g}}$, $a^{(p+1)}$

$W_{\hat{g}}(C_{p+1})$ can be attached to the boundary, no longer gauge invariant.

Should attach a charged operator $\psi_{\hat{g}}(\partial C_{p+1})$ on the physical boundary



$$U_g(\Sigma_{d-p-1})W_{\hat{g}}(C_{p+1})\psi_{\hat{g}} = \exp\left(\frac{2\pi i n\hat{n}}{N}\langle\Sigma_{d-p-1}, C_{p+1}\rangle_{M_{d+1}}\right)\psi_{\hat{g}}$$

- After squeezing

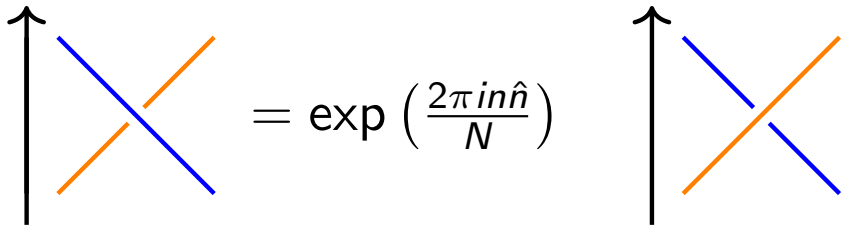
$$U_g(\Sigma_{d-p-1})\psi_{\hat{g}} = \exp\left(\frac{2\pi i n\hat{n}}{N}\langle\Sigma_{d-p-1}, \partial C_{p+1}\rangle_{M_d}\right)\psi_{\hat{g}}$$

- U_g generates p-form symmetry $G^{(p)} = \mathbb{Z}_N$ in M_d, acting on $\psi_{\hat{g}}$!

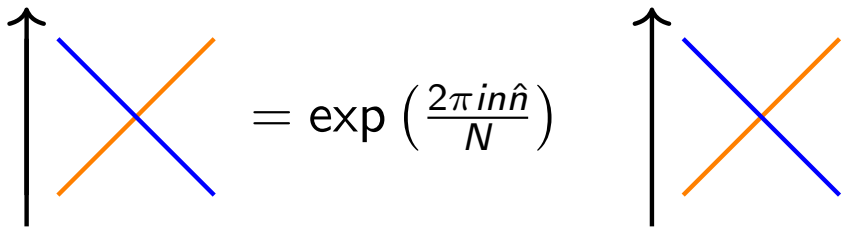
Topological boundary conditions

(2) Dirichlet b.c. for U_g, $\hat{a}^{(d-p-1)}$, Neumann b.c. for $W_{\hat{g}}$, $a^{(p+1)}$

- $W_{\hat{g}}(C_{p+1})$ generates the dual $\widehat{G}^{(d-p-2)}$ symmetry
- Swapping b.c. $\equiv$ Gauging!
- Dirac quantization: U and W cannot both have Dirichlet b.c.
- Another perspective (x^{d+1} is perpendicular to the plane)



- After squeezing to M_d:



- Partition function not well defined when $\exp\left(\frac{2\pi i n\hat{n}}{N}\right) \neq 1$!

Gauging continuous symmetries

• For continuous symmetries G, two types of gauging

(1) Gauging w/ non-flat gauge field A: new local dynamics (not a SymTFT, see Sym-Th (Apruzzi, Bedogna, Dondi 24')

(2) Gauging w/ flat connection: SymTFT!

• Proposals for continuous SymTFT: (Brennan, Sun 24')(Bonetti, del Zotto, Minasian 24')(Antinucci, Benini 24')

• No satisfactory construction of operator spectrum, top. b.c. etc for continuous non-abelian 0-form symmetry G!

