Topological vertex formalism with orientifold plane

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Some formal aspects of Field theories and Holography, 2025/4/19

§1. Introduction

Nekrasov partition function for a 5d N=1 SUSY gauge theory (on S^1)

- **Index of ADHM quantum mechanics**

(Integrate over instanton moduli)

$$Z_{\text{inst}}(\epsilon_1, \epsilon_2, a_i, m) = \text{Tr}\left[(-1)^{\mathsf{F}} e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-a_i C_i} e^{-m\cdot F}\right] .$$

ϵ_1, ϵ_2 : Omega deformation parameter

J_1, J_2 : Cartans of spacetime symmetry SO(4)

J_R : Cartan of $\text{SU}(2)_R$

a_i : Coulomb branch parameters

m_i : Mass parameters

- **Topological string partition function**

Topological string partition function

(A-model)

Counts BPS particle of the effective field theory of string theory compactified on X

$$Z(\{Q\};q) = \mathrm{PE}\left(\sum_{C\in H_2(X,\mathbb{Z}_2)} \sum_j (-1)^{2j}\, N_{j,C}\, Q_C \left(\frac{q^{-2j}+\cdots+q^{2j}}{(q^{1/2}-q^{-1/2})^2} \right) \right)$$

Gopakumar-Vafa invariant

(Number of BPS particles with (left) spin j and “charge” C)

$Q_C = e^{-T_C}$ T_C : Kahler parameter of cycle C = BPS mass for charge C

$q = e^{-g_s}$ g_s : string Coulpling = Omega deformation parameters $(\epsilon_1 = -\epsilon_2 = \hbar)$

5d N=1 (e.g. SU(N) linear quiver) gauge theory

↓

Type IIB string theory with 5-brane web

↓ **5-brane web = Toric diagram** [Leung, Vafa '97]

M theory on toric Calabi-Yau 3-fold

↓

Topological A-model on toric Calabi-Yau 3-fold

↓ **Topological vertex**
[Aganagic, Klemm, Marino, Vafa '03]

Topological string partition function
= 5d Nekrasov's partition function

5d N=1 (e.g SO(N)/Sp(N)) Gauge theory

↓

Type IIB string theory with 5-brane web (e.g. with orientifold plane)

↓ **?**

M theory on non-toric Calabi-Yau 3-fold

↓

Topological A-model on non-toric Calabi-Yau 3-fold

↓ **Topological vertex ?**

Topological string partition function = 5d Nekrasov's partition function

5d N=1 (e.g SO(N)/Sp(N)) Gauge theory

↓

Type IIB string theory with 5-brane web
(e.g. with orientifold plane)

↓

M theory on non-toric Calabi-Yau 3-fold

↓

Topological A-model on non-toric Calabi-Yau 3-fold

"Topological vertex”

↓

Topological string partition function

= **5d Nekrasov’s partition function**

Topological vertex formalism

M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, C. Vafa ’03
A. Iqbal, C. Kozcaz, C. Vafa '07
H. Awata, H. Kanno '08

Systematic method to compute
topological string partition function
based on (dual) toric diagram

Topological vertex formalism

Systematic method to compute
5d Nekrasov partition function
based on 5-brane web diagram

Various generalizations have been studied motivated by this idea.

Generalization for topological vertex formalism motivated by 5-brane web diagrams

1. Periodic (5d N=1*, elliptic quiver, little string, M-string…)

[T. Hollowood, A.Iqbal, C. Vafa '03]
[B. Haghighat, A. Iqbal, C. Kozcaz, G. Lockhart, C. Vafa '13]
[B. Haghighat, C. Kozcaz, G. Lockhart, C. Vafa ’13]
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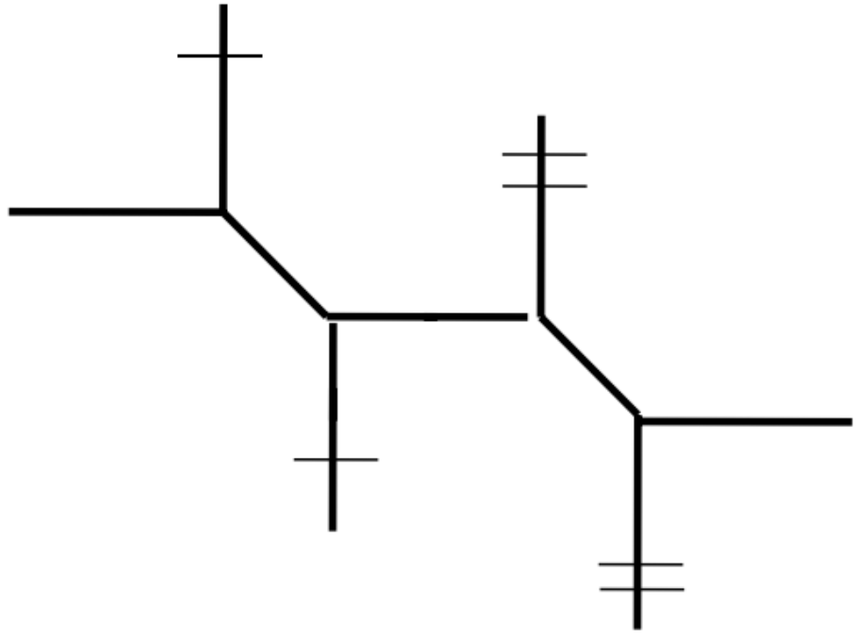
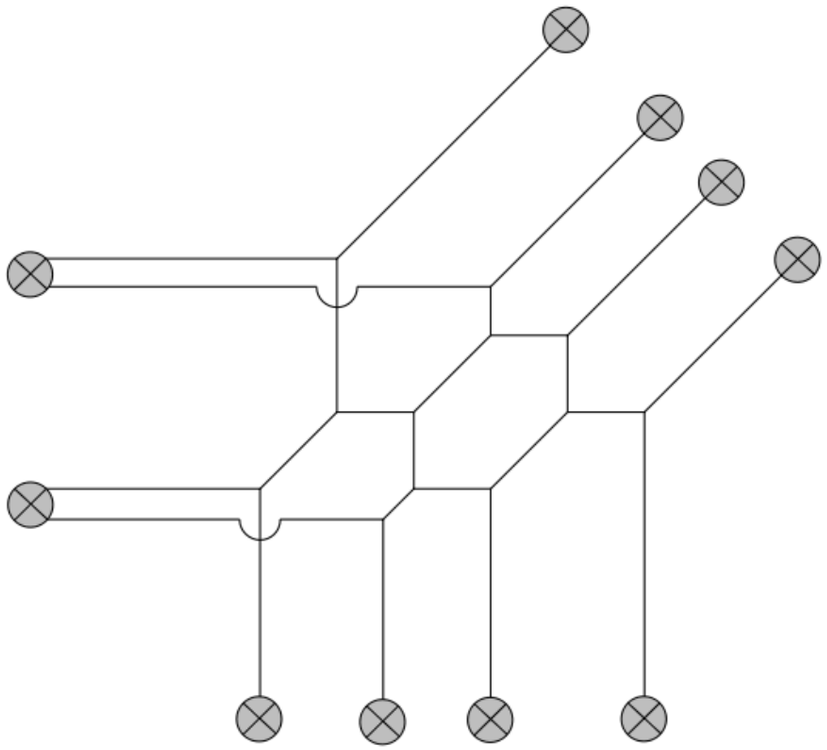


