Wall-croosing and TBA equations for deformed supersymmetric quantum mechanics

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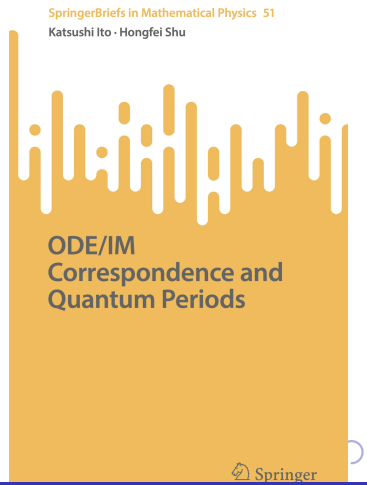
April 18, 2025
Some formal aspects of Field theories and Holography
Based on [Ito-Shu '24, Ito-Shu-Yang to appear]



Non-perturbative methods in QM/QFT

Integrable model

1. The system can be solved exactly in principle, if there are enough independent conserved charges.

e.g. non-linear sigma model in $AdS_5 \times S^5$ background, $\mathcal{N} = 4$ SYM in planar limit.

Resurgence theory has seen a resurgence in recent years

1. The perturbative series in physics are almost always divergent.
2. The resurgence theory convert them into meaningful objects, and will connect the perturbative series and the non-perturbative series.

e.g. Exact WKB method in QM.

Connections of those two methods: TBA/WKB correspondence (ODE/IM correspondence)

Schrödinger equation in hep-th

The ODE

$$\Big(-(-\hbar)^{r+1} \frac{d^{r+1}}{dx^{r+1}} + V(x) - E \Big) \psi(x) = 0$$

with potential $V(x)$ appears in many areas of physics:

- Schrödinger equation of 1D quantum mechanics when $r = 1$.
- Quantized Seiberg-Witten curve of $\mathcal{N} = 2$ SYM in the NS limit of Ω background [Nekrasov-Shatashvili '09].
- Minimal surface in the scattering amplitude/Wilson loop duality [Alday-Maldacena-Sever-Viera, Hatsuda-Ito-Sakai-Satoh '10].
- Black hole perturbation $\leftarrow$ See Rui-Dong's Talk.

- The TBA/WKB correspondence is a very powerful method to study these ODEs.

Outline

- Quick review on Exact WKB method
- TBA/WKB correspondence
- TBA/WKB correspondence for deformed SUSY QM
- Wall-crossing of TBA equations for deformed SUSY QM

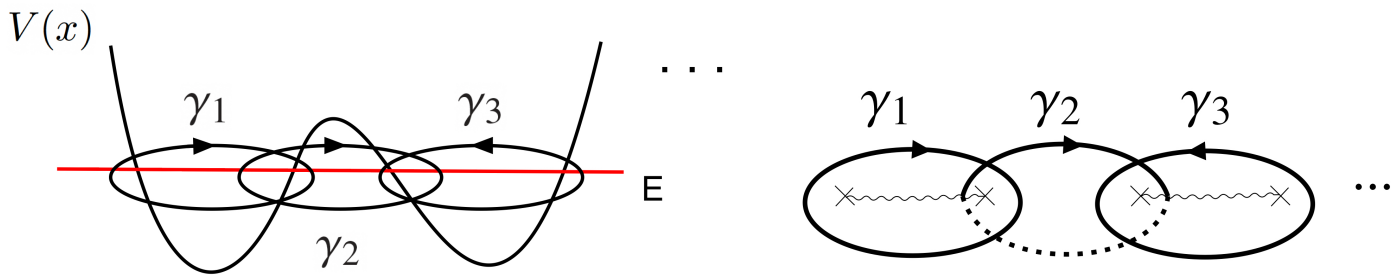
Quick review on Exact WKB Method

Exact WKB method

- WKB ansatz for the Schrödinger eq $\psi(x) = \exp\left[\frac{1}{\hbar}\int^x P(x')dx'\right]$

$$\text{Ricatti equation:} \qquad P^2(x) - i\hbar\frac{dP(x)}{dx} = Q_0(x) := V(x) - E$$

- WKB curve Σ_{WKB} (Riemann surface): $y^2 = Q_0(x) = V(x) - E$.
- WKB periods: $\Pi_{\gamma_i} = \oint_{\gamma_i} P(x)dx$, $\gamma_i \in H_1(\Sigma_{\mathrm{WKB}})$.
- Quantization condition $\mathcal{Q}(\hbar, \Pi_{\gamma_1}, \Pi_{\gamma_2}, \cdots, \Pi_{\gamma_r}) = 0$ [Voros et al.].
(Bohr–Sommerfeld condition: $\oint_\gamma \sqrt{E - V(x)}dx = \hbar(n + \frac{1}{2})$).



