Symmetry, SymTFT and von Neumann Algebra

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April 18, 2025

Some Formal Aspects of Field Theories and Holography @ PCFT

Work with Qiang Jia @ KAIST (Typos are mine)

Overview

1. Minute Physics: von Neumann Algebra
2. Uncharged Non-local Operators from Symmetry
3. Uncharged Non-local Operators from SymTFT
4. Violate Additivity and Haag Dualtity of SvNA
5. Discussion and Outlook

Minute Physics: von Neumann Algebra

What is von Neumann algebra?

- An algebraic approach to QFT [Haag-Kastler, Haag]
- QFT is described/defined by the set of operators represented on $\mathcal{H}$.
- Subsystem is defined by $\langle\psi|a|\psi\rangle$, $\forall a \in \mathcal{A}$ an algebra.
- The algebra is typically *von Neumann algebra*: set of operators satisfy:

Definition 1

$$\mathcal{A} = \mathcal{A}'' \text{ (double commutant)}$$

Where:

Definition 2

$$\mathcal{A}' = \{a' \in \mathcal{B}(\mathcal{H}) | [a', a] = 0,\ \forall a \in \mathcal{A}\}$$

Why von Neumann algebra?

- Locality is the key.

Two basic requirements:

Additivity

$$\mathcal{A}(R_1 \cup R_2) = (\mathcal{A}(R_1) \cup \mathcal{A}(R_2))''$$

Haag duality

$$\mathcal{A}(R') = \mathcal{A}(R)'$$

Perspectives on von Neumann algebra

- Do we include non-local operators in the "local" algebra $\mathcal{A}$?
- Some people don't [Casini et al.]
- Some people do [Shao et al.]
- We respect those don't but stand with those do, because:
  - It lets us detect (violation of) additivity and Haag duality.
  - Locality is still expected even in the presence of non-local operators. [Harlow-Ooguri]

What von Neumann algebra do we study?

- We wish to check additivity and Haag duality.

... but locality is expected for $\mathcal{A}$ even when non-local operators are included, so how?

Surprisingly, locality of a subsector of $\mathcal{A}$ becomes tricky.

We shall restrict to:

Symmetric sector von Neumann algebra (SvNA)

$$\mathcal{A}_{\mathcal{F}} := \{a | [a, U(N)] = 0, \ \forall N \in \mathcal{F}\}$$

Uncharged Non-local Operators from Symmetry

What do we mean by symmetry?

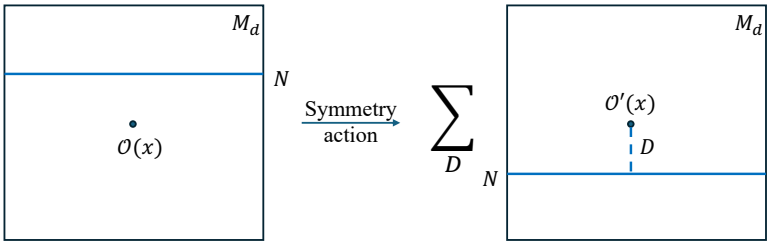
A slogan

Symmetry "=" Set of actions + QFT

- Set of actions:
  - Group G: $N_i N_j = N_k$
  - Fusion algebra $\mathcal{F}$: $N_i N_k = f_{ijk} N_k$
  - Whatever as long as they compose consistently
- QFT (where more physics sneaks in):
  - Representing symmetries on $\mathcal{H}$
  - Commute with H
  - Locality/Splittability [Doplicher et al., Harlow-Ooguri]

What does symmetry do to charged operators?

- Focus on 0-form symmetry, in general we have:



Symmetry could be a group, a ring, an algebra, etc., as said earlier.

Bi-local operator

- Assume: $\exists \mathcal{O}(x)$ such that $N\mathcal{O}(x) = \rho(N)\mathcal{O}(x)N$, $\rho(N) \in GL(V)$ for some $N \in \mathcal{F}$

  $\Rightarrow \exists$ Bi-local operator $\mathrm{tr}\mathcal{O}(x)\overline{\mathcal{O}}(y)$, uncharged under N

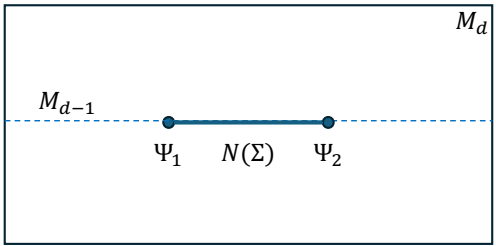
  ... but is it in $\mathcal{A}_{\mathcal{F}}$?

