Quantum-Corrected Wilson Loop Correlators and Super-Yang-Mills Theory at Infrared

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Some Formal Aspects of Field theories and Holography
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Based on arXiv:2412.11107 with
Xiao-Long Liu, Cong-Yuan Yue, Wenni Zheng

Quark Confinement

- Why we never observe a free quark directly?

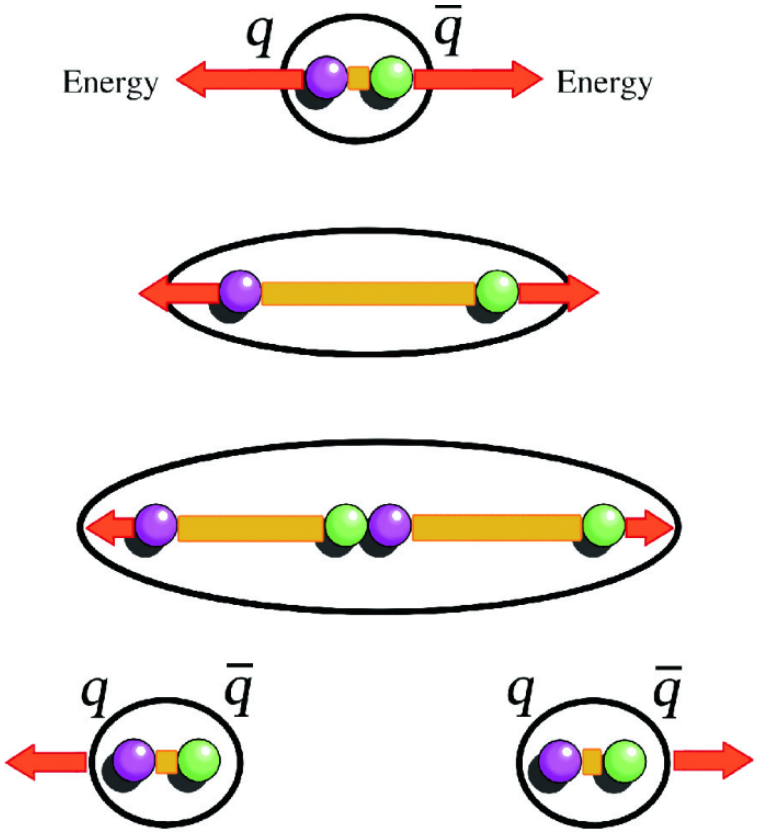
Answer: quark (color) confinement

Related problem: Yang-Mills mass gap problem

(one of the Millennium Problems in mathematics)

Qualitative Explanation

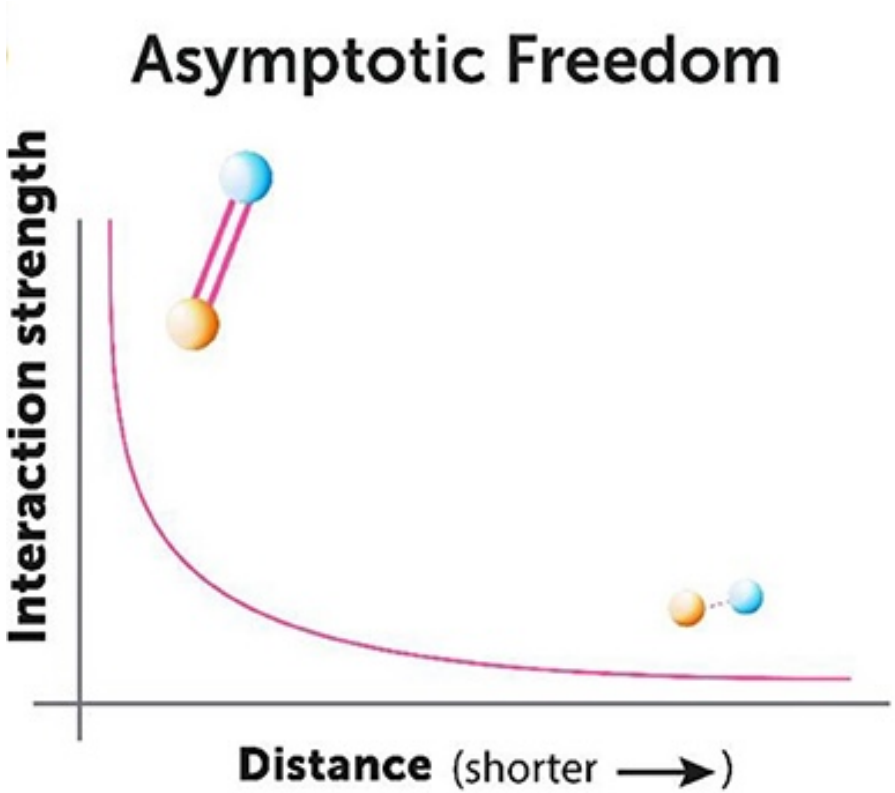
- How do we qualitatively explain confinement?