Borel resummation of WKB periods

- $\Pi_\gamma = \sum_{n\geq 0} \hbar^{2n} \Pi_\gamma^{(n)}$: $\Pi_\gamma^{(n)} \sim (2n)!, \quad n \gg 1$ [Voros '81, Sato et.al].
- Borel transform of the WKB period (See also Jie's talk):

$$\hat{\Pi}_\gamma(\xi) = \sum_{n\geq 0} \frac{1}{(2n)!} \Pi_\gamma^{(n)} \xi^{2n}$$

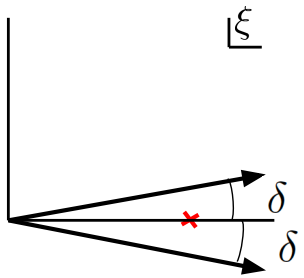
- The Borel resummation of the WKB period (Laplace transformation)

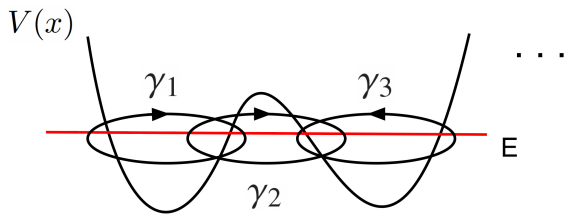
$$s[\Pi_\gamma](\hbar) = \frac{1}{\hbar} \int_0^\infty e^{-\xi/\hbar} \hat{\Pi}_\gamma(\xi) d\xi$$

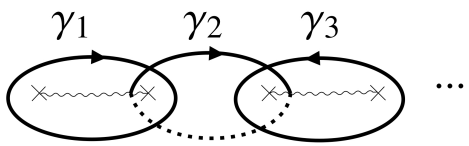
- If the integral converges when $\hbar$ is small enough, the quantum period is said to be Borel summable.

- Borel non-summable: singularity in the ξ-plane:

$$\text{discontinuity}: \quad \text{disc}[\Pi_\gamma] = s_\delta[\Pi_\gamma] - s_{-\delta}[\Pi_\gamma]$$







$$\text{e.g.} \quad \text{disc}[\Pi_{\gamma_1}] = -i\hbar \log\left(1 + \exp[-\frac{i}{\hbar}\Pi_{\gamma_2}(\hbar)]\right), \quad \hbar \in \mathbb{R}$$

[Delabaere-Dillinger-Pham ' 93-99, Iwaki-Nakanishi '14]

Voros' Riemann-Hilbert problem [Voros' 83]

Voros' problem: asymptotic series and the discontinuity formula $\Longrightarrow$ Exact WKB periods.

TBA/WKB correspondence

TBA equations provide solution to Voros' Riemann–Hilbert problem.

TBA/WKB correspondence

Inspired by [Gaiotto-Moore-Neitzke '08, '09], [Gaiotto '14], $\cdots$, the ODE/IM correspondence has been generalized to the Schrödinger equation with

- $V(x) = x^{r+1} + \sum_{a=1}^{r+1} u_a x^{r+1-a} + \frac{u_{r+2}}{x} + \hbar^2 \frac{l(l+1)}{x^2}$ [Ito-Marino-HS'18, Ito-HS '19, Gabai-Yin'21, Ito-Yang '24].

- $V(x) = \cosh(x)$: associated with pure $\mathcal{N} = 2$ $SU(2)$ SYM [Zamolodchikov, Grassi-Gu-Marino'19, Fioravanti-Gregori'19].

- Higher order ODE [Ito-Kuroda-Kondo-HS, Ito-Kondo-HS '21].

These models can be obtained by quantization of the Seiberg-Witten curve of $\mathcal{N} = 2$ SYM.

- Solutions of Schrödinger eq $\to$ Y-system + the asymptotic behavior $\leftrightarrow$ Thermodynamics Bethe Ansatz equations [Ito-Mariño-HS'18].

- Our TBA can reproduce the asymptotic series of WKB period $\Pi_{\gamma_i} = \sum_n \hbar^n \Pi_i^{(n)}$.

