$OSp(1|4)$ covariant formalism of $N = 1$ AdS_4 supergravity

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Outline

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Motivation

- A lesson from AdS/CFT:
  Global symmetry in the boundary field theory
  $\Rightarrow$ Gauge symmetry in the bulk gravity theory
- For a CFT at the d-dimensional boundary, we have the conformal symmetry $SO(2,d)$ as the global symmetry.
- Thus we would expect the local $SO(2,d)$ gauge symmetry in the $D = d+1$ dimensional bulk theory.
- For AdS_3, the bulk gravity can be re-written as the $SO(2,2) = SL(2,R) \times SL(2,R)$ Chern-Simons theory.
- In our works, the AdS_{d+1} gravity is reformulated as a $SO(2,d)$ gauge theory in a manifestly gauge covariant way.
- Can we construct SUGRA action in a manifestly SUSY gauge covariant way?

Einstein Gravity

- Written in terms of vielbein, the Einstein-Hilbert action is

$$S[e] = \int \epsilon_{a_1 \cdots a_D} \left[\Theta^{a_1 a_2} + \frac{(D-2)}{D\,\ell^2} e^{a_1} \wedge e^{a_2} \right] \wedge e^{a_3} \wedge \cdots \wedge e^{a_D}$$

  where the spin connection $\omega^a{}_b$ is decided by the torsion free condition

$$\mathrm{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$$

  and the curvature is decided by the spin connection

$$\Theta^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

- The Einstein equation is

$$\left(\Theta^{[a_1 a_2} + \ell^{-2} e^{[a_1} \wedge e^{a_2} \right) \wedge e^{a_3} \wedge \cdots \wedge e^{a_{D-1}]} = 0$$

Palatini's 1-st Order Formula

- Palatini: Treating the spin connection $\omega^a{}_b$ as independent variables

$$S[e,\omega] = \int \epsilon_{a_1\cdots a_D}\left[\Theta^{a_1 a_2} + \frac{(D-2)}{D\,\ell^2} e^{a_1}\wedge e^{a_2}\right]\wedge e^{a_3}\wedge\cdots\wedge e^{a_D}$$

- The EOM's are

$$\left(\Theta^{[a_1 a_2} + \ell^{-2} e^{[a_1}\wedge e^{a_2}\right)\wedge e^{a_3}\wedge\cdots\wedge e^{a_{D-1}]} = 0$$

$$\mathrm{D}e^{[a_1}\wedge e^{a_2}\wedge\cdots\wedge e^{a_{D-2}]} = 0$$

- If the vielbein e^a is not degenerate, the torsion free condition will be automatically implied by the 2nd EOM

Uplift to $SO(2,d)$ via ruler

- Basic idea of $SO(1,d) \to SO(2,d)$:

$$A^a{}_b \sim \omega^a{}_b\,, \qquad A^a{}_\bullet \sim \ell^{-1} e^a$$

- In the covariant treatment, we impose a ruler field $Y^{\hat{\alpha}}$ in the vector representation of $SO(2,d)$ to point out the way of $SO(2,d) \to SO(1,d)$
- Ruler field satisfies the constraints $Y^{\hat{\alpha}} Y_{\hat{\alpha}} = -\ell^2$. Thus no new DOF is introduced.
- The space time metric is induced by the ruler field $Y^{\hat{\alpha}}$

$$ds^2 = \mathrm{D}_M Y^{\hat{\alpha}}\, \mathrm{D}_N Y_{\hat{\alpha}}\, dx^M dx^N\,,$$

  where $\mathrm{D} = \mathrm{d} + A$ is the $SO(2,d)$ covariant derivative.
- The Palatini EOM's can be unified $SO(2,d)$ covariantly as

$$F^{[\hat{\alpha}_1 \hat{\alpha}_2} \wedge \mathrm{D} Y^{\hat{\alpha}_3} \wedge \cdots \wedge \mathrm{D} Y^{\hat{\alpha}_d]} = 0\,.$$

$SO(2,d)$ covariant action

- We can realize the previous uplift in terms of a gauge invariant action constructed by $SO(2,d)$ covariant fields

$$\int \epsilon_{a_1\cdots a_D}\left[\Theta^{a_1 a_2}+\frac{(D-2)}{D\,\ell^2}e^{a_1}\wedge e^{a_2}\right]\wedge e^{a_3}\wedge\cdots\wedge e^{a_D}$$
$$\sim\int \epsilon_{\hat{\alpha}_1\cdots\hat{\alpha}_{D+1}}\left[F^{\hat{\alpha}_1\hat{\alpha}_2}-\frac{2}{D\,\ell^2}\mathrm{D}Y^{\hat{\alpha}_1}\wedge\mathrm{D}Y^{\hat{\alpha}_2}\right]\wedge\mathrm{D}Y^{\hat{\alpha}_3}\wedge\cdots\wedge\mathrm{D}Y^{\hat{\alpha}_D}Y^{\hat{\alpha}_{D+1}}$$

- For $D=3$, it is equivalent to the $SL(2)\times SL(2)$ C-S formalism up to a total derivative term. The Y field is implicitly imposed when one decides the vielbein from the gauge field $e^a=A^a_L+A^a_R$.
- For $D=4$, it is equivalent to the MacDowell-Mansouri-Stelle-West action

$$S'_{(d+1)}\sim\int_M \epsilon_{\hat{\alpha}_0\cdots\hat{\alpha}_4}F^{\hat{\alpha}_0\hat{\alpha}_1}\wedge F^{\hat{\alpha}_2\hat{\alpha}_3}Y^{\hat{\alpha}_4}\,.$$

up to a closed form which is just the Gauss-Bonnet term.