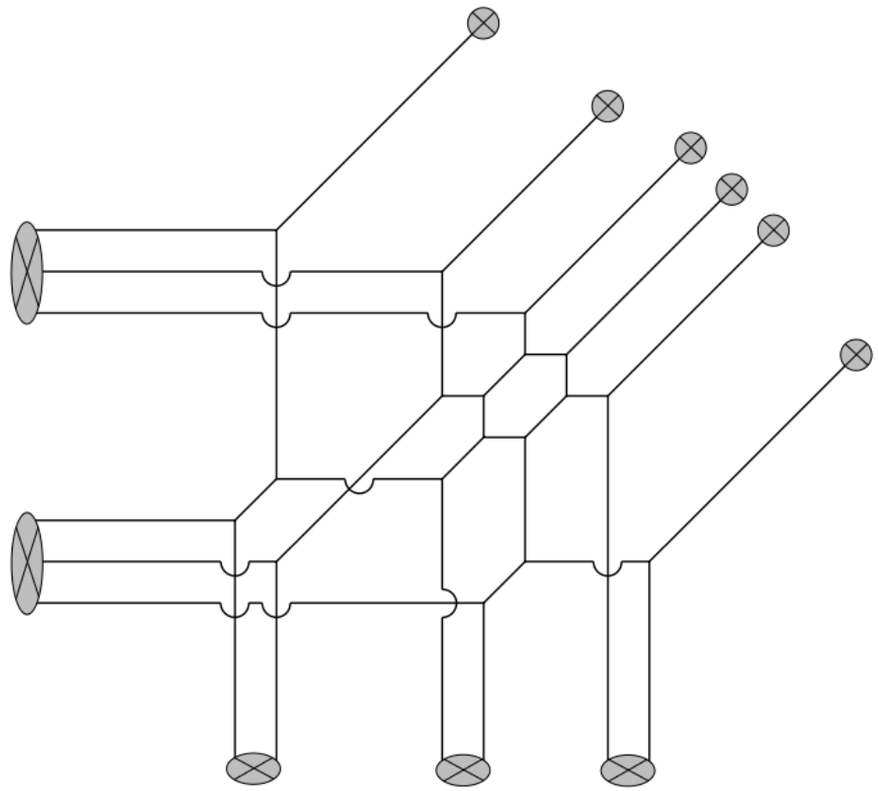
Generalization for topological vertex formalism motivated by 5-brane web diagrams

2. With 7-branes (5d Rank 1 SCFT with E_7, E_8 symmetry)

[H. Hayashi, H. Kim, T. Nishinaka '13]
[H. Hayashi, G. Zoccarato '14]