- It also reproduces the discontinuities of the WKB periods. Namely, our TBA equations provide solutions to Voros' Riemann-Hilbert problem of exact WKB periods.

$$\log Y_k = \frac{1}{\hbar} s[\Pi_{\gamma_k}](\hbar).$$

- Quantization condition $\mathcal{Q}(\Pi_{\gamma_1}, \Pi_{\gamma_2}, \cdots, \Pi_{\gamma_r}) = 0$ [Voros et al.]. Our method provides a powerful method to solve spectral problems.

From the viewpoint of $\mathcal{N} = 2$ SYM

Exact WKB	$N = 2$ SYM
WKB curve	SW curve
Schrödinger eq	quantum SW curve in NS limit
WKB periods Π_γ	BPS particle with $\vec{a}\,\vec{n}_e + \vec{a}_D\vec{n}_m$
TBA equations	GMN TBA in conformal limit
Wall-crossing of TBA	Wall-crossing of BPS states.
$\vdots$	$\vdots$

In principle, we can read the BPS spectrum in each chamber, and write down the corresponding GMN TBA equation [GMN '08, Gaiotto '14].

TBA/WKB correspondence for deformed SUSY QM

(May) beyond but highly related to the GMN

Schrödinger equation of deformed SUSY QM

$V(x) = (W'(x))^2 + \hbar m W''(x)$:

- $m = \pm 1/2$: the SUSY QM [Witten '81].
- $m \neq \pm 1/2$: deformed SUSY QM, a bosonic field x(t) and N_f Grassmann-valued fields with a superotential W.

To make sure the Schrödinger equation is invariant under the rotation:

$$Y_s(e^{\frac{\pi i}{2}}\hbar)Y_s(e^{-\frac{\pi i}{2}}\hbar) = \big(1+Y_{s-1}\big)\big(1+Y_{s+1}\big)(\hbar)$$

$$\xrightarrow{\hbar-\text{extension}} Y_s(e^{\frac{\pi i}{2}}\hbar, e^{-\frac{\pi i}{2}}m)Y_s(e^{-\frac{\pi i}{2}}\hbar, e^{\frac{\pi i}{2}}m) = \big(1+Y_{s-1}\big)\big(1+Y_{s+1}\big)(\hbar, m)$$

It is thus natural to introduce Y-functions:

$$Y_{a,s}(\hbar) = Y_s(\hbar, e^{a\frac{\pi i}{2}}m).$$

$\mathbb{Z}_4$ extended TBA

Evaluating driving term of Y-functions can be using the WKB method, one can convert the Y-system into the TBA equations

$$\begin{aligned}\log \widetilde{Y}_{a,s} = &- |m_s| e^{\theta} + m_{a,s}^{(\frac{1}{2})} + K_{+;s,s-1} \star \widetilde{L}_{a+1,s-1} + K_{+;s,s+1} \star \widetilde{L}_{a+1,s+1} \\ &+ K_{-;s,s-1} \star \widetilde{L}_{a+3,s-1} + K_{-;s,s+1} \star \widetilde{L}_{a+3,s+1}, \quad a \equiv a+4,\end{aligned}$$

where $\star$ is the convolution, $m_s = |m_s| e^{i\phi_s}$, $e^{\theta} = 1/\hbar$.

$$\begin{aligned}\widetilde{Y}_{a,s} =& Y_{a,s}(\theta - i\phi_s), \quad \widetilde{L}_{a,s} = \log\left(1 + \widetilde{Y}_{a,s}\right), \\ K_{\pm} =& \frac{1}{4\pi}\left(\frac{1}{\cosh\theta} \pm i\frac{\sinh\theta}{\cosh\theta}\right), \quad K_{\pm;s',s} = K_{\pm}(\theta - i\phi_{s'} + i\phi_s).\end{aligned}$$

- The pole of the kernel locates at $\phi_s - \phi_{s\pm1} = \pm\frac{\pi}{2}$.
- Minimal chamber: $|\phi_s - \phi_{s+1}| < \pi/2$ for $^{\forall}s$.