Gauging continuous symmetries

- Dual symmetry after gauging G w/ flat connection: Wilson loop $W_R(\mathcal{C}) = \mathrm{Tr}_R \mathcal{P} \exp i \oint_{\mathcal{C}} A_R$ is topological, generates a dual $\mathrm{Rep}(G)^{(d-2)}$ symmetry!
- Fourier transformation perspective:
- Moduli space of flat G-connection:

$$\widetilde{\mathcal{A}} = \frac{\mathrm{Hom}(\pi_1(M_d), G)}{G}$$

- e.g. if $\pi_1(M_d) = \mathbb{Z}$, $\widetilde{\mathcal{A}} = Cl(G)$
- Tannaka duality between monoidal categories:

$$L^2(Cl(G)) \overset{T}{\cong} L^2(G^\vee)$$

- $G^\vee$ is the isomorphism classes of irreps of G

Non-abelian SymTFT

- Non-abelian BF-theory

$$S_{sym} = \int_{M_{d+1}} \mathrm{Tr}(B \wedge F)$$

- $F = dA - iA \wedge A$; $B \in \mathfrak{g}$ is a $(d-1)$-form, non-compact gauge field
- Gauge transformations:

$$A \to gAg^{-1} + igdg^{-1} \ , \ B \to gBg^{-1}$$

$$B \to B + D_A K$$

- Equation of motions:

$$F = 0 \ , \ D_A B = 0 \, .$$

- Gauge invariant topological operator generating $Rep(G)^{(d-2)}$ symmetry for Neumann b.c. of A:

$$W_R(\mathcal{C}) = \mathrm{Tr}_R \mathcal{P} \exp\left(i \oint_{\mathcal{C}} A_R \right)$$

Symmetry operator

- Difficulty in constructing non-abelian symmetry generator $U_\alpha(\Sigma_{d-1})$ in M_{d+1}:
- A topological cod-2 operator in M_{d+1} cannot be non-abelian!
- Let's try to write down an operator of form

$$U_\alpha(\Sigma_{d-1}) = \exp\left(i\int_{\Sigma_{d-1}}(\alpha, B)\right)$$

- α, $B \in \mathfrak{g}$; $(.,.)$ is a bilinear inner product
- To prove it's topological

$$d(\alpha, B) = D_A(\alpha, B) = (D_A\alpha, B) + (\alpha, D_A B) = (D_A\alpha, B)$$

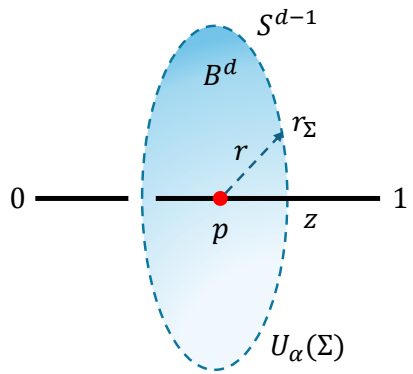
- Hence we require $D_A\alpha = 0$, i.e. α is a covariantly constant section of the adjoint bundle $\mathfrak{ad}(P)$, $U_\alpha(\Sigma_{d-1})$ is topological

Symmetry operator

- However, only true at the classical level!
- To compute the linking correlation function ($W_R(\mathcal{C})$ is untraced)

$$\langle U_\alpha(\Sigma) W_R(\mathcal{C})\rangle$$
$$= \int [DB][DA] \exp(i \int_{M_{d+1}} \mathrm{Tr}(B \wedge F)) \exp(i \int_\Sigma (\alpha, B)) \mathcal{P} \exp(i \int_{\mathcal{C}} A_R)$$

Integrate over all flat A where $D_A \alpha = 0$ no longer satisfies!