E_7



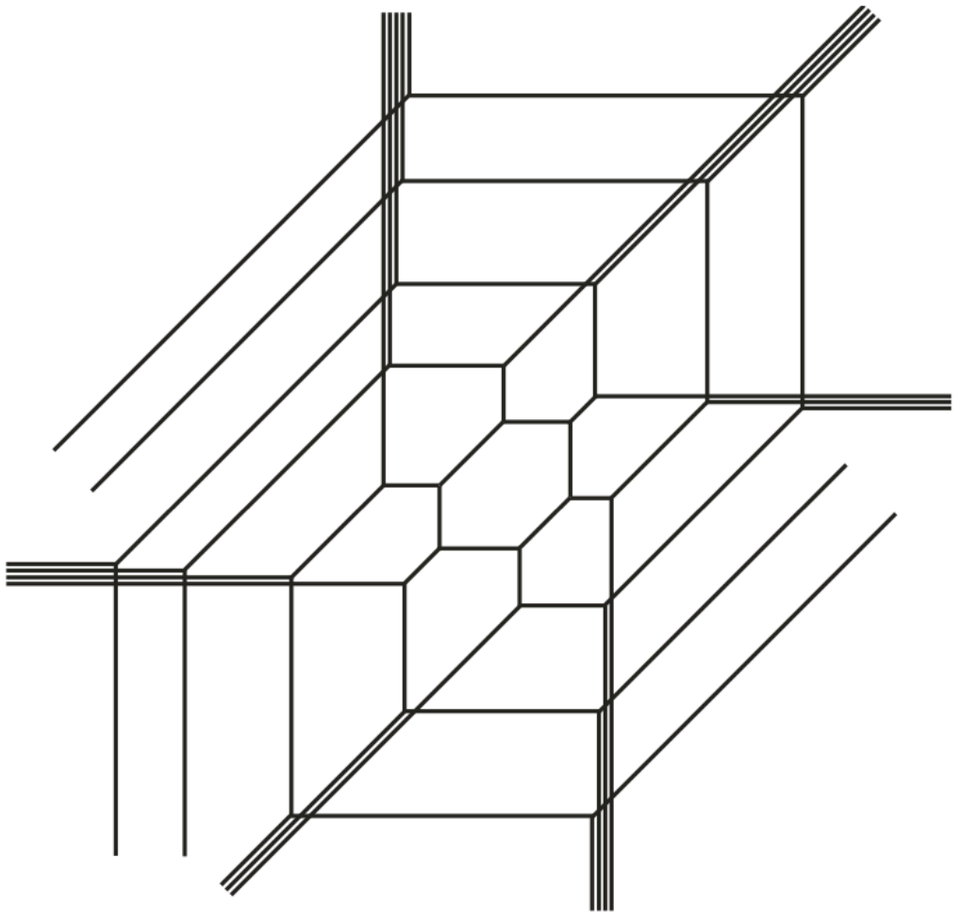
E_8

cf [F. Benini, S. Benvenuti, Y. Tachikawa '09]

Generalization for topological vertex formalism motivated by 5-brane web diagrams

3. "Tao diagram" (E-string, 6d SU(N) + 2N flavor)

[S-S. Kim, M. Taki, F. Yagi '15]
[H. Hayashi, S. Kim, K. Lee, M. Taki, F. Yagi '15]

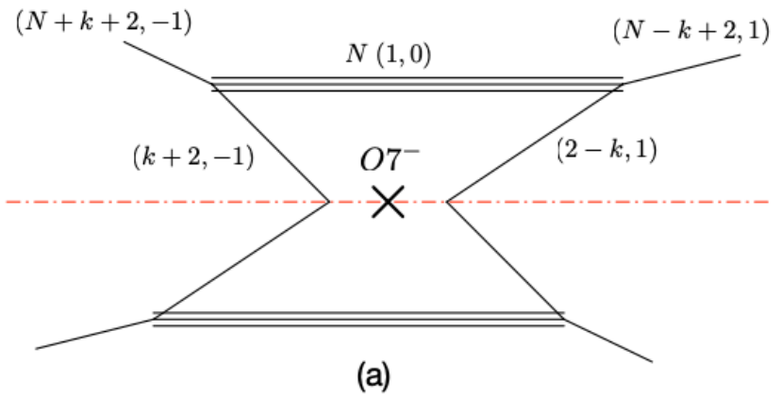


E-string theory

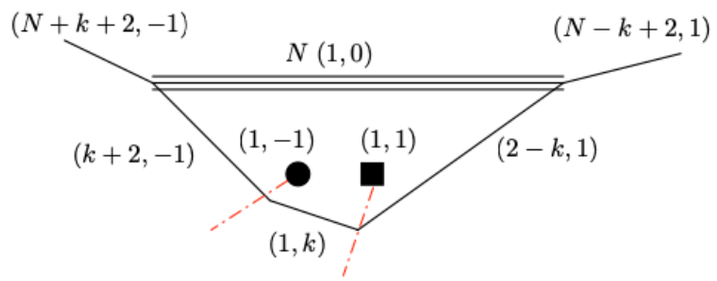
Generalization for topological vertex formalism motivated by 5-brane web diagrams

4. With O7⁻ -plane (decomposed into (1,1) 7-brane and (1,-1) 7-brane [Sen '96])
(5d SU(N)+AS, Sp(N), Sp(N)+AS) (AS: hypermultiplet in AntiSymmetric representation)

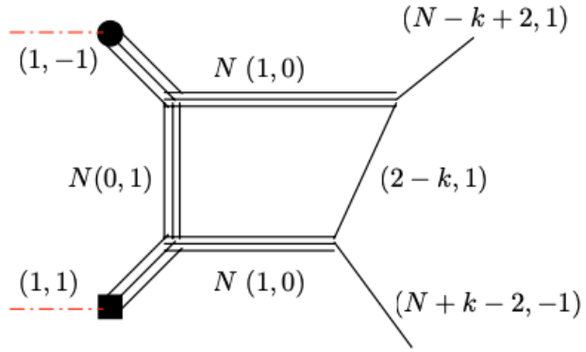
[H. Hayashi, G. Zoccarato '16]
[S. Cheng, S-S. Kim '18]



(a)



(b)



(c)

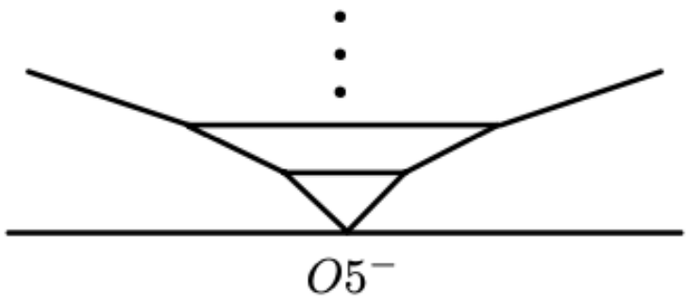
Sp(N)

[O.Bergman, G.Zafrir '15]