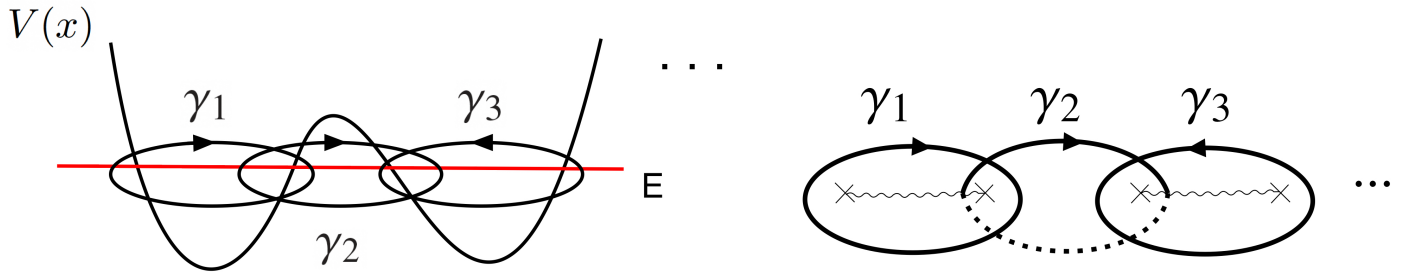
Minimal chamber: $W(x) = \frac{x^3}{3} - \frac{u_2 x}{2}, u_2 = \frac{1}{2}, 0 < E < 1/32$

- $(W')^2$ recovers the double well potential, $s = 1, 2, 3$.
- $m_1 = m_3$, $m_{a,2}^{(\frac{1}{2})} = 0$, $m_{a,1}^{(\frac{1}{2})} = -m_{a,3}^{(\frac{1}{2})}$.
- This leads to the identification $Y_{a,1} = Y_{a+2,3}$, $Y_{a,2} = Y_{a+2,2}$.
- $3_{\mathbb{Z}_4}$ TBA system (12-TBA) reduces to two independent symmetric D_3-type TBA systems (D_3 GMN TBA systems in minimal chamber)

$$\log \hat{Y}_{a,1} = -m_1 e^{\theta} + K \star \left[\log \left(1 + \hat{Y}_{a,2}\right) \right], \qquad a = 1, 2$$

$$\log \hat{Y}_{a,2} = -m_2 e^{\theta} + K \star \left[\log \left(1 + e^{m_{0,1}} \hat{Y}_{a,1}\right) + \log \left(1 + e^{-m_{0,1}} \hat{Y}_{a,1}\right) \right],$$

where $K(\theta) = \frac{1}{\cosh(\theta)}$, $\hat{Y}_{a,1} = e^{-m_{0,1}^{(1/2)}} Y_{a,1} = e^{m_{0,1}^{(1/2)}} Y_{a+2,1}$, $\hat{Y}_{a,2} = Y_{a,2}$



At $\theta \to \infty$, we can expand the kernel of the TBA equation as

$$-\log Y_{a,s}(\theta) \sim m_s e^{\theta} - m_{a,s}^{(\frac{1}{2})} + \sum_{n=1}^{\infty} m_{a,s}^{(n)} e^{(1-2n)\theta}$$

$$\Pi_{\gamma_i} \sim \Pi_{\gamma_i}^{(0)} + \Pi_{\gamma_i}^{(1/2)} + \sum_{n=1} \Pi_{\gamma_i}^{(n)} \hbar^{2n}.$$

n	$m_{0,1}^{(n)}$	$\Pi_{\gamma_1}^{(n)}/i$	$m_{1,2}^{(n)}$	$\Pi_{\gamma_2}^{(n)}$
0	0.103932897990	0.103932897990	0.146983313914	0.146983313914
1	-0.595015127256	0.595015127355	10.917186649450	10.917186650161
2	$1.135557990286 \cdot 10^2$	$1.135557990512 \cdot 10^2$	$-1.258067561652 \cdot 10^3$	$-1.258067561659 \cdot 10^3$
3	$-7.852584078608 \cdot 10^4$	$7.852584080303 \cdot 10^4$	$1.299686482674 \cdot 10^6$	$1.299686482683 \cdot 10^6$

Figure: Comparison of the coefficients in the expansion based on the TBA equation and the WKB periods at $m = 1/2$ with $u_2 = 1/2$ and $E = 1/64$.

Moreover, we can also confirm the discontinuity structure of the WKB periods from TBA, which leads to the identifications

$$-\log Y_{0,1}(\theta + \frac{\pi i}{2} \pm i0) = \frac{1}{\hbar} s_{\pm 0}(\Pi_{\gamma_1})(\hbar), \quad -\log Y_{1,2}(\theta) = \frac{1}{\hbar} s_0(\Pi_{\gamma_2})(\hbar).$$

Wall-crossing of TBA equations

The Schrödinger equation appears as the quantum Seiberg-Witten curve of $\mathcal{N} = 2$ SYM.