String theory was originally invented to solve this problem!

Asymptotic Freedom

- Asymptotic freedom:

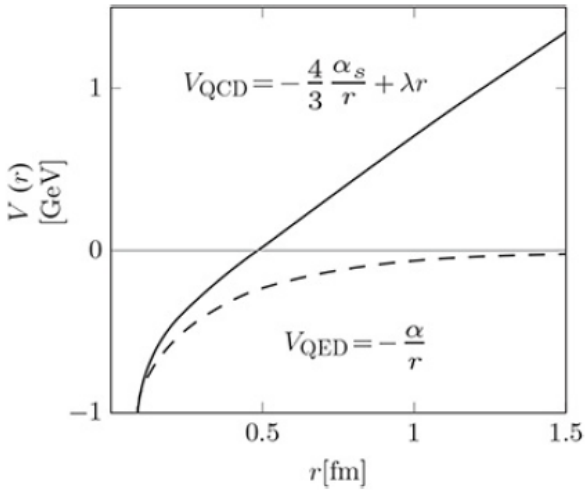


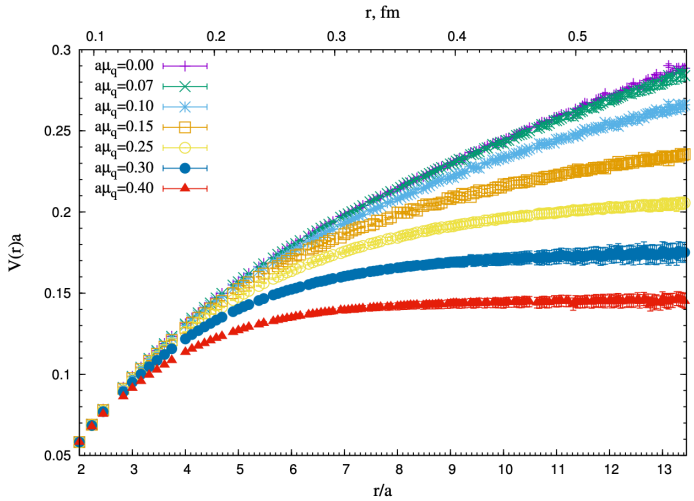
2004 Nobel Prize in Physics:

David J. Gross, H. David Politzer, Frank Wilczek

Quantitative Expectation

- How do we quantitatively describe confinement?
  The potential between a quark and an anti-quark pair:





- Can we theoretically compute this potential?
  Wilson loop $\langle W \rangle$ + quantum-corrected AdS/CFT

Wilson Loop

- Given a closed loop $\mathcal{C}$ in spacetime.
- A Wilson loop in Yang-Mills theory is defined as

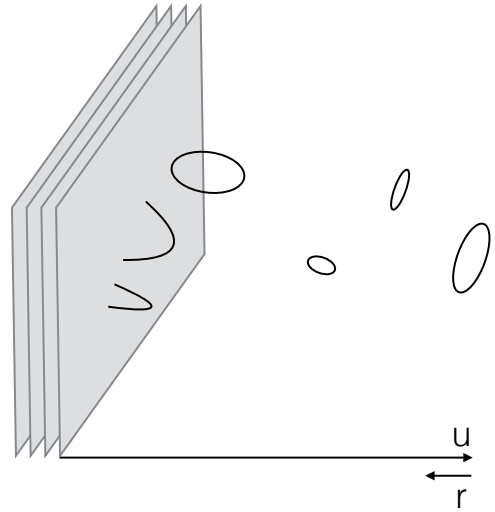
$$W(\mathcal{C}) \equiv \frac{1}{N} \operatorname{Tr} \left(P\, e^{i \int_{\mathcal{C}} dx^{\mu} A_{\mu}} \right) .$$

- $W(\mathcal{C})$ is a gauge-invariant quantity, and captures many important features of Yang-Mills theory.
- All the QCD observables can be reformulated in terms of $\langle W \rangle$, $\langle WW \rangle$, $\cdots$