$SO(2,d)$ covariant action

- The bulk covariant action of AdS gravity

$$S^{\rm cov}_{(d+1)} = \int_M \epsilon_{\hat\alpha_1\cdots\hat\alpha_{d+1}}\left[F^{\hat\alpha_0\hat\alpha_1} - \frac{2}{(d+1)\ell^2}\mathrm{D}Y^{\hat\alpha_0}\wedge\mathrm{D}Y^{\hat\alpha_1}\right]\wedge\mathrm{D}Y^{\hat\alpha_2}\wedge\cdots\wedge\mathrm{D}Y^{\hat\alpha_d}Y^{\hat\alpha_{d+1}}\,.$$

  implies the EOM's

$$F^{[\hat\alpha_1\hat\alpha_2}\wedge\mathrm{D}Y^{\hat\alpha_3}\wedge\cdots\wedge\mathrm{D}Y^{\hat\alpha_d]} = 0\,,$$
$$\epsilon_{\hat\alpha\hat\beta\hat\alpha_1\cdots\hat\alpha_d}Y^{\hat\beta}\mathrm{D}(F^{\hat\alpha_1\hat\alpha_2}\wedge\mathrm{D}Y^{\hat\alpha_3}\wedge\cdots\wedge\mathrm{D}Y^{\hat\alpha_d}) = 0\,.$$

- The torsion free condition is equivalent to

$$[\mathrm{D}_\mu,\mathrm{D}_\nu]Y^{\hat\alpha} = (F_{\mu\nu})^{\hat\alpha}{}_{\hat\beta}Y^{\hat\beta} = 0\,.$$

- Providing the 1st EOM, the 2nd EOM will be automatically satisfied. In fact, it vanishes only when the torsion free condition is satisfied.
- Thus the new EOM will not introduce any further constraints, and the above system is equivalent to the original Einstein gravity classically.

$SO(2,d)$ covariant action

- Holographic $SO(2,d)$ anomaly

$$\mathcal{I}_{(d+2)} = \mathrm{d}\mathcal{L}^{\mathrm{cov}}_{(d+1)} \sim \epsilon_{\hat{\alpha}_0\cdots\hat{\alpha}_{d+1}} F^{\hat{\alpha}_0\hat{\alpha}_1} \wedge F^{\hat{\alpha}_2\hat{\alpha}_3} \wedge \mathrm{D}Y^{\hat{\alpha}_4} \wedge \cdots \wedge \mathrm{D}Y^{\hat{\alpha}_{d+1}} .$$

with the descendent formula

$$\begin{aligned} \mathcal{I}_{d+2} &= \mathrm{d}\mathcal{I}_{d+1} , \\ \delta_u \mathcal{I}_{d+1} &= \mathrm{d}\mathcal{I}_{d} , \\ &\vdots \end{aligned}$$

- Abstractly, the AdS gravity is realized as a gauge theory for the AdS fiber bundle E

$$\mathrm{AdS}_{d+1} = \frac{SO(2,d)}{SO(1,d)} \longrightarrow E \xrightarrow{\pi} \mathcal{M}_{d+1}$$

$$\{\mathbf{y}^{\hat{\alpha}}\} \to \{\mathbf{y}^{\hat{\alpha}}, x^{\mu}\} \xrightarrow{\pi} \{x^{\mu}\}$$

upon the manifold $\mathcal{M}_{d+1}$ with coordinates $\{x^{\mu}\}$.

- The ruler field is a section $\mathbf{y}^{\hat{\alpha}} = Y^{\hat{\alpha}}(x)$ of the AdS bundle.

Generalize to SUGRA?

- Can we generalize the previous story to the supersymmetry case?
- Is it possible to construct a manifestly supersymmetry covariant action for AdS SUGRA?
- For simplicity, let us consider the N=1 AdS_4 SUGRA which is invariant under the local supergroup $OSp(1|4;\mathrm{R})$.
- In this case, we expect to have a gauge theory for the super fiber bundle E

$$\mathrm{AdS}^{4|4} \longrightarrow \mathbb{E} \xrightarrow{\pi} \mathcal{M}_4$$

where the fiber space $\mathrm{AdS}^{4|4}$ is the rigid $\mathcal{N}=1$ AdS_4 superspace

$$\mathrm{AdS}^{4|4} = \frac{OSp(1|4;\mathrm{R})}{SO(1,3)}\,.$$

Spinor conventions

- The $SO(2,3)$ spinor space is same as $SO(1,3)$ spinor space. A spinor $\theta^{\hat{A}}$ has 4 components $\hat{A} = 1, \cdots, 4$.
- The $SO(2,3)$ gamma-matrices are related to $SO(1,3)$ gamma matrices as following

$$\hat{\Gamma}^{\hat{\alpha}} = (\hat{\Gamma}^{\hat{a}}, \hat{\Gamma}^{\hat{\bullet}}) = (i\Gamma\Gamma^{\hat{a}}, i\Gamma)$$
$$\text{where} \quad \Gamma = -i\Gamma^{\hat{0}}\Gamma^{\hat{1}}\Gamma^{\hat{2}}\Gamma^{\hat{3}}.$$

- The $SO(2,3)$ charge conjugation operator C is same as $SO(1,3)$ charge conjugation operator

$$C^T = -C\,, \qquad (C\hat{\Gamma}^{\hat{\alpha}})^T = -C\hat{\Gamma}^{\hat{\alpha}}\,, \qquad (C\hat{\Gamma}^{\hat{\alpha}_1\hat{\alpha}_2})^T = C\hat{\Gamma}^{\hat{\alpha}_1\hat{\alpha}_2}\,.$$