Wall-crossing of the BPS particles

In the $\mathcal{N}=2$ SYM, the BPS particle with charge γ is given by

$$\Pi_\gamma = \vec{a}\,\vec{n}_e + \vec{a}_D\vec{n}_m, \quad M_\gamma = |\Pi_\gamma|$$

where $\vec{a}$ is the vev of the scalars in the Cartan subalgebra, $\vec{a}_D$ is its dual.

The decay process $\gamma_0 \to \sum_{i=1}^{p} \gamma_i$

- Conservation of charges: $\Pi_{\gamma_0} = \sum_{i=1}^{p} \Pi_{\gamma_i}$
- Conservation of masses: $|\Pi_{\gamma_0}| = \sum_{i=1}^{p} |\Pi_{\gamma_i}|$

These two constraints are simultaneously satisfied when

$$\arg \Pi_{\gamma_i} = \arg \Pi_{\gamma_0} \;\to\; \mathrm{Im}\big(\Pi_{\gamma_0}(u)/\Pi_{\gamma_i}(u)\big) = 0, \cdots$$

This is known as the marginal stability walls.

Wall-crossing of TBA equations

- Marginal stability walls are located at the pole in the kernel of TBA.
- Crossing the wall, one needs to pick up the contribution of the pole and modify the TBA equations, i.e. wall-crossing of the TBA.
- We will consider the wall-crossing for general $\mathbb{Z}_4$ extended TBA, and then impose the symmetry due to the double well potential.

General procedure of wall-crossing for $r_{\mathbb{Z}_4}$-TBA: $\phi_2 - \phi_1$ cross $\pi/2$

$$\log \widetilde{Y}_{a,1} = -|m_1|e^{\theta} + m_{a,1}^{(\frac{1}{2})} + K_{+;1,2} \star \widetilde{L}_{a+1,2} + K_{-;1,2} \star \widetilde{L}_{a+3,2} + L_{a+3,2}(\theta - i\phi_1 - \frac{i\pi}{2})$$

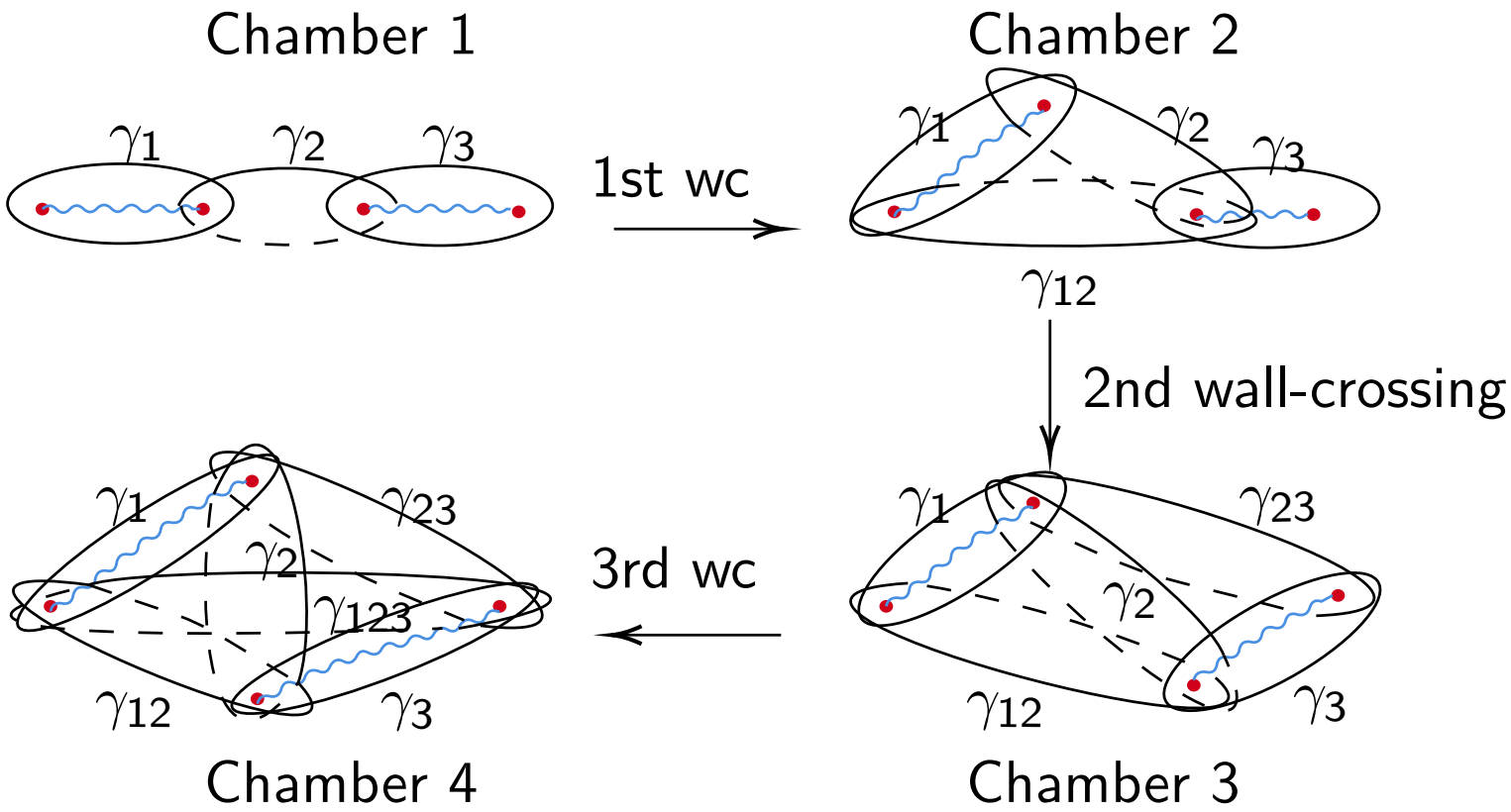
$$\log \widetilde{Y}_{a,2} = -|m_2|e^{\theta} + m_{a,2}^{(\frac{1}{2})} + K_{+;2,1} \star \widetilde{L}_{a+1,1} + K_{+;2,3} \star \widetilde{L}_{a+1,3} + K_{-;2,1} \star \widetilde{L}_{a+3,1} + K_{-;2,3} \star \widetilde{L}_{a+3,3} + L_{a+1,1}(\theta - i\phi_2 + \frac{i\pi}{2}), \cdots .$$