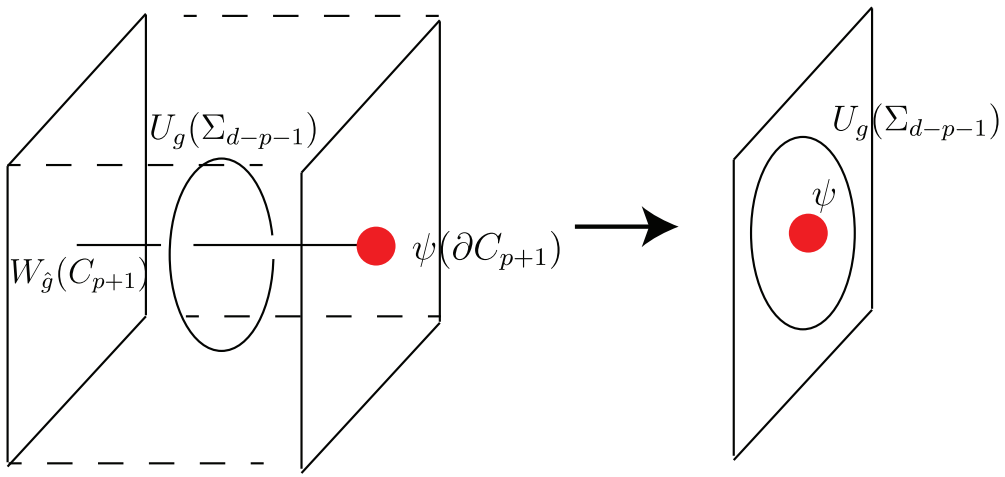


$$\langle U_\alpha(\Sigma) W_R(\ell)\rangle = e^{-i\alpha_R(p)} \langle W_R(\ell)\rangle \quad (r_\Sigma \to 0)$$

is dependent on the position of p! $e^{-i\alpha_R(p)}$ is a Lie group element of G in the rep. R

Symmetry operator

- Hence $U_\alpha(\Sigma_{d-1})$ is not topological, as expected!



- After squeezing (taking $p = 0$), get the non-abelian symmetry action!

$$U_\alpha(\Sigma)\psi_R(x) = e^{-i\alpha_R(0)}\psi_R(x)$$

Topological operator

- What if we want to construct a topological operator? Define

$$\tilde{U}_{\tilde{\alpha}}(\Sigma) = \int_G dg \exp\left(i \int_\Sigma (\tilde{\alpha}, B) \right)$$

Here $\tilde{\alpha}$ is the parallel transport of $g\alpha(p)g^{-1}$ by the holonomy of A:

$$\tilde{\alpha}(p') = e^{i\int_p^{p'} A} g\alpha(p)g^{-1} e^{-i\int_p^{p'} A}$$

- $\tilde{\alpha}$ lives in the same conjugacy class! $\tilde{\alpha} \in Cl(G)$ labels the $\tilde{U}_{\tilde{\alpha}}(\Sigma)$
- Can prove

$$\langle \tilde{U}_{\tilde{\alpha}}(\Sigma) W_R(\mathcal{C}) \rangle = \frac{\chi_R(e^{-i\tilde{\alpha}})}{\dim(R)} \langle W_R(\mathcal{C}) \rangle$$

Dirac pairing

- Hence in the non-abelian G SymTFT, the Dirac pairing function is the normalized character!

$$(e^{-i\tilde{\alpha}}, R) = \frac{\chi_R(e^{-i\tilde{\alpha}})}{\dim(R)}$$

- For example top. b.c. for $G = SU(2)$
- Top. operators: W_R, $R = j \in \frac{1}{2}\mathbb{Z}$; $\tilde{U}_\theta$, $\theta \in CI(SU(2)) = [0, \pi]$
- 2×2 special unitary matrices can be diagonalized into

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad \theta \in [0, \pi]\,.$$

- Normalized character

$$\frac{\chi_j(e^{-i\theta})}{\dim(\chi_j)} = \frac{\sin(2j+1)\theta}{(2j+1)\sin\theta}\,.$$