- For N=1, we work with the Majorana spinor

$$\overline{\Theta}_{\hat{A}} = C_{\hat{A}\hat{B}}\Theta^{\hat{B}}\,.$$

The N=1 AdS_4 superspace

- Recently, we found that the N=1 AdS_4 SUSY can be realized on the superspace $\{\mathbf{y}^{\hat{\alpha}}, \theta^{\hat{A}}, \mathbf{w}\}$ as the non-linear super coordinate transformation

$$\begin{pmatrix} \delta\mathbf{y}^{\hat{\alpha}} \\ \delta\theta^{\hat{A}} \\ \delta\mathbf{w} \end{pmatrix} = \begin{pmatrix} u^{\hat{\alpha}}{}_{\hat{\beta}}\mathbf{y}^{\hat{\beta}} + \mathrm{i}(2\ell+\mathbf{w})^{-1}\mathbf{y}_{\hat{\beta}}(\overline{\epsilon}\hat{\Gamma}^{\hat{\beta}\hat{\alpha}}\theta) \\ \frac{1}{4}(\not{u}\theta)^{\hat{A}} + (1+\ell^{-1}\mathbf{w})\epsilon^{\hat{A}} \\ \mathrm{i}(\overline{\epsilon}\theta) \end{pmatrix}$$

  where $\not{u} = u_{\hat{\alpha}_1\hat{\alpha}_2}\Gamma^{\hat{\alpha}_1\hat{\alpha}_2}$.
- Providing the constraints

$$(\ell+\mathbf{w})^2 - \mathrm{i}\ell(\overline{\theta}\theta) = \ell^2\,,$$

  one can verify that $[\delta_{\mathrm{I}}, \delta_{\mathrm{II}}]$ gives rise to the correct anti-commutation relation for SUSY algebra

$$\{\overline{Q}_A, \overline{Q}_B\} \sim (C\hat{\Gamma}^{\hat{\alpha}\hat{\beta}})_{AB}M_{\hat{\alpha}\hat{\beta}}$$

- The norm of $\mathbf{y}^{\hat{\alpha}}$ is fixed $\mathbf{y}^{\hat{\alpha}}\mathbf{y}_{\hat{\alpha}} = -\ell^2$ as the bosonic ruler.

The N=1 AdS_4 superspace

- By requiring that $\mathbf{w} = 0$ when $\theta = 0$, the constraint

$$(\ell + \mathbf{w})^2 - \mathrm{i}\ell(\bar{\theta}\theta) = \ell^2 \,,$$

  implies that

$$\mathbf{w} = \tfrac{\mathrm{i}}{2}(\bar{\theta}\theta) + \tfrac{1}{8\ell}(\bar{\theta}\theta)^2 \,.$$

- In 1970s, [B. W. Keck, J. Phys. A 8, 1819 (1975); B. Zumino, Nucl. Phys. B 127, 189(1977); E. A. Ivanov and A. S. Sorin, J. Phys. A 13 (1980) 1159] have found that

$$\delta\theta = [1 + \tfrac{1}{12}\lambda(5\bar{\theta}\theta - \bar{\theta}\kappa^a\theta \,.\, \kappa_a) - (\tfrac{1}{6}\lambda\bar{\theta}\theta)^2]\alpha$$

$$\delta x^a = \tfrac{1}{2}\lambda\bar{\alpha}\mathrm{i}\sigma^{ab}\theta \,.\, (1 - \tfrac{1}{12}\lambda\bar{\theta}\theta) \,.\, x_b$$

  by considering the action of the super group $OSp(1|4;R)$ on the coset space $OSp(1|4;R)/SO(1,3)$.
- We can recover their results under the identification

$$\theta = (1 - \tfrac{1}{3}\ell^{-1}\mathbf{w} + \tfrac{1}{6}\ell^{-2}\mathbf{w}^2)\theta \,, \quad x = \mathbf{y} \,.$$

SUSY irreps 1: super spinor rep

- A key observation is that θ itself forms a non-linear irrep of the AdS SUSY

$$\delta\theta^{\hat{A}} = \tfrac{1}{4}(\not{\psi}\theta)^{\hat{A}} + \left[1 + \tfrac{\mathrm{i}}{2\ell}(\overline{\theta}\theta) + \tfrac{1}{8\ell^2}(\overline{\theta}\theta)^2\right]\epsilon^{\hat{A}}$$

- Furthermore, in terms of the redundant super spinor notation

$$\mathbb{\theta}^{\hat{\mathbb{A}}} = \begin{pmatrix} \theta^{\hat{A}} \\ \hat{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} \theta^{\hat{A}} \\ \ell + \mathbf{w} \end{pmatrix} = \begin{pmatrix} \theta^{\hat{A}} \\ \ell + \frac{\mathrm{i}}{2}(\overline{\theta}\theta) + \frac{1}{8\ell}(\overline{\theta}\theta)^2 \end{pmatrix},$$

this irrep is translated to a 5-dimensional linear irrep

$$\delta\mathbb{\theta}^{\hat{\mathbb{A}}} = \begin{pmatrix} \frac{1}{4}(\not{\psi})^{\hat{A}}{}_{\hat{B}} & \ell^{-1}\epsilon^{\hat{A}} \\ \mathrm{i}\overline{\epsilon}_{\hat{B}} & 0 \end{pmatrix} \begin{pmatrix} \theta^{\hat{B}} \\ \hat{\mathbf{w}} \end{pmatrix} = \mathbb{u}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}}\mathbb{\theta}^{\hat{\mathbb{B}}}.$$

- This is in fact the fundamental rep of the super group $OSp(1|4;R)$.