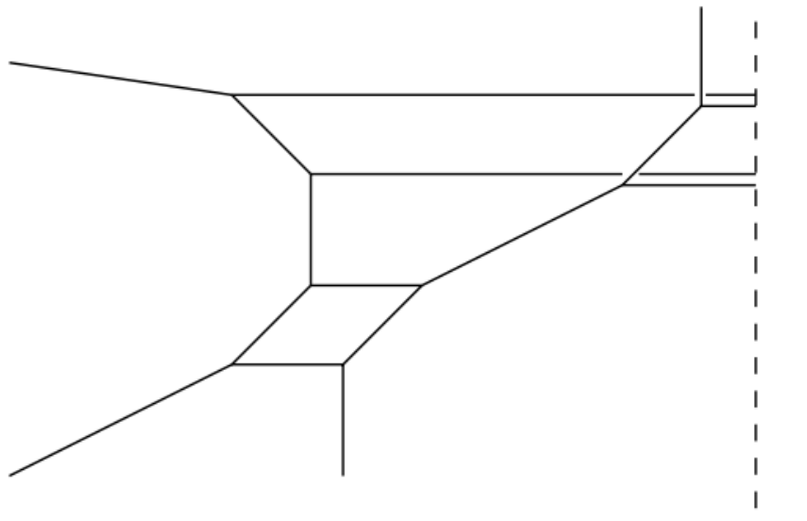
Generalization for topological vertex formalism motivated by 5-brane web diagrams

5. With O5⁻-plane or ON⁻ -plane (5d Sp(N), E-string, D-type quiver)

[S-S. Kim, F. Yagi ’17]
[J. Bourgine, M. Fukuda, Y. Matsuo, R. Zhu ’17]
[S-S. Kim, X. Wei '22]



Sp(N) = $SU(N+1)_{N+3}$

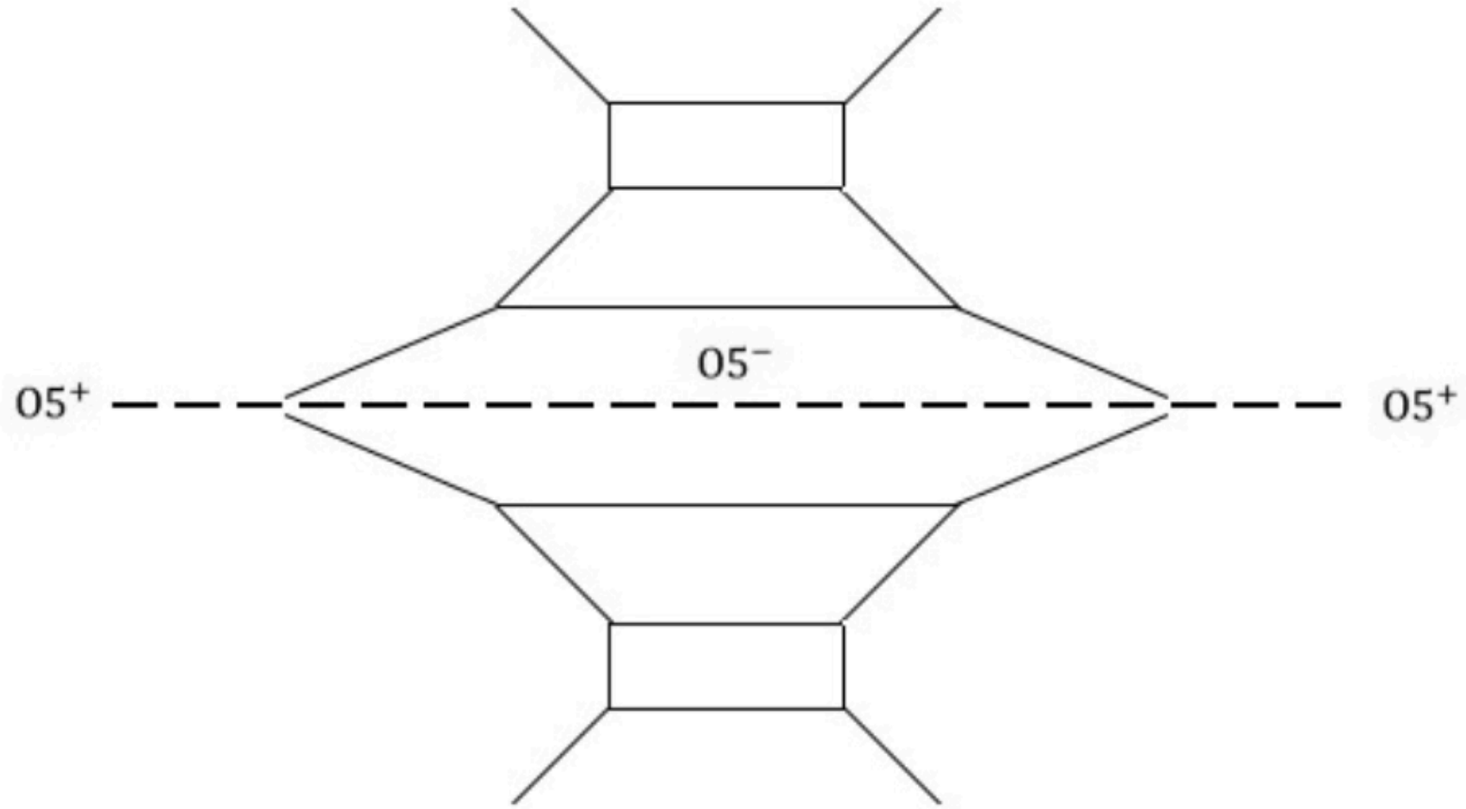


D_2 quiver

Generalization for topological vertex formalism motivated by 5-brane web diagrams

5’. With O5$^+$-plane (and with O5$^-$-plane with finite length) (5d SO(N))

[H. Hayashi, R. Zhu ’20]
[S. Nawata, R. Zhu ’21]