To obtain a closed TBA system, we introduce new Y-functions: $Y^{\mathrm{n}}_{a,1}(\theta) = \frac{Y_{a,1}(\theta)}{1+Y_{a+3,2}(\theta-\frac{i\pi}{2})}$, $Y^{\mathrm{n}}_{a,2}(\theta) = \frac{Y_{a,2}(\theta)}{1+Y_{a+1,1}(\theta+\frac{i\pi}{2})}$, $Y^{\mathrm{n}}_{a,12}(\theta) = \frac{Y_{a,1}(\theta)Y_{a+3,2}(\theta-\frac{i\pi}{2})}{1+Y_{a,1}(\theta)+Y_{a+3,2}(\theta-\frac{i\pi}{2})}$

$$
\begin{aligned}
\log \widetilde{Y}^{\mathrm{n}}_{a,1} =& -|m_1|e^{\theta} + m^{(\frac{1}{2})}_{a,1} \\
&+ K_{+;1,2} \star \widetilde{L}^{\mathrm{n}}_{a+1,2} + K^{+}_{+;1,12} \star \widetilde{L}^{\mathrm{n}}_{a+2,12} + K_{-;1,2} \star \widetilde{L}^{\mathrm{n}}_{a+3,2} + K^{+}_{-;1,12} \star \widetilde{L}^{\mathrm{n}}_{a,12} \\
\log \widetilde{Y}^{\mathrm{n}}_{a,2} =& -|m_2|e^{\theta} + m^{(\frac{1}{2})}_{a,2} \\
&+ K_{+;2,1} \star \widetilde{L}^{\mathrm{n}}_{a+1,1} + K_{+;2,12} \star \widetilde{L}^{\mathrm{n}}_{a+1,12} + K_{-;2,1} \star \widetilde{L}^{\mathrm{n}}_{a+3,1} + K_{-;2,12} \star \widetilde{L}^{\mathrm{n}}_{a+3,12} \\
&+ K_{+;2,3} \star \widetilde{L}_{a+1,3} + K_{-;2,3} \star \widetilde{L}_{a+3,3} \\
\log \widetilde{Y}_{a,3} =& -|m_3|e^{\theta} + m^{(\frac{1}{2})}_{a,3} + K_{+;3,2} \star \widetilde{L}^{\mathrm{n}}_{a+1,2} + K^{+}_{+;3,12} \star \widetilde{L}^{\mathrm{n}}_{a+2,12} \\
&+ K_{-;3,2} \star \widetilde{L}_{a+3,2} + K^{+}_{-;3,12} \star \widetilde{L}^{\mathrm{n}}_{a,12} + K_{+;3,4} \star \widetilde{L}_{a+1,4} + K_{-;3,4} \star \widetilde{L}_{a+3,4} \\
&\vdots \\
\log \widetilde{Y}^{\mathrm{n}}_{a,12}(\theta) =& -|m_{12}|e^{\theta} + m^{(\frac{1}{2})}_{a,1} + m^{(\frac{1}{2})}_{a+3,2} + K_{+;12,2} \star \widetilde{L}^{\mathrm{n}}_{a+1,2} + K_{-;12,2} \star \widetilde{L}^{\mathrm{n}}_{a+3,2} \\
&+ K^{-}_{+;12,1} \star \widetilde{L}^{\mathrm{n}}_{a,1} + K^{-}_{-;12,1} \star \widetilde{L}^{\mathrm{n}}_{a+2,1} + K^{-}_{+;12,3} \star \widetilde{L}_{a,3} + K^{-}_{-;12,3} \star \widetilde{L}_{a+2,3}
\end{aligned}
$$