SUSY irreps 1: super spinor rep

- A natural metric on the super spinor space is

$$\mathbb{C}_{\hat{\mathbb{A}}\hat{\mathbb{B}}} = \begin{pmatrix} -\mathrm{i}\ell C_{\hat{A}\hat{B}} & 0 \\ 0 & 1 \end{pmatrix},$$

  which is SUSY invariant under

$$\delta\mathbb{C}_{\hat{\mathbb{A}}\hat{\mathbb{B}}} = -(-1)^{\mathfrak{f}[\hat{\mathbb{B}}](\mathfrak{f}[\hat{\mathbb{A}}_1]+\mathfrak{f}[\hat{\mathbb{A}}])}\mathbb{C}_{\hat{\mathbb{A}}_1\hat{\mathbb{B}}}\mathfrak{u}^{\hat{\mathbb{A}}_1}{}_{\hat{\mathbb{A}}} - \mathbb{C}_{\hat{\mathbb{A}}\hat{\mathbb{B}}_1}\mathfrak{u}^{\hat{\mathbb{B}}_1}{}_{\hat{\mathbb{B}}} = 0\,.$$

- $OSp(1|4;R)$ can be defined by the super linear transformation which keeps $\mathbb{C}_{\hat{\mathbb{A}}\hat{\mathbb{B}}}$ intact.
- The super co-spinor is

$$\theta_{\hat{\mathbb{A}}} = \mathbb{C}_{\hat{\mathbb{A}}\hat{\mathbb{B}}}\theta^{\hat{\mathbb{B}}} = \begin{pmatrix} -\mathrm{i}\ell\overline{\theta}_{\hat{A}} & \hat{\mathbf{w}} \end{pmatrix},$$

  which transforms as

$$\delta\theta_{\hat{\mathbb{A}}} = -\theta_{\hat{\mathbb{B}}}\mathfrak{u}^{\hat{\mathbb{B}}}{}_{\hat{\mathbb{A}}}\,.$$

- Thus the inner product is SUSY invariant

$$\theta_{\hat{\mathbb{A}}}\theta^{\hat{\mathbb{A}}} = \mathbb{C}_{\hat{\mathbb{A}}\hat{\mathbb{B}}}\theta^{\hat{\mathbb{B}}}\theta^{\hat{\mathbb{A}}} = -\mathrm{i}\ell(\overline{\theta}\theta) + \hat{\mathbf{w}}^2 = \ell^2$$

SUSY irreps 1: super spinor rep

- The right inverse superspinor metric $\mathbb{C}^{\hat{\mathbb{A}}\hat{\mathbb{B}}}$ is defined by

$$\mathbb{C}_{\hat{\mathbb{A}}\hat{\mathbb{C}}}\mathbb{C}^{\hat{\mathbb{C}}\hat{\mathbb{B}}} = \mathbb{I}_{\hat{\mathbb{A}}}{}^{\hat{\mathbb{B}}}$$

  where

$$\mathbb{I}_{\hat{\mathbb{A}}}{}^{\hat{\mathbb{B}}} = \begin{pmatrix} -\delta_{\hat{A}}{}^{\hat{B}} & 0 \\ 0 & 1 \end{pmatrix} .$$

- The explicit formula is

$$\mathbb{C}^{\hat{\mathbb{A}}\hat{\mathbb{B}}} = \begin{pmatrix} -\mathrm{i}\ell^{-1}(\hat{C}^{-1})^{\hat{A}\hat{B}} & 0 \\ 0 & 1 \end{pmatrix}$$

Super bi-spinor rep: $5 \otimes 5 = 1 \oplus 10 \oplus 14$

- The tensor product of 2 super spinor rep is the reducible bi-spinor rep $\Phi^{\hat{\mathbb{A}}\hat{\mathbb{B}}}$.
- It can be decomposed to
  - 1-dimensional trace part $\varphi = \mathbb{C}_{\hat{\mathbb{B}}\hat{\mathbb{A}}}\Phi^{\hat{\mathbb{A}}\hat{\mathbb{B}}}$;
  - The $[(4+1)\times(4+1)]_{\square\square} = (1\times 1) + (4\times 1) + \frac{4\times(4-1)}{2} - 1 = 10$ dimensional symmetric traceless part

$$\begin{aligned}\Phi^{\hat{\mathbb{A}}\hat{\mathbb{B}}}_{\square\square} &= \frac{1}{2}\left[\Phi^{\hat{\mathbb{A}}\hat{\mathbb{B}}} + (-1)^{\mathfrak{f}[\hat{\mathbb{A}}]\mathfrak{f}[\hat{\mathbb{B}}]}\Phi^{\hat{\mathbb{B}}\hat{\mathbb{A}}}\right] + \frac{1}{3}\mathbb{C}_{\hat{\mathbb{B}}_1\hat{\mathbb{A}}_1}\Phi^{\hat{\mathbb{A}}_1\hat{\mathbb{B}}_1}\mathbb{C}^{\hat{\mathbb{A}}\hat{\mathbb{B}}} \\ &= (-1)^{\mathfrak{f}[\hat{\mathbb{A}}]\mathfrak{f}[\hat{\mathbb{B}}]}\Phi^{\hat{\mathbb{B}}\hat{\mathbb{A}}}_{\square\square}\end{aligned}$$

  - The $[(4+1)\times(4+1)]_{\square} = (4\times 1) + \frac{4\times(4+1)}{2} = 14$ dimensional antisymmetric part

$$\Phi^{\hat{\mathbb{A}}\hat{\mathbb{B}}}_{\square} = \frac{1}{2}\left[\Phi^{\hat{\mathbb{A}}\hat{\mathbb{B}}} - (-1)^{\mathfrak{f}[\hat{\mathbb{A}}]\mathfrak{f}[\hat{\mathbb{B}}]}\Phi^{\hat{\mathbb{B}}\hat{\mathbb{A}}}\right] = -(-1)^{\mathfrak{f}[\hat{\mathbb{A}}]\mathfrak{f}[\hat{\mathbb{B}}]}\Phi^{\hat{\mathbb{B}}\hat{\mathbb{A}}}_{\square}\,.$$