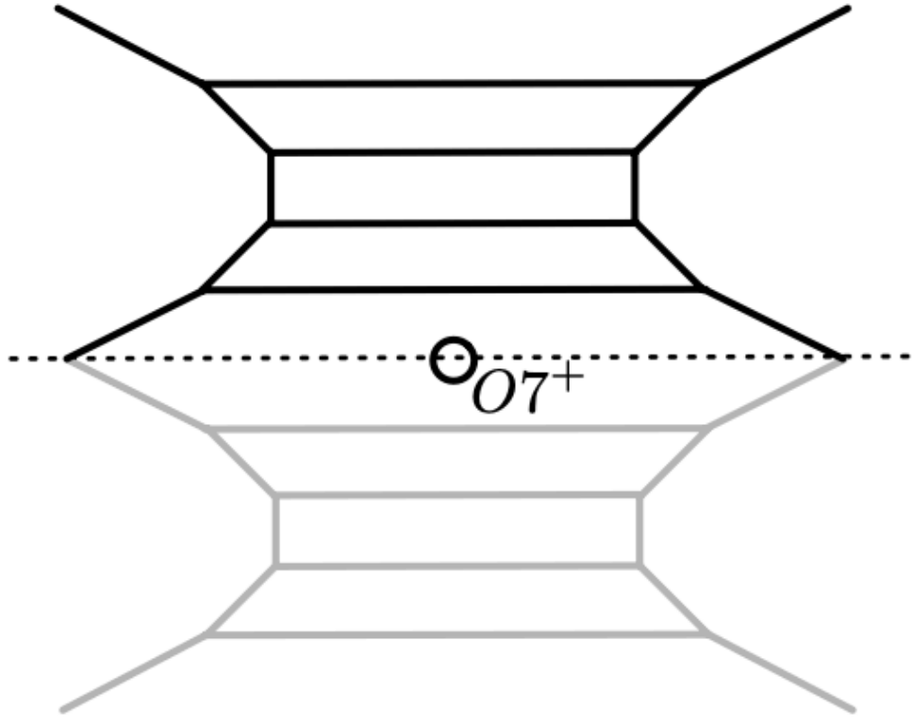


SO(2N)

Generalization for topological vertex formalism motivated by 5-brane web diagrams

6. With $O7^+$ - plane (5d SO(N)) ← NEW!

[S-S. Kim, X. Li, F. Yagi, R. Zhu '24]

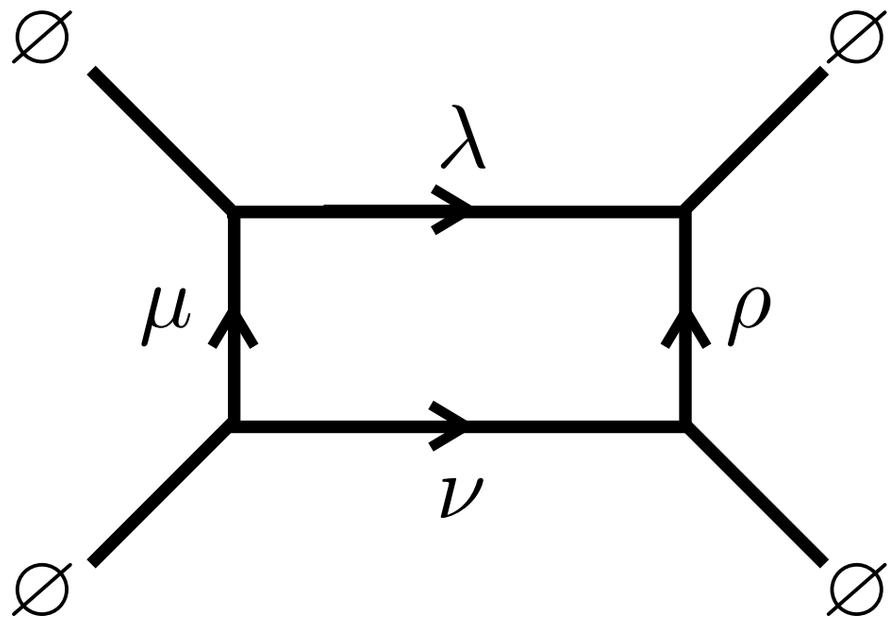


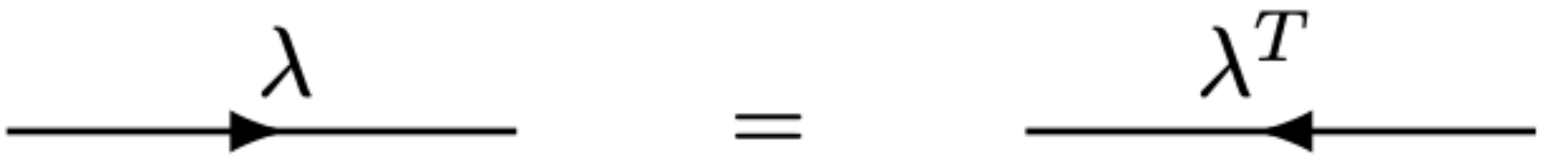
SO(2N)