A closed TBA system with $(r+1)_{\mathbb{Z}_4}$-TBA in the new chamber $(\gamma_1, \gamma_2, \cdots)_{\mathbb{Z}_4} \to (\gamma_1, \gamma_2, \cdots, \gamma_1+\gamma_2)_{\mathbb{Z}_4}$

Wall-crossing for quartic potential $(W')^2$



For SUSY QM with superpotential $W(x) = \frac{x^3}{3} - \frac{u_2 x}{2}$, $(W')^2$ is always double well potential, which only locate at chamber 1 (minimal chamber) and chamber 4 (maximal chamber).

TBA in maximal chamber

$W(x) = \frac{x^3}{3} - \frac{u_2 x}{2}$, $u_2 = \frac{1}{2}$, $E > 1/32$. For a given $a = 0, 1$

$$
\begin{aligned}
\log \widetilde{\widehat{Y}}^{(3)}_{a,1} =& -|m_1|e^{\theta} + K_{1,2} \star \widetilde{L}^{(3)}_{a+1,2} + K^{+}_{1,312} \star \widetilde{L}^{(3)}_{a,312} \\
&+ K^{+}_{1,12} \star \log\left(1 + e^{-m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,12}\right) + K^{+}_{1,12} \star \log\left(1 + e^{m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,12}\right) \\
\log \widetilde{\widehat{Y}}^{(3)}_{a,12} =& -|m_{12}|e^{\theta} + K_{12,2} \star \widetilde{L}^{(3)}_{a+1,2} + K^{-}_{12,312} \star \widetilde{L}^{(3)}_{a,312} \\
&+ K^{-}_{12,1} \star \log\left(1 + e^{m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,1}\right) + K^{-}_{12,1} \star \log\left(1 + e^{-m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,1}\right) \\
\log \widetilde{Y}^{(3)}_{a+1,2} =& -|m_2|e^{\theta} + 2K_{2,312} \star \widetilde{L}^{(3)}_{a,312} \\
&+ K_{2,1} \star \log\left(1 + e^{-m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,1}\right) + K_{2,1} \star \log\left(1 + e^{m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,1}\right) \\
&+ K_{2,12} \star \log\left(1 + e^{-m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,12}\right) + K_{2,12} \star \log\left(1 + e^{m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,12}\right) \\
\log \widetilde{Y}^{(3)}_{a,312} =& -|m_{312}|e^{\theta} + 2K_{312,2} \star \widetilde{L}^{(3)}_{a+1,2} \\
&+ K^{-}_{312,1} \star \log\left(1 + e^{m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,1}\right) + K^{-}_{312,1} \star \log\left(1 + e^{-m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,1}\right) \\
&+ K^{+}_{312,12} \star \log\left(1 + e^{-m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,12}\right) + K^{+}_{312,12} \star \log\left(1 + e^{m^{(\frac{1}{2})}_{a,1}} \widetilde{\widehat{Y}}^{(3)}_{a,12}\right) \\
& a \to a + 1.
\end{aligned}
$$

Two independent D_3 GMN TBA system in maximal chamber [Ito-Yang '24].