SUSY irreps 2: Adjoint rep

- The 14 dimensional antisymmetric super bi-spinor rep is

$$\Phi^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} = \begin{pmatrix} \frac{1}{4} s^{\hat{\alpha}\hat{\beta}} (\hat{\Gamma}_{\hat{\alpha}\hat{\beta}})^{\hat{A}}{}_{\hat{B}} & \ell^{-1} \kappa^{\hat{A}} \\ \mathrm{i} \overline{\kappa}_{\hat{B}} & 0 \end{pmatrix}$$

- It contains an $SO(2,3)$ antisymmetric 2-tensor $s^{\hat{\alpha}\hat{\beta}}$ and a Majorana spinor $\kappa^{\hat{A}}$. This is in fact the $\frac{5(5-1)}{2} + 4 = 14$ dimensional adjoint rep

$$\begin{pmatrix} \delta s^{\hat{\alpha}\hat{\beta}} \\ \delta \kappa^{\hat{A}} \end{pmatrix} = \begin{pmatrix} -\frac{\mathrm{i}}{\ell} (\overline{\epsilon} \hat{\gamma}^{\hat{\alpha}\hat{\beta}} \kappa) \\ -\frac{1}{4} (\not{s} \epsilon)^{\hat{A}} \end{pmatrix} .$$

SUSY irreps 3: Super vector rep

- The 10-dimensional symmetric traceless representation is given by

$$(\Phi_{\square\!\square})^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} = \begin{pmatrix} \frac{1}{4}\phi\delta^{\hat{A}}{}_{\hat{B}} + \frac{1}{4}v^{\hat{\alpha}}(\hat{\Gamma}_{\hat{\alpha}})^{\hat{A}}{}_{\hat{B}} & \ell^{-1}\overline{\zeta}^{\hat{A}} \\ -\mathrm{i}\zeta_{\hat{B}} & \phi \end{pmatrix}$$

  In terms of $SO(2,3)$ reps, it contains $5_v + 4_s + 1 = 10$ components.

- We shall name it as the supervector representation $\mathbb{v}^{\hat{\mathbb{\alpha}}} = \{v^{\hat{\alpha}}, \zeta^{\hat{A}}, \phi\}$. It transforms as

$$\delta\mathbb{v}^{\hat{\mathbb{\alpha}}} = \begin{pmatrix} \delta v^{\hat{\alpha}} \\ \delta\zeta^{\hat{A}} \\ \delta\phi \end{pmatrix} = \begin{pmatrix} u^{\hat{\alpha}}{}_{\hat{\beta}} & 2\mathrm{i}\ell^{-1}(\overline{\epsilon}\hat{\Gamma}^{\hat{\alpha}})_{\hat{B}} & 0 \\ -\frac{1}{4}(\hat{\Gamma}_{\hat{\beta}}\epsilon)^{\hat{A}} & \frac{1}{4}(\psi)^{\hat{A}}{}_{\hat{B}} & \frac{3}{4}\epsilon^{\hat{A}} \\ 0 & 2\mathrm{i}\ell^{-1}(\overline{\epsilon})_{\hat{B}} & 0 \end{pmatrix} \begin{pmatrix} v^{\hat{\beta}} \\ \zeta^{\hat{B}} \\ \phi \end{pmatrix} = \mathbb{u}^{\hat{\mathbb{\alpha}}}{}_{\hat{\mathbb{\beta}}}\mathbb{v}^{\hat{\mathbb{\beta}}}.$$

SUSY irreps 3: Super vector rep

- In the supervector space, the symmetric super 2-tensor

$$\mathbb{n}_{\hat{\mathfrak{a}}\hat{\mathfrak{b}}} = \begin{pmatrix} \hat{\eta}_{\hat{\alpha}\hat{\beta}} & 0 & 0 \\ 0 & 8\mathrm{i}\ell^{-1}\hat{C}_{\hat{A}\hat{B}} & 0 \\ 0 & 0 & -3 \end{pmatrix} = (-1)^{\mathfrak{f}[\hat{\mathfrak{a}}]\mathfrak{f}[\hat{\mathfrak{b}}]}\mathbb{n}_{\hat{\mathfrak{b}}\hat{\mathfrak{a}}}$$

  is an invariant super tensor, satisfying

$$\delta\mathbb{n}_{\hat{\mathfrak{a}}\hat{\mathfrak{b}}} = -(-1)^{\mathfrak{f}[\hat{\mathfrak{b}}](\mathfrak{f}[\hat{\mathfrak{a}}_1]+\mathfrak{f}[\hat{\mathfrak{a}}])}\mathbb{n}_{\hat{\mathfrak{a}}_1\hat{\mathfrak{b}}}\mathbb{u}^{\hat{\mathfrak{a}}_1}{}_{\hat{\mathfrak{a}}} - \mathbb{n}_{\hat{\mathfrak{a}}\hat{\mathfrak{b}}_1}\mathbb{u}^{\hat{\mathfrak{b}}_1}{}_{\hat{\mathfrak{b}}} = 0\,.$$

Super ruler

- Initially, we have found that the bosonic coordinate $\mathbf{y}^{\hat{\alpha}}$ transforms as

$$\delta \mathbf{y}^{\hat{\alpha}} = \iota^{\hat{\alpha}}{}_{\hat{\beta}} \mathbf{y}^{\hat{\beta}} + \mathrm{i}(2\ell + \mathbf{w})^{-1} \mathbf{y}_{\hat{\beta}} (\overline{\boldsymbol{\epsilon}} \hat{\Gamma}^{\hat{\beta}\hat{\alpha}} \theta)$$