§2. Topological vertex

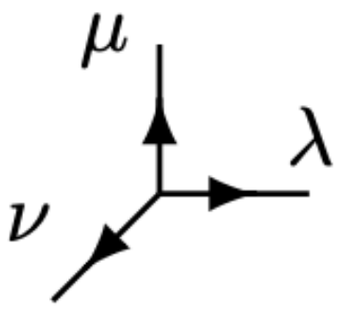
Topological vertex formalism

Assign different Young diagrams to each edge with specifying the orientation





Assign Vertex factor (topological vertex) to each vertex



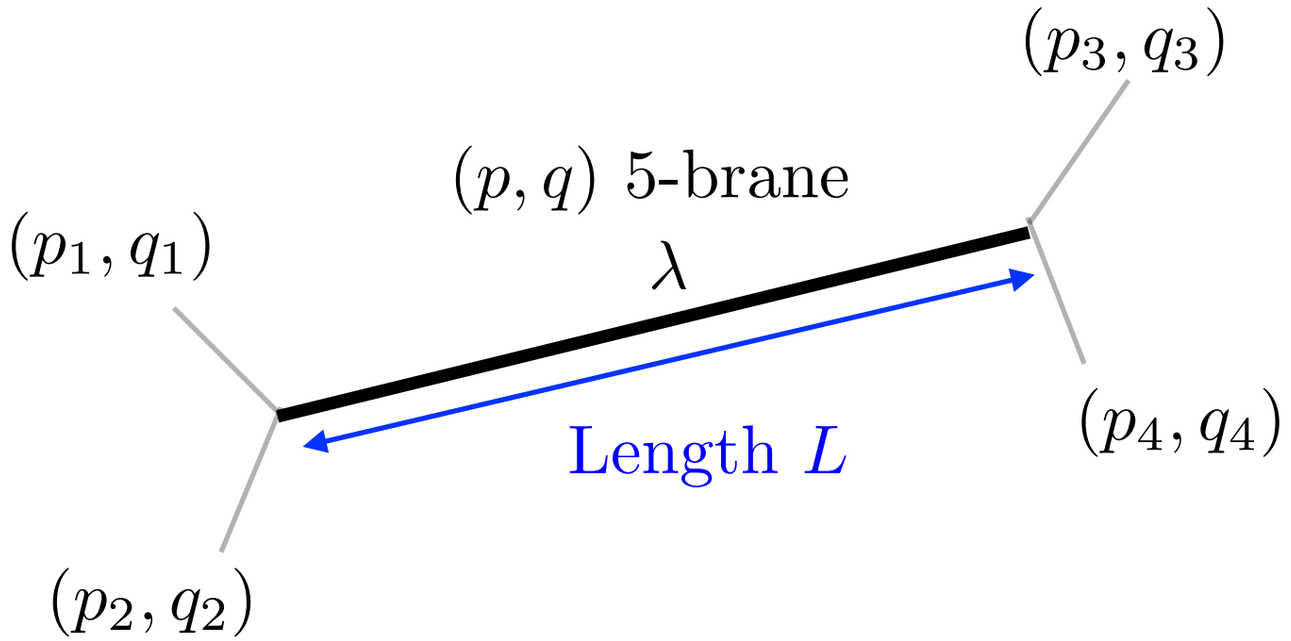
$$= \quad C_{\nu\mu\lambda}$$

$$C_{\nu\mu\lambda} = q^{\frac{\kappa(\mu)}{2}+\frac{\kappa(\lambda)}{2}} s_\lambda(q^{-\rho}) \sum_\sigma s_{\nu^T/\sigma}(q^{-\rho-\lambda}) s_{\mu/\sigma}(q^{-\rho-\lambda^T})$$

s_{Y_1/Y_2}, : Skew Schur function $\qquad \rho = \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \right\}$

$$\kappa(\lambda) := 2 \sum_{(i,j)\in\lambda} (j-i)$$

Assign edge factor to each edge



$$= (-Q)^{|\lambda|} f_\lambda(q)^n$$

$$Q = \exp\left(-\frac{L}{\sqrt{p^2+q^2}}\right), \qquad f_\lambda = (-1)^{|\lambda|} q^{\frac{1}{2}\kappa(\lambda)} \qquad n = p_1 p_4 - q_1 p_4 = p_2 q_3 - p_3 q_2$$

Multiply all the factors and sum over all the possible Young diagrams

$$Z = \sum_{\lambda,\mu,\nu\cdots} \prod (\text{Edge factor}) \cdot \prod (\text{Vertex factor}).$$