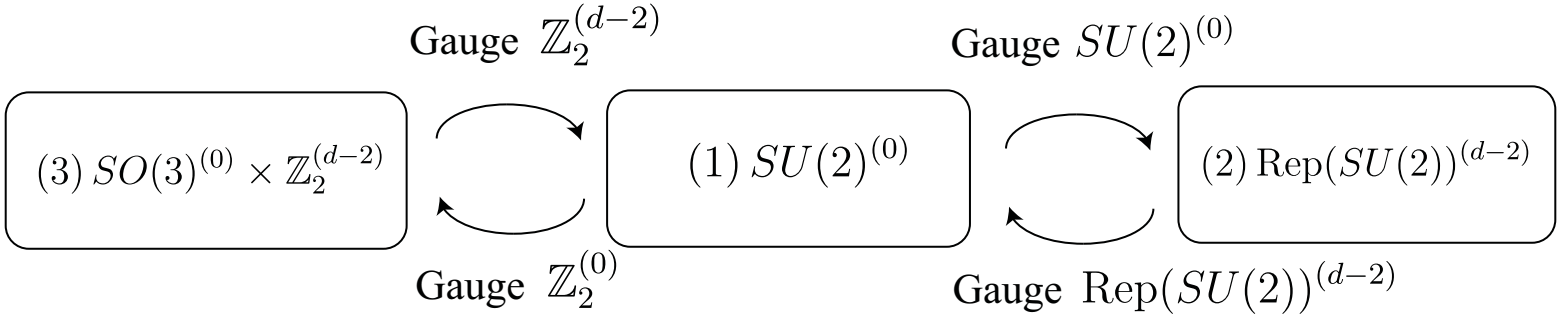
Topological b.c. for $G = SU(2)$

- All Dirac pairings between Dirichlet b.c. operators $= 1$

(1) Dirichlet b.c. to all W_j, Neumann b.c. to all $\tilde{U}_{\tilde{\alpha}}$: $SU(2)^{(0)}$ symmetry

(2) Neumann b.c. to all W_j, Dirichlet b.c. to all $\tilde{U}_{\tilde{\alpha}}$: $Rep(SU(2))^{(d-2)}$ symmetry

(3) Dirichlet b.c. to all W_j ($j \in \mathbb{Z}$) and $\tilde{U}_0$, $\tilde{U}_\pi$: $SO(3)^{(0)} \times \mathbb{Z}_2^{(d-2)}$ symmetry



Anomalies

- Adding twist terms in the BF-action

$$S_{sym} = \int_{M_{d+1}} \mathrm{Tr}(B \wedge F) + I(A, B)$$

- Obstruct the gauging/ some topological boundary conditions
- e.g. $d = 4$, adding a Chern-Simons term

$$S_{sym} = \int_{M_5} \mathrm{Tr}(B \wedge F) + CS_5(A)\,,$$

- Action is no longer gauge invariant under

$$A \to gAg^{-1} + igdg^{-1}\ ,\ B \to gBg^{-1}$$

- g Gauge variation $\to$ new boundary term, does not vanish when giving A Neumann b.c.
- Hence the Neumann b.c. for A, i.e. gauging G is obstructed!

Interpretation in AdS/CFT

- Justification of BF-action in holography! ($U(1)$ case: (DeWolfe, Higginbotham, 20'))
- We start from AdS_5 YM action

$$S_{YM} = \int d^5x\sqrt{-g}\left(-\frac{1}{4}tr(F_{\mu\nu}F^{\mu\nu})\right)$$

- AdS metric: $ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\eta_{ij}dx^i dx^j$
- Expand the solution of e.o.m. for A near the AdS boundary $r \to \infty$

$$A_i(x^i, r) = \alpha_i(x^i)L + \gamma_i(x^i)\frac{L^5 \log r}{r^2} + \beta_i(x^i)\frac{L^5}{r^2} + \dots$$

- Define $a_i = \alpha_i L$, and covariant derivative $\mathcal{D}_i = \partial_i - i[a_i, \cdot]$,
- e.o.m. for β_i:

$$\mathcal{D}^i\beta_i = 0$$

- Covariant conservation eq. for non-abelian current J_i after assigning (1) a_i ∼source (2) $\beta_i \sim J_i$ =current!

Interpretation in AdS/CFT

- Evaluate δS_{YM} by expanding A with $\alpha, \beta \ldots$

$$\delta S_{YM} = 2L^3 \int d^4x Tr(\delta\alpha_i \beta^i) + (\ldots),$$

- response of source α_i at the boundary

$$\langle J^i(x_i) \rangle = \frac{\delta S}{\delta \alpha_i} = 2L^3 \beta^i$$

- Same results can be derived starting from the BF action as well!

$$S_{BF} = \int_{M_5} \mathrm{Tr}(B \wedge F)$$

Outlooks

- We constructed the SymTFT for continuous non-abelian flavor symmetry, with a detailed computation of operators
- Details of anomalies
- Match with categorical language $Z(Vec_G)$ (ongoing work w/ Jia, Luo, Tian, Zhang)
- Derivation of SymTFT action from string theory/holography
- Find applications in CMT etc.
- Thanks!