- Can we embed it into a linear rep to formulate a SUSY generalization of the bosonic ruler?
- We find that under the fermionic transformation

$$\begin{aligned}
\delta[(2\ell + \mathbf{w})^{-1} \mathbf{y}^{\hat{\alpha}}] &= -\mathrm{i}(2\ell + \mathbf{w})^{-2} (\overline{\boldsymbol{\epsilon}} \hat{\Gamma}^{\hat{\alpha}} \not{\mathbf{y}} \theta)\,, \\
\delta[(2\ell + \mathbf{w})^{-2} (\not{\mathbf{y}} \theta)^{\hat{A}}] &= \frac{3\mathrm{i}}{16\ell^3} (\overline{\theta} \not{\mathbf{y}} \theta) \boldsymbol{\epsilon}^{\hat{A}} + \frac{1}{2} \ell^{-1} (2\ell + \mathbf{w})^{-1} (\not{\mathbf{y}} \boldsymbol{\epsilon})^{\hat{A}}\,, \\
\delta(\overline{\theta} \not{\mathbf{y}} \theta) &= 8\ell^2 (2\ell + \mathbf{w})^{-2} (\overline{\boldsymbol{\epsilon}} \not{\mathbf{y}} \theta)\,,
\end{aligned}$$

where $\not{\mathbf{y}} = \mathbf{y}^{\hat{\alpha}} \Gamma_{\hat{\alpha}}$.

Super ruler

- We recognize that the super ruler

$$\mathbb{y}^{\hat{\alpha}} = \begin{pmatrix} (1+\frac{1}{2\ell}\mathbb{w})^{-1}\mathbb{y}^{\hat{\alpha}} \\ -\ell^2(2\ell+\mathbb{w})^{-2}(\not{\mathbb{y}}\theta)^{\hat{A}} \\ -\frac{\mathrm{i}}{4\ell}(\bar{\theta}\not{\mathbb{y}}\theta) \end{pmatrix} = \begin{pmatrix} [1-\frac{\mathrm{i}}{4\ell}(\bar{\theta}\theta)-\frac{1}{8\ell^2}(\bar{\theta}\theta)^2]\mathbb{y}^{\hat{\alpha}} \\ -\frac{1}{4}[1-\frac{\mathrm{i}}{2\ell}(\bar{\theta}\theta)](\not{\mathbb{y}}\theta)^{\hat{A}} \\ -\frac{\mathrm{i}}{4\ell}(\bar{\theta}\not{\mathbb{y}}\theta) \end{pmatrix}$$

  forms a super vector rep.

- In terms of bi-spinor notation,

$$\mathbb{y}^{\hat{A}}{}_{\hat{B}} = \begin{pmatrix} -\frac{\mathrm{i}}{4\ell}(\bar{\theta}\not{\mathbb{y}}\theta)\delta^{\hat{A}}{}_{\hat{B}} + \left[1-\frac{\mathrm{i}}{4\ell}(\bar{\theta}\theta)-\frac{1}{8\ell^2}(\bar{\theta}\theta)^2\right]\mathbb{y}^{\hat{A}}{}_{\hat{B}} & -\ell^{-1}\left[1-\frac{\mathrm{i}}{2\ell}(\bar{\theta}\theta)\right](\not{\mathbb{y}}\theta)^{\hat{A}} \\ \mathrm{i}\left[1-\frac{\mathrm{i}}{2\ell}(\bar{\theta}\theta)\right](\bar{\theta}\mathbb{y})_{\hat{B}} & -\mathrm{i}\ell^{-1}(\bar{\theta}\not{\mathbb{y}}\theta) \end{pmatrix}$$

- It is noticed that

$$\mathbb{y}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}}\theta^{\hat{\mathbb{B}}} = 0\,, \qquad \mathbb{y}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{C}}}\mathbb{y}^{\hat{\mathbb{C}}}{}_{\hat{\mathbb{B}}} = \theta^{\hat{\mathbb{A}}}\bar{\theta}_{\hat{\mathbb{B}}} - \ell^2\delta^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}}\,.$$

- The supertrace $\mathrm{Str}(\mathbb{y}\mathbb{y}) = \bar{\theta}_{\hat{\mathbb{A}}}\theta^{\hat{\mathbb{A}}} - \ell^2\mathbb{I}_{\hat{\mathbb{A}}}{}^{\hat{\mathbb{A}}} = 4\ell^2$, is equivalent to $\eta_{\hat{\alpha}\hat{\beta}}\mathbb{y}^{\hat{\beta}}\mathbb{y}^{\hat{\alpha}} = -16\ell^2$

$OSp(1|4)$ *connection and curvature*

- Now we can consider a fiber bundle for which each fiber is a super rep space, e.g. the super spinor bundle on which the local section is $\mathbb{\Theta} = \mathbb{\Theta}(x)$.
- The SUGRA is just the gauge theory for the x-dependent local AdS SUSY transformation

$$\mathbb{u}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}}(x) = \begin{pmatrix} \frac{1}{4}(\psi(x))^{\hat{A}}{}_{\hat{B}} & \ell^{-1}\epsilon^{\hat{A}}(x) \\ \mathrm{i}\overline{\epsilon}_{\hat{B}}(x) & 0 \end{pmatrix}.$$

- To compensate the x-derivative terms on $\mathbb{u}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}}(x)$, we introduce the 1-form SUSY gauge field on $\mathcal{M}_4$

$$\mathbb{A}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} = \begin{pmatrix} \frac{1}{4}(\not{A})^{\hat{A}}{}_{\hat{B}} & \ell^{-1}\psi^{\hat{A}} \\ \mathrm{i}\overline{\psi}_{\hat{B}} & 0 \end{pmatrix}$$

which transforms as

$$\delta\mathbb{A}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} = -\mathrm{d}\mathbb{u}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} - \mathbb{A}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{C}}}\mathbb{u}^{\hat{\mathbb{C}}}{}_{\hat{\mathbb{B}}} + \mathbb{u}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{C}}}\mathbb{A}^{\hat{\mathbb{C}}}{}_{\hat{\mathbb{B}}}$$