*** Similarity with Feynman rule**

Young diagrams ↔ Momentum

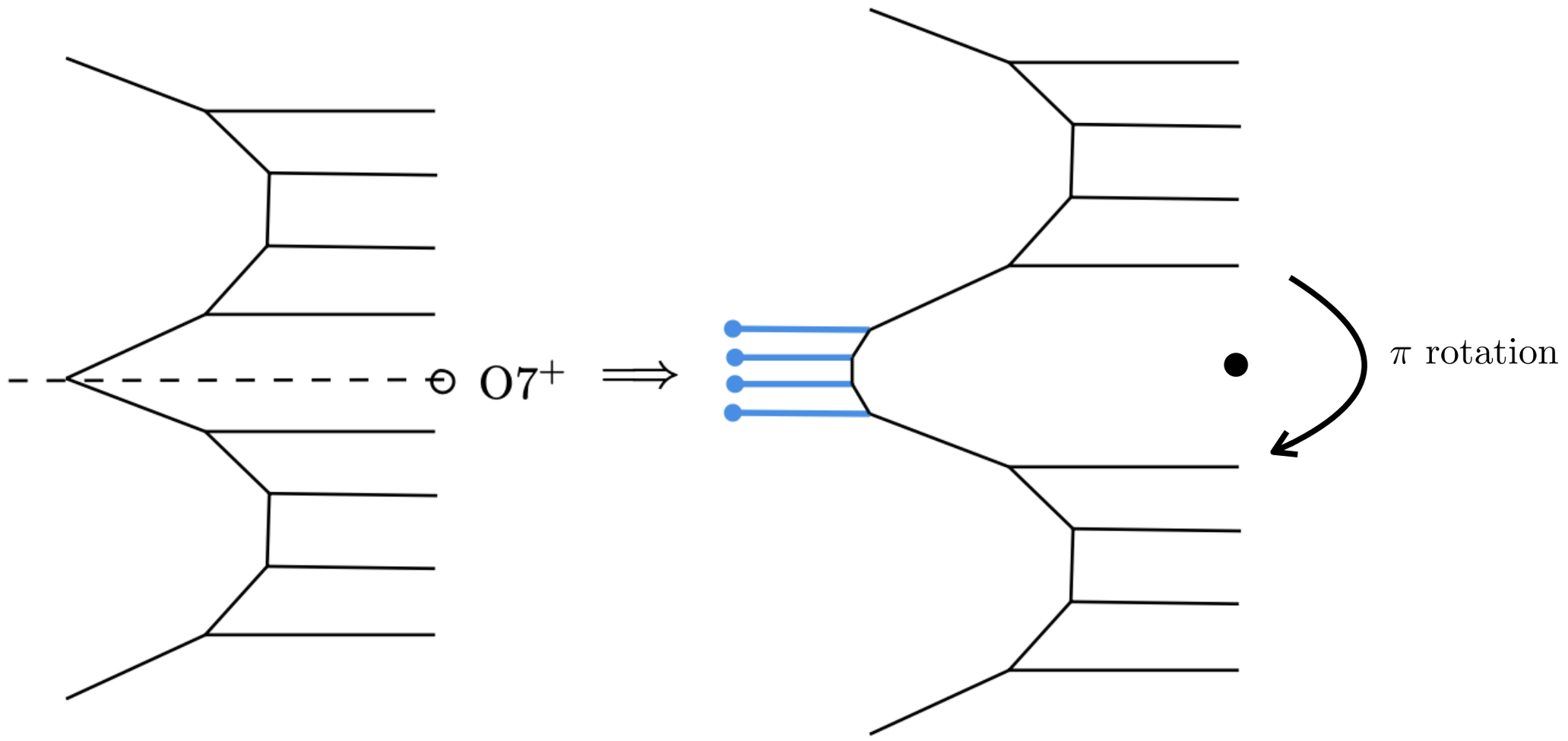
Edge factor ↔ Propagators

Vertex factor ↔ Couling constant

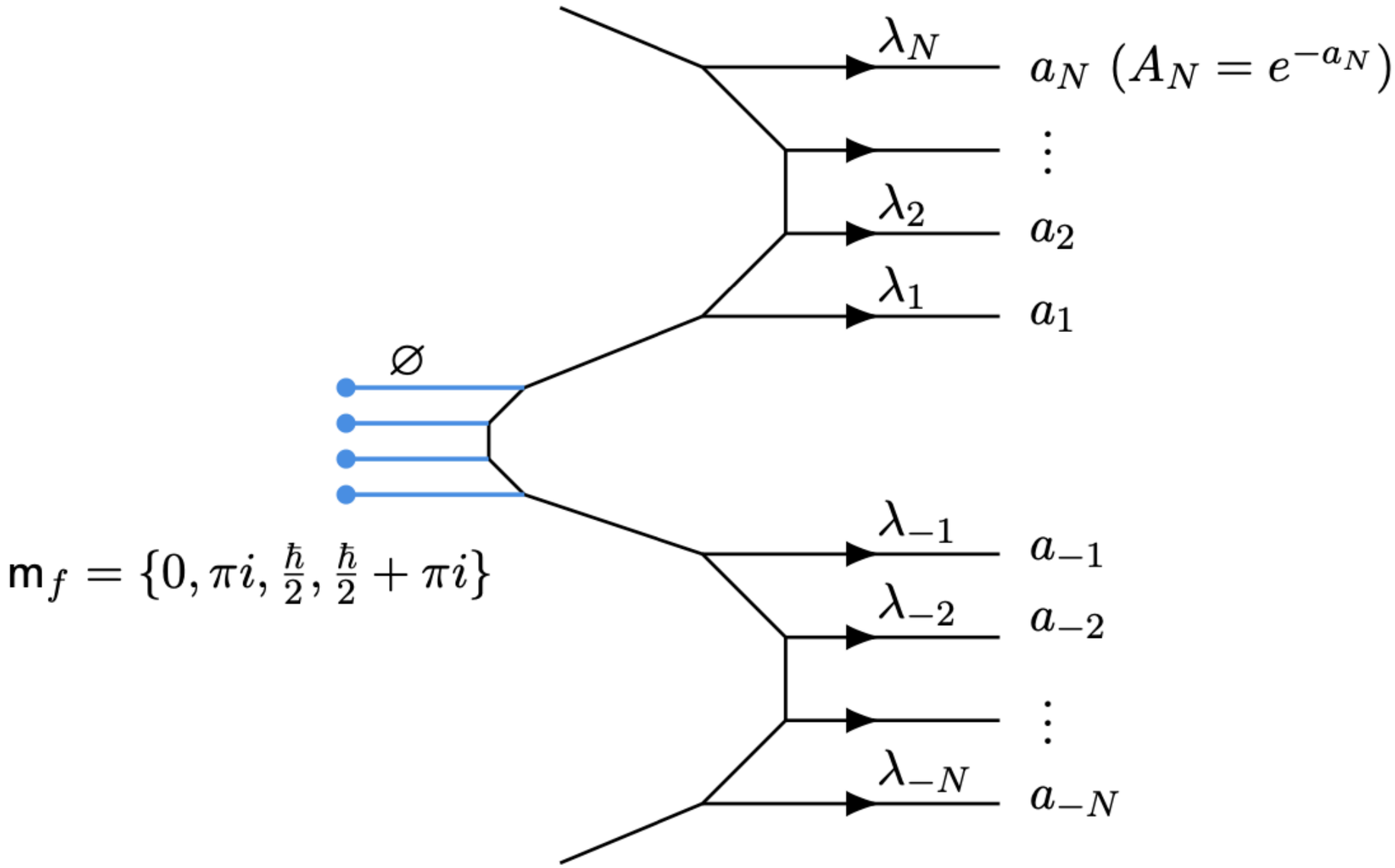
Topological vertex with O7+ -plane is based on the idea:

$$O7^+ \sim \mathbb{Z}_2 \text{ orbifold } + 4D7$$

5d N=1 SO(2N) gauge theory



Assignment of Young diagrams and parameters



$$a_{-s} := -a_s, \quad \lambda_{-s} := \lambda_s^T, \qquad (s = 1, 2, \cdots, N)$$

Include only vertex factors and edge factors in the fundamental region

§3. 5d SO(2N) Nekrasov partition function

Nekrasov partition function for 5d N=1 SO(2N) gauge theory from 5-brane web with O7$^+$-plane

$$Z^{\text{inst}}_{\text{SO}(2N)} = \sum_{\vec{\lambda}} Q_I^{|\vec{\lambda}|} \frac{\displaystyle\prod_{\substack{-N\le s\le N \\ s\neq 0}} \prod_{f=1}^{4} \tilde{n}_{\varnothing\lambda_s}(a_s - \mathsf{m}_f;\hbar)}{\displaystyle\prod_{\substack{-N\le s\le N \\ s\neq 0}} \prod_{t=1}^{N} \tilde{n}_{\lambda_s\lambda_t}(a_t - a_s;\hbar)}$$