AdS/CFT Correspondence

- Maldacena (1997):

$$\text{4D } U(N)\ \mathcal{N}=4 \text{ SYM} \quad \Leftrightarrow \quad \text{IIB string theory on } AdS_5 \times S^5$$



- Gubser, Klebanov, Polyakov (1998); Witten (1998):

$$Z_{CFT} = Z_{AdS}\,, \quad Z_{AdS} \approx Z_{SUGRA} \text{ when } \lambda \gg 1 \text{ and } N \gg 1$$

$$N^2 = \frac{\pi}{2}\frac{L^3}{G_5}\,, \quad \lambda \equiv g_{YM}^2 N = \left(\frac{L}{\ell_s}\right)^4$$

- Rey, Yee (1998); Maldacena (1998):
  The AdS/CFT cannot produce the linear term in $V_{q\bar{q}}$ at $T = 0$.

Outline

1. Introduction:
2. AdS_2 JT gravity and Schwarzian theory
3. Embed AdS_2 into higher-dimensional AdS black holes
4. Quantum gravity corrected and SYM confinement
5. Summary and Prospect

Quantum Gravity

In a quantum gravity (QG) theory, we should be able to evaluate:

- The partition function:

$$Z = \int [Dg_{\mu\nu}]\,[D\phi]\, e^{-S_{grav}[g_{\mu\nu}]-S_{matter}[\phi,\, g_{\mu\nu}]}$$

- The correlation functions, e.g., for $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$

$$\langle h_{\lambda\xi} h_{\rho\sigma} \rangle = \int [Dh_{\mu\nu}]\, h_{\lambda\xi} h_{\rho\sigma}\, e^{-S_{grav}[\bar{g}_{\mu\nu}+h_{\mu\nu}]}$$

$$\text{where} \qquad S_{grav} = \int d^D x \sqrt{g}\,(R + \Lambda)$$

  We also want to compute the matter correlators $\langle \phi\phi \rangle$.

- There are still many open issues (e.g., renormalization and black hole singularity).
- The AdS/CFT correspondence (originally motivated by the **strongly-coupled Yang-Mills theory in the infrared**) could shed some light on it.

JT Gravity to Schwarzian

- Start from the JT gravity:

  ('14 Almhieiri, Polchinski; '16 Maldacena, Stanford, Yang)

$$I_{JT} = -\frac{1}{16\pi G}\left[\int d^2x\,\sqrt{g}\,\phi\,(R+2) + 2\int_{bdy}\phi_b\,K\right]$$

- Plugging the bulk equations of motion back into the action:

$$I = -\frac{1}{8\pi G}\int du\,\phi_r(u)\,Sch(t,u)\,, \quad Sch(t,u) \equiv -\frac{1}{2}\frac{t''^2}{t'^2} + \left(\frac{t''}{t'}\right)'\,.$$

  where t is reparametrization zero modes of boundary time u:

$$t(u) = u + \varepsilon(u)\,,$$

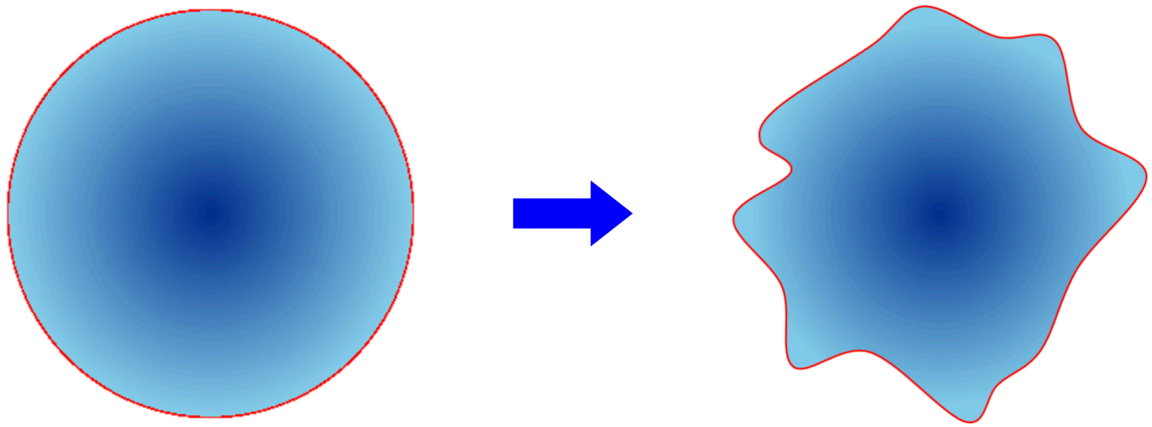
  and the boundary value of dilaton is

$$\phi_b = \phi|_{bdy} = \frac{\phi_r(u)}{\epsilon}\,.$$

Bulk vs Boundary in Schwarzian Theory

- ('14 Almhieiri, Polchinski; '16 Maldacena, Stanford, Yang)

  The only physical degree of freedom is the boundary's reparametrization $t(u)$.