$OSp(1|4)$ connection and curvature

- The 2-form SUSY gauge field strength

$$\mathbb{F}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} = \mathrm{d}\mathbb{A}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} + \mathbb{A}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{C}}} \wedge \mathbb{A}^{\hat{\mathbb{C}}}{}_{\hat{\mathbb{B}}} = \begin{pmatrix} \frac{1}{4}(\mathcal{F}^{\hat{\alpha}_1\hat{\alpha}_2})^{\hat{A}}{}_{\hat{B}} & \ell^{-1}\mathrm{D}\psi^{\hat{A}} \\ \mathrm{i}\mathrm{D}\overline{\psi}_{\hat{B}} & 0 \end{pmatrix}$$

$$\mathcal{F}^{\hat{\alpha}_1\hat{\alpha}_2} = F^{\hat{\alpha}_1\hat{\alpha}_2} - \frac{\mathrm{i}}{2}\ell^{-1}(\overline{\psi} \wedge \hat{\gamma}^{\hat{\alpha}_1\hat{\alpha}_2}\psi)$$

  forms the adjoint rep

$$\delta\mathbb{F}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} = \mathbb{u}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{C}}}\mathbb{F}^{\hat{\mathbb{C}}}{}_{\hat{\mathbb{B}}} - \mathbb{F}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{C}}}\mathbb{u}^{\hat{\mathbb{C}}}{}_{\hat{\mathbb{B}}}\,.$$

- The super ruler field is a section of the super ruler bundle

$$\mathbb{y}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} = \mathbb{Y}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}}(x) \quad \Leftrightarrow \quad \{\mathbb{y}^{\hat{\alpha}} = Y^{\hat{\alpha}}(x)\,, \quad \theta^{\hat{A}} = \Theta^{\hat{A}}(x)\}$$

  The local SUSY gauge transformation can always take it to the super Einstein gauge $Y^a = 0\,, Y^\bullet = \ell, \Theta^{\hat{A}} = 0$.
- Acting the super covariant derivative on it, we obtain

$$\mathbb{D}\mathbb{Y}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} = \mathrm{d}\mathbb{Y}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} + \mathbb{A}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{C}}}\mathbb{Y}^{\hat{\mathbb{C}}}{}_{\hat{\mathbb{B}}} - \mathbb{Y}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{C}}}\mathbb{A}^{\hat{\mathbb{C}}}{}_{\hat{\mathbb{B}}}\,.$$

  which is also in the super vector rep.

$N{=}1$ AdS_4 SUGRA

- The SUGRA action can be constructed in the similar way as the pure bosonic case

$$S = \int_{\mathcal{M}_4} I_4 = \mathrm{STr}(\mathscr{F} \wedge \mathscr{F}\, \mathbb{Y}) = \mathrm{STr}(\mathbb{F} \wedge \mathbb{F}\, \mathbb{Y}) + \frac{\lambda(\lambda-2)}{\ell^2}(\overline{\Theta}\mathbb{F} \wedge \mathbb{Y}\mathbb{F}\Theta)$$

where

$$\mathscr{F}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} = \mathbb{F}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{B}}} - \frac{\lambda}{\ell^2}[\mathbb{F}^{\hat{\mathbb{A}}}{}_{\hat{\mathbb{C}}}\Theta^{\hat{\mathbb{C}}}\overline{\Theta}_{\hat{\mathbb{B}}} + \Theta^{\hat{\mathbb{A}}}\overline{\Theta}_{\hat{\mathbb{C}}}\mathbb{F}^{\hat{\mathbb{C}}}{}_{\hat{\mathbb{B}}}]$$

It is manifestly local SUSY invariant by construction.

- Expanding I_4 by the powers of $\Theta(x)$, we get

$$\begin{aligned} I_4 \;=\; & \frac{1}{4}\epsilon_{\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3\hat{\alpha}_4\hat{\alpha}_5}\mathcal{F}^{\hat{\alpha}_1\hat{\alpha}_2} \wedge \mathcal{F}^{\hat{\alpha}_3\hat{\alpha}_4}\, Y^{\hat{\alpha}_5} + \frac{\mathrm{i}(1-\lambda)^2}{\ell}(\mathrm{D}\overline{\psi}\mathcal{Y} \wedge \mathrm{D}\psi) \\ & + \frac{\mathrm{i}\lambda(2-\lambda)}{2\ell}(\overline{\Theta}\mathcal{F}\mathcal{Y} \wedge \mathrm{D}\psi) + \frac{\mathrm{i}}{2\ell}(\overline{\Theta}\mathcal{Y}\mathcal{F} \wedge \mathrm{D}\psi) + O(\Theta^2)\,. \end{aligned}$$

$N{=}1$ AdS_4 SUGRA

- At the point of super Einstein gauge

$$\{Y^{\hat{\bullet}}(x) = \ell, \quad Y^{\hat{a}}(x) = 0\,, \quad \Theta^{\hat{A}}(x) = 0\}\,,$$

the Lagrangian density I_4 becomes

$$\begin{aligned} I_4^{(0)} &= \frac{1}{4}\epsilon_{\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3\hat{\alpha}_4\hat{\alpha}_5}\mathcal{F}^{\hat{\alpha}_1\hat{\alpha}_2}\wedge\mathcal{F}^{\hat{\alpha}_3\hat{\alpha}_4}Y^{\hat{\alpha}_5} + \frac{\mathrm{i}(1-\lambda)^2}{\ell}(\mathrm{D}\overline{\psi}\Gamma\wedge\mathrm{D}\psi) \\ &= \frac{\ell}{4}\epsilon_{\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4}\mathcal{F}^{\hat{a}_1\hat{a}_2}\wedge\mathcal{F}^{\hat{a}_3\hat{a}_4} + \mathrm{i}(1-\lambda)^2(\mathrm{D}\overline{\psi}\boldsymbol{\gamma}\wedge\mathrm{D}\psi)\,, \end{aligned}$$