Nekrasov factor:

$$\tilde{n}_{\lambda\nu}(a;\hbar) := \prod_{(i,j)\in\lambda} 2\sinh\frac{1}{2}\Big(a+\hbar\big(1-i-j+(\lambda)_i+(\nu^T)_j\big)\Big) \prod_{(i,j)\in\nu} 2\sinh\frac{1}{2}\Big(a+\hbar\big(i+j-1-(\lambda^T)_j-(\nu)_i\big)\Big)$$

$$a_{-s} := -a_s, \quad \lambda_{-s} := \lambda_s^T, \qquad (s = 1, 2, \cdots, N)$$

$$\mathsf{m}_f = \{0, \pi i, \tfrac{\hbar}{2}, \tfrac{\hbar}{2} + \pi i\}$$

Agrees with the computation based on topological vertex formalism for 5-brane web with O5-plane

[H. Hayashi, R. Zhu '20]
[S. Nawata, R. Zhu '21]

$$Z^{\text{inst}}_{\text{SO}(2N)} = \sum_{\vec{\lambda}} \prod_{s=1}^{N} (-Q_{Bs})^{|\lambda_s|} f_{\lambda_s}^{5-2s} s_{\lambda_s}(q^{-\rho}) s_{\lambda_s^T}(q^{-\rho}) \times \prod_{1\le s<t\le N} N^{-2}_{\lambda_s\lambda_t}(A_t A_s^{-1}; q)$$
$$\times M^2_{\vec{\lambda}}(A_1, A_2, \cdots, A_N) \ ,$$

$$N_{\lambda\nu}(Q;q) := \prod_{(i,j)\in\lambda} \left(1 - Q q^{1-i-j+(\lambda)_i+(\nu^T)_j}\right) \prod_{(m,n)\in\nu} \left(1 - Q q^{m+n-(\lambda^T)_n-(\nu)_m-1}\right)$$

$$M_{\vec{\lambda}}(A_1, A_2, \cdots, A_N) := \frac{\langle 0| \mathbb{O}(q) \prod_{s=1}^{N} \Gamma_-(A_s q^{-\rho-\lambda_s}) |0\rangle}{\langle 0| \mathbb{O}(q) \prod_{s=1}^{N} \Gamma_-(A_s q^{-\rho}) |0\rangle}$$

$$\Gamma_\pm(\vec{x}) := \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_i x_i^n J_{\pm n}\right) \qquad \mathbb{O}(q) := \exp\left(\sum_{n=1}^{\infty} -\frac{1+q^n}{2n(1-q^n)} J_{2n} + \frac{1}{2n} J_n J_n\right)$$

$$[J_n, J_m] = n\, \delta_{n+m,0}$$

Agrees with the computation based on ADHM method

[S. Nawata, R. Zhu '21]

$$Z^k_{SO(2N)} = \sum_{\vec{\lambda}} \prod_{(i,j)\in\lambda_s} \prod_{s=1}^{N} \frac{\sinh \phi_{i,j}(s)}{\prod_{t=1}^{N} \sinh^2 \frac{N_{i,j}(s,t)}{2} \cdot \sinh^2 \frac{\phi_{i,j}(s)+a_t}{2}}$$

$$\times \prod_{s\le t}^{N} \prod_{\substack{(i,j)\in\lambda_s,\\ (m,n)\in\lambda_t \\ (i,j)<(m,n)}} \frac{\sinh^4 \frac{\phi_{i,j}(s)+\phi_{m,n}(t)}{2}}{\sinh^2 \frac{\phi_{i,j}(s)+\phi_{m,n}(t)+\hbar}{2} \sinh^2 \frac{\phi_{i,j}(s)+\phi_{m,n}(t)-\hbar}{2}}$$

$$\phi_{i,j}(s) = a_s + (i-j)\hbar$$

$$N_{i,j}(s,t) = a_s - a_t - \hbar(i+j-(\lambda_s)_i - (\lambda_t^T)_j + 1)$$

$$\lambda^{(s)} \ni (i,j) < (m,n) \in \lambda^{(t)} \quad \text{if} \quad \begin{cases} s < t \\ s = t, i < m \\ s = t, i = m, j < n \end{cases}$$

Consistent with the observation in ADHM construction

[Kimyeong Lee, Norton Lee '24]

$$\text{“}\mathcal{Z}_{SU(2N)}[\lambda] = \left(\mathcal{Z}_{SO(2N)}[\lambda]\right)^2\text{”}$$

$$Z^{\text{inst}}_{\text{SO}(2N)} = \sum_{\vec{\lambda}} Q_I^{|\vec{\lambda}|} \left(\frac{\prod_{\substack{-4\le f\le 4 \\ f\neq 0}} \prod_{\substack{-N\le t\le N \\ t\neq 0}} \tilde{n}_{\varnothing\lambda_t}(a_t - \mathsf{m}_f;\hbar)}{\prod_{\substack{-N\le s\le N \\ s\neq 0}} \prod_{\substack{-N\le t\le N \\ t\neq 0}} \tilde{n}_{\lambda_s\lambda_t}(a_t - a_s;\hbar)} \right)^{\frac{1}{2}}$$

Summand of 5d SU(2N) Nekrasov partition function with

$$a_{-s} := -a_s, \quad \lambda_{-s} := \lambda_s^T, \qquad (s = 1, 2, \cdots, N)$$

$$\mathsf{m}_f = \{0, \pi i, \tfrac{\hbar}{2}, \tfrac{\hbar}{2} + \pi i\}$$

Summary

- **Topological vertex formalism has been generalized for various 5-brane web diagrams.**

- **We proposed new topological vertex formalism for 5-brane web diagrams with O7+-plane.**

- **5d SO(2N) Nekrasov partition function is correctly reproduced.**