Monomial potential: $(W')^2 = x^4$, $E > 0$

$$
\begin{aligned}
|m_1| =& |m_3| = |m_{12}| = |m_{32}| = \frac{1}{\sqrt{2}}|m_2| = \frac{1}{\sqrt{2}}|m_{312}|, \\
\phi_1 =& \phi_3 = -\frac{\pi}{4}, \quad \phi_2 = \pi, \quad \phi_{12} = \phi_{34} = \frac{\pi}{4}, \quad \phi_{312} = 0, \\
m^{(0)}_{a,2} =& 0, \quad m^{(0)}_{a,1} = -m^{(0)}_{a,3},
\end{aligned}
$$

$6_{\mathbb{Z}_4}$-TBA reduces to 2 independent D_3-type TBA of [Zamolodchikov '91]

$$
\begin{aligned}
\log \widehat{\widetilde{Y}}^{(3)}_{a,1} =& -|m_1|e^{\theta} - \frac{\sqrt{2}}{\pi}\frac{\cosh(\theta-\theta')}{\cosh\left(2(\theta-\theta')\right)} \star \widetilde{L}^{(3)}_{a+1,2} \\
& - \frac{1}{2\pi\cosh(\theta-\theta')} \star \log\left(1 + e^{m^{(\frac{1}{2})}_{a,1}} \widehat{\widetilde{Y}}^{(3)}_{a,1}\right) - \frac{1}{2\pi\cosh(\theta-\theta')} \star \log\left(1 + e^{-m^{(\frac{1}{2})}_{a,1}} \widehat{\widetilde{Y}}^{(3)}_{a,1}\right) \\
\log \widetilde{Y}^{(3)}_{a+1,2} =& -|m_2|e^{\theta} - \frac{1}{\pi\cosh(\theta-\theta')} \star \widetilde{L}^{(3)}_{a+1,2} \\
& - \frac{\sqrt{2}}{\pi}\frac{\cosh(\theta-\theta')}{\cosh\left(2(\theta-\theta')\right)} \star \log\left(1 + e^{m^{(\frac{1}{2})}_{a,1}} \widehat{\widetilde{Y}}^{(3)}_{a,1}\right) - \frac{\sqrt{2}}{\pi}\frac{\cosh(\theta-\theta')}{\cosh\left(2(\theta-\theta')\right)} \star \log\left(1 + e^{-m^{(\frac{1}{2})}_{a,1}} \widehat{\widetilde{Y}}^{(3)}_{a,1}\right)
\end{aligned}
$$

In a similar way, we can expand the TBA equation and WKB periods at large θ (small $\hbar$).

n	$m_{0,1}^{(n)}$	$\Pi_{\gamma_1}^{(n)}/i$	$m_{1,2}^{(n)}$	$\Pi_{\gamma_2}^{(n)}$
1	$-e^{\frac{\pi i}{4}}0.041934908215746$	$e^{\frac{\pi i}{4}}0.041934908215746$	-0.083869816431491	-0.083869816431492
2	$-e^{-\frac{\pi i}{4}}0.065578751956822$	$-e^{-\frac{\pi i}{4}}0.065578751956827$	-0.131157503913644	-0.131157503913654

Figure: Comparison of the coefficients in the expansion based on the TBA equation and the WKB periods at $m = 2/5$, $u_2 = 0, E = 1/2$.

$$-\log Y_{0,1}^{(3)}(\theta + \frac{\pi i}{2} \pm i0) = \frac{1}{\hbar} s_{\pm 0}(\Pi_{\gamma_1})(\hbar), \quad -\log Y_{1,2}^{(3)}(\theta) = \frac{1}{\hbar} s_0(\Pi_{\gamma_2})(\hbar).$$

Summary

- TBA/WKB correspondence for one-dimensional SUSY QM.
- $\log Y \leftrightarrow s[\Pi_\gamma]$, which allows us to compute the WKB periods/BPS states exactly, and is useful to solve the spectral problem.
- Our model may beyond the GMN formalism, but highly related to it.
- Our TBA system is more complicated than usual GMN TBA. New poles will appear in $\log(1+Y)$, e.g. when m crosses $1/2$, where new contribution due to this type of pole should be included [Dorey-Tateo '96].

Discussions

- Different integrable systems are unified by the same ODE but with different moduli parameters.
- Z_4-extension and Wall-crossing in quantum integrable model?
- For the SUSY QM with superpotential $W = \frac{x^3}{3} - \frac{u_2 x}{2}$, the TBA system reduces to two independent D_3 GMN TBA-systems. Any interpretation from Seiber-Witten theory?

Thanks for your attention !