- At this point, the infinitesimal fermionic translation is reduced to

$$\begin{aligned} &\delta Y^{\hat{\alpha}} = 0\,, && \delta\Theta^{\hat{A}} = \boldsymbol{\epsilon}^{\hat{A}}\,, \\ &\delta e^{\hat{a}} = -\mathrm{i}(\overline{\boldsymbol{\epsilon}}\hat{\Gamma}^{\hat{a}}{}_{\hat{\bullet}}\psi)\,, && \delta\omega^{\hat{a}}{}_{\hat{b}} = -\frac{\mathrm{i}}{\ell}(\overline{\boldsymbol{\epsilon}}\hat{\Gamma}^{\hat{a}}{}_{\hat{b}}\psi)\,, && \delta\psi^{\hat{A}} = -\mathrm{D}\boldsymbol{\epsilon}^{\hat{A}}\,. \end{aligned}$$

N=1 AdS_4 SUGRA

- Under the reduced fermionic translation, the transformation of $I_4^{(0)}$ is totally fixed by the original supersymmetry invariance of I_4 . The result is simply the $I_4^{(1)}$ term

$$-\delta I_4^{(0)} = \frac{\mathrm{i}\lambda(2-\lambda)}{2\ell}(\overline{\epsilon}\not{\mathcal{F}}\not{Y}\wedge \mathrm{D}\psi) + \frac{\mathrm{i}}{2\ell}(\overline{\epsilon}\not{Y}\not{\mathcal{F}}\wedge \mathrm{D}\psi)\,,$$

- Especially, when $\lambda = 1 \pm \sqrt{2}$,

$$\delta S_4^{(0)} \;=\; \frac{\mathrm{i}}{\ell}\int_{\mathcal{M}_4} \mathcal{F}^{\hat{\alpha}}{}_{\hat{\beta}} Y^{\hat{\beta}} \wedge (\overline{\epsilon}\hat{\Gamma}_{\hat{\alpha}}\mathrm{D}\psi)\,.$$

  Thus the reduced fermionic translation of $I_4^{(0)}$ shall vanish by satisfying the super torsion free condition

$$\mathcal{F}^{\hat{\alpha}}{}_{\hat{\beta}} Y^{\hat{\beta}} = \ell \mathcal{F}^{\hat{a}}{}_{\hat{\bullet}} = 0\,.$$

N=1 AdS_4 SUGRA

- On the other hand, the super torsion free condition is equivalent to the equation of motion derived by varying $\omega^{\hat{a}}{}_{\hat{b}}$ in $S_4^{(0)}$

$$\frac{\delta S_4^{(0)}}{\delta \omega^{\hat{a}}{}_{\hat{b}}} = 0 \qquad \Leftrightarrow \qquad \mathcal{F}^{\hat{a}}{}_{\hat{\bullet}} = 0\,.$$

- Therefore, in the 2nd order formalism where the spin connection $\omega^{\hat{a}}{}_{\hat{b}}$ is integrated out, the action $S_4^{(0)}$ is invariant under the transformation

$$\delta e^{\hat{a}} = -\mathrm{i}(\overline{\epsilon}\hat{\Gamma}^{\hat{a}}{}_{\hat{\bullet}}\psi)\,, \qquad \delta\psi^{\hat{A}} = -\mathrm{D}\epsilon^{\hat{A}}\,.$$

- This exactly reproduces the 1.5-order argument for the supersymmetry invariance of N=1 AdS_4 supergravity.

N=1 AdS_4 SUGRA

- In fact, when $\lambda = 1 \pm \sqrt{2}$, the action at the Super Einstein gauge

$$S_4^{(0)} = \int_{\mathcal{M}_4} \left[\frac{1}{4} \epsilon_{\hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3 \hat{\alpha}_4 \hat{\alpha}_5} \mathcal{F}^{\hat{\alpha}_1 \hat{\alpha}_2} \wedge \mathcal{F}^{\hat{\alpha}_3 \hat{\alpha}_4} \Gamma^{\hat{\alpha}_5} + \frac{2\mathrm{i}}{\ell} (\mathrm{D}\overline{\psi} \Gamma \wedge \mathrm{D}\psi) \right]$$
$$= \int_{\mathcal{M}_4} \left[\frac{\ell}{4} \epsilon_{\hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4} \mathcal{F}^{\hat{a}_1 \hat{a}_2} \wedge \mathcal{F}^{\hat{a}_3 \hat{a}_4} + 2\mathrm{i} (\mathrm{D}\overline{\psi} \Gamma \wedge \mathrm{D}\psi) \right]$$

reproduces the MacDowell-Mansouri-Stelle-West formula for N=1 AdS_4 SUGRA [see e.g. P. v. Nieuwenhuizen hep-th/0408137]

$$\mathcal{L} = R_{\mu\nu}{}^{mn} R_{\rho\sigma}{}^{pq} \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnpq} + a(\bar{R}_{\mu\nu})_\alpha (\gamma_5)^\alpha{}_\beta (R_{\rho\sigma})^\beta \epsilon^{\mu\nu\rho\sigma}$$

with the correct coefficient $a = 8\ell^{-1}$.

Summary

- In this talk, we have discussed
  - Non-linear realization of AdS_4 SUSY in superspace
  - SUSY linear irreps
  - SUSY ruler in the super vector rep
  - SUSY gauge covariant description of AdS_4 N=1 SUGRA
- What next?
  - Super holographic anomaly
  - N>1 , $D \neq 4$
  - Is it possible to construct a $OSp(1;32|\mathbb{R})$ gauge theory as AdS_{11} SUGRA?

Thank You!