  Have to check against all of $\mathcal{F}$ to ensure no "tail" is generated, a formidable task :(

  $\Rightarrow$ SymTFT will be useful.

Bi-twist operator

- Bi-twist operator is $\Psi_1 N(\Sigma)\Psi_2$

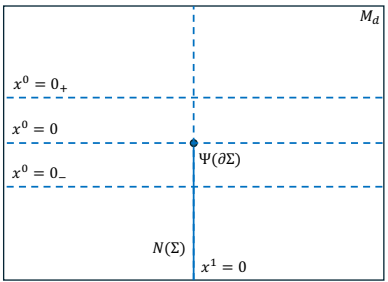


- $\Sigma \subset M_{d-1}$
- Ψ_1 and Ψ_2 supported on $\partial\Sigma := \Sigma_1 \cup \Sigma_2$, $\Sigma_1 \cap \Sigma_2 = \emptyset$

But why exists?

Existence of bi-twist

- Any requirement for its existence?



$$\phi(x^0 = 0_-, x^1 = 0_+, \vec{x}) = N \cdot \phi(x^0 = 0_-, x^1 = 0_-, \vec{x})$$
$$\phi(x^0 = 0_+, x^1 = 0_+, \vec{x}) = \phi(x^0 = 0_+, x^1 = 0_-, \vec{x})$$
$$\Rightarrow$$
$$\phi(x^0 = 0, x^1 = 0, \vec{x}) = N \cdot \phi(x^0 = 0, x^1 = 0, \vec{x})$$
$$\Rightarrow$$
$$\phi(0, 0, \vec{x}) \in \mathrm{Rep}(C_{\mathcal{F}}(N))$$

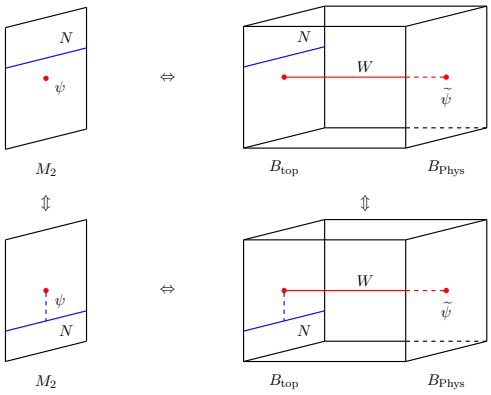
$\Rightarrow$ Drinfeld center? [Five guys, ongoing]

- Given existence, still remains to check if it is in $\mathcal{A}_{\mathcal{F}}$, a formidable task :(

Uncharged Non-local Operators from SymTFT

SymTFT

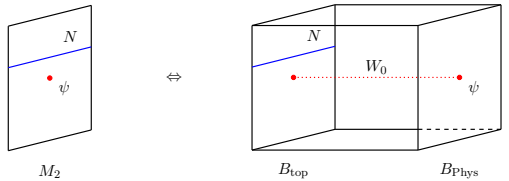
- Symmetry and dynamics are separated.



- Operators in $\mathcal{T}_{\mathcal{F}} \Rightarrow$ Extended operators in SymTFT
- Symmetries are put on B_{top}.
- Symmetry action $\Rightarrow$ Generator sweeps through the end locus of W

SvNA from SymTFT

- To end, or not to end, that is the question.

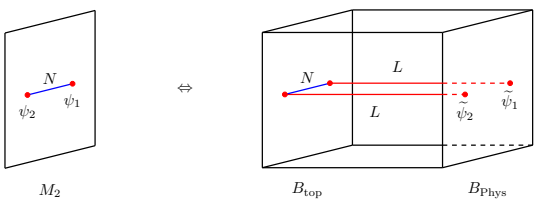


- N sweeps through and does nothing
- W_0 can end $\Leftrightarrow$ "Tail" of W_0 is 1
- Generally $L = \bigoplus n_\alpha W_\alpha$
- L endable if $1 \in \text{Tail}(L)$
- $\mathcal{L} := \bigoplus_{\max\{\text{endable and commute}\}} L$

[Cong wt al.][Bhardwaj et al.]

SvNA from SymTFT

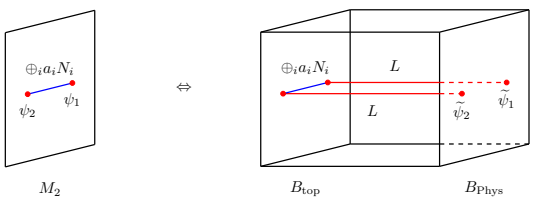
- Bi-operator in (3d) SymTFT



- L can have "tail" N
- Generally L has "tail" $\oplus a_i N_i$
- Generally N noninvariant under $\mathcal{F}$
- Generally $\widetilde{\psi}_1 L N L \widetilde{\psi}_2$ noninvariant under $\mathcal{F}$
- Generally $\widetilde{\psi}_1 N \widetilde{\psi}_2 \notin \mathcal{A}_{\mathcal{F}}$

Uncharged bi-locals from SymTFT

- What are the uncharged bi-locals?



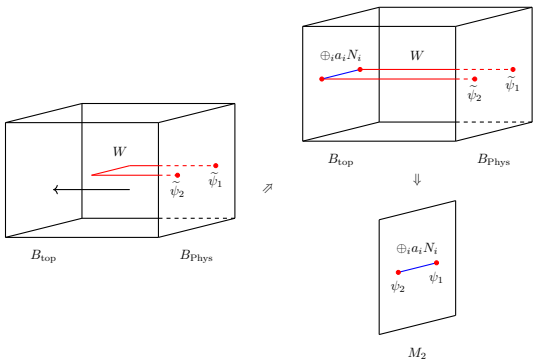
- Bi-local if $1 \in \text{Tail}(L)$
- But is $\psi_1\psi_2$ truly bi-local?
- Generally NO since 1 may change $\psi_1 1 \psi_2 \to \psi_1' M \psi_2'$
- To avoid, we require $\exists L$, $\text{Tail}(L) = 1$
- Define: $\widetilde{\mathcal{L}} := \bigoplus_{\text{Tail}(L)=1} L$

Necessary condition for uncharged bi-local

$$\widetilde{\mathcal{L}} \neq 0 \Rightarrow \exists \psi_1\psi_2 \in \mathcal{A}_{\mathcal{F}}$$

Uncharged bi-twists from SymTFT in 2d

- What are the uncharged bi-twists? (2d QFT)



- Now W not necessarily in $\mathcal{L}$
- Bi-twist if tail $\neq 1$
- But is it in $\mathcal{A}_{\mathcal{F}}$?
- Drag tail into bulk $\Leftrightarrow$ Yes it is
- E.g. $\widetilde{U}$ in [Five guys, 2503.04546]

Necessary condition for uncharged bi-twist (2d QFT)

Draggable W always exists, just need to include all tails.

Uncharged bi-twists from SymTFT in general dimension

- What are the uncharged bi-twists? (General dimension)
- In general dim, line W becomes hypersurface, but that's fine
- Just fix any $N \in C(\mathcal{F})$ on B_{top} and make sure $\Psi(\partial\Sigma)$ consistent

Therefore,

Necessary condition for uncharged bi-twist (General dim)

Uncharged bi-twist exists if $C(\mathcal{F})$ non-trivial.

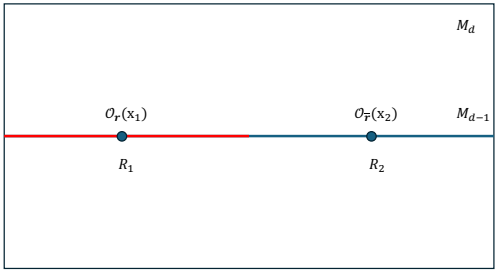
A spoiler:

Not all uncharged bi-twists serve to break Haag duality.
Sharper requirements about $\mathcal{L}$ can be made.

Violate Additivity and Haag Dualtity of SvNA

Additivity of SvNA

- If $\exists$ bi-local $\mathcal{O}_1\mathcal{O}_2 \in \mathcal{A}_\mathcal{F}$



- $\mathcal{O}_1\mathcal{O}_2 \notin (\mathcal{A}_\mathcal{F}(R_1) \cup \mathcal{A}_\mathcal{F}(R_2))''$
- $\mathcal{O}_1\mathcal{O}_2 \in \mathcal{A}_\mathcal{F}(R_1 \cup R_2)$

$$\mathcal{A}_\mathcal{F}(R_1 \cup R_2) \subsetneq (\mathcal{A}_\mathcal{F}(R_1) \cup \mathcal{A}_\mathcal{F}(R_2))''$$

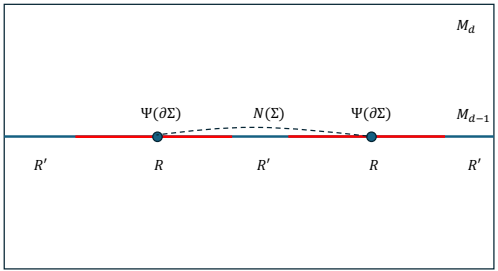
(E.g., something like $N(R_1)$ is in $(\mathcal{A}_\mathcal{F}(R_1) \cup \mathcal{A}_\mathcal{F}(R_2))'$, but can fail to commute with $\mathcal{O}_1$)

Conclusion 1

Additivity of $\mathcal{A}_\mathcal{F}$ is violated.

Haag duality of SvNA

- If $\exists$ bi-twist $\Psi_1 N(\Sigma)\Psi_2 \in \mathcal{A}_{\mathcal{F}}$ (always exists because $\exists$ Draggable)



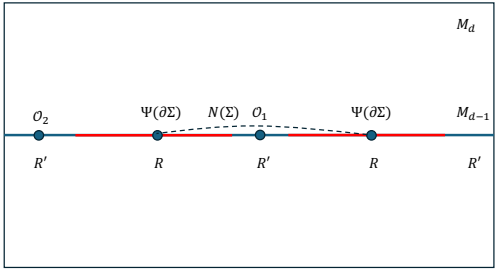
- $\Psi_1 N(\Sigma)\Psi_2 \notin \mathcal{A}_{\mathcal{F}}(R)$ by def
- $\Psi_1 N(\Sigma)\Psi_2 \in \mathcal{A}_{\mathcal{F}}(R')'$

$$\mathcal{A}_{\mathcal{F}}(R) \subsetneq \mathcal{A}_{\mathcal{F}}(R')'$$

But, what if there is bi-local in $\mathcal{A}_{\mathcal{F}}$, whose charged component in middle R' does NOT commute with $\Psi_1 N(\Sigma)\Psi_2$?

Haag duality of SvNA

- If there is also bi-local, then ...



- $\Psi_1 N(\Sigma)\Psi_2 \notin \mathcal{A}_{\mathcal{F}}(R)$ by def
- Q: $\Psi_1 N(\Sigma)\Psi_2 \in \mathcal{A}_{\mathcal{F}}(R')'$?

$\mathcal{O}_1\mathcal{O}_2 \in \mathcal{A}_{\mathcal{F}}(R') \Rightarrow M$ acts linearly on $\mathcal{O}_i$

$\Leftrightarrow \mathcal{O}_i$ can develop NO tail by def

Let $N = \overline{M}M$ answers Q so still

$$\mathcal{A}_{\mathcal{F}}(R) \subsetneq \mathcal{A}_{\mathcal{F}}(R')'$$

Conclusion 2

Haag duality of $\mathcal{A}_{\mathcal{F}}$ is violated.

Haag duality of SvNA, sharpened

A recap the spoiler:

Not all uncharged bi-twists serve to break Haag duality.
Sharper requirements about $\mathcal{L}$ can be made.

- What if $\mathcal{F}$ is invertible, say a group?

$\Rightarrow$ Haag duality cannot be violated by bi-twist, because:

- Any N fails to commute with at least one charged local operator $\mathcal{O}(x)$ while $\mathcal{O}(x)\overline{\mathcal{O}}(y) \in \mathcal{A}_{\mathcal{F}}$
- $N\overline{N}$ does not help, since it is 1, and the corresponding operator is simply no longer a bi-twist, in other words, $M\overline{M} \equiv 1, \forall M \in \mathcal{F}$ for invertible $\mathcal{F}$

We want to AVOID:

Invertible $\mathcal{F} \Leftarrow \widetilde{\mathcal{L}} = \mathcal{L}$ ($\Rightarrow$: no known counterexamples)

Two theorems from SymTFT perspective

- Since $\widetilde{\mathcal{L}} \neq \emptyset \Rightarrow \mathcal{O}_1\mathcal{O}_2 \in \mathcal{A}_{\mathcal{F}}$

Theorem 1

Additivity of $\mathcal{A}_{\mathcal{F}}$ is violated if $\widetilde{\mathcal{L}} \neq 0$.

- Since $\widetilde{\mathcal{L}} \neq \mathcal{L} \Rightarrow \Psi_1 N(\Sigma)\Psi_2 \in \mathcal{A}_{\mathcal{F}}$

Theorem 2

Haag duality of $\mathcal{A}_{\mathcal{F}}$ is violated if $\widetilde{\mathcal{L}} \neq \mathcal{L}$.

Discussion and Outlook

Discussion and outlook

A lesson

Symmetry affects von Neumann algebra, and vice versa.

- Outlook:
  - To understand the appearance of Drinfeld center better. [Five guys, ongoing]
  - To learn the implications of higher form and higher group symmetries on vNA.
  - To explore the consequences of vNA on symmetries.

Thanks