Schwarzian as Toy Model QG

- We assume $\phi_r(u)$ to be a constant denoted by $\bar{\phi}_r$.
- Defining $C \equiv \bar{\phi}_r/(8\pi G)$, we obtain the Schwarzian action:

$$I = -C \int du\, Sch(t,u) = -C \int du \left[Sch(\tau,u) + \frac{1}{2}\tau'^2 \right],$$

where $t = \tan(\tau/2)$. Recall that $t = u + \varepsilon(u)$.

- The 2-point correlation functions can be computed:

  ('16 Maldacena, Stanford, Yang; '19 Qi, Sin, Joon)

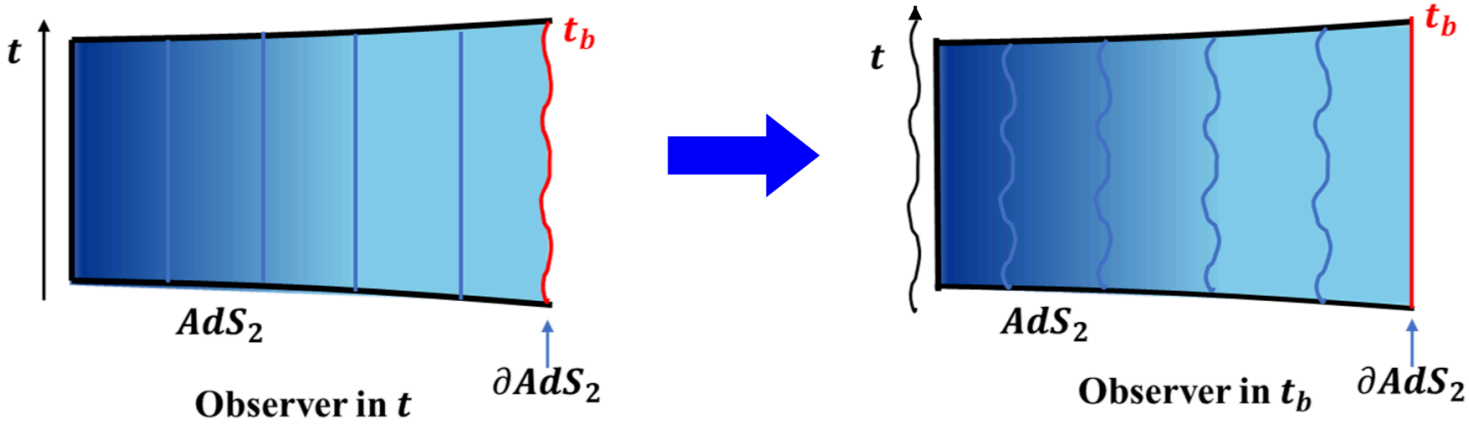
$$\langle \varepsilon(u)\, \varepsilon(0) \rangle = \frac{1}{2\pi C} \left[-\frac{(|u|-\pi)^2}{2} + (|u|-\pi)\sin|u| + a + b\cos u \right]$$

  which plays the role of $\langle h_{\lambda\xi} h_{\rho\sigma} \rangle$ in this toy model QG.

- For a matter field χ on the fluctuating AdS_2 spacetime background, its 2-point function $\langle \chi(u)\, \chi(0) \rangle$ can also be computed, which plays the role of $\langle \phi\phi \rangle$.

Embedding of AdS_2 into $AdS_{D\geq 4}$

- ('19 Blommaert, Mertens, Verschelde; '20 Iliesiu, Turiaci; '24 Liu, Yue, **JN**, Zheng)



- By requiring continuity and smoothness conditions, we can embed AdS_2 with quantum fluctuations into $AdS_{D\geq 4}$.

Thermal Fluctuations vs Quantum Fluctuations

- For near-extremal black holes with AdS_2 near-horizon regions, we can introduce quantum gravity fluctuations of AdS_2 and embed them into higher-D spacetime.
- This provides a toy model of quantum gravity:

$$\int [Dg_{\mu\nu}]\, e^{-S[g_{\mu\nu}]} \quad \Rightarrow \quad \int D\varepsilon\, e^{-S_{Sch}[\varepsilon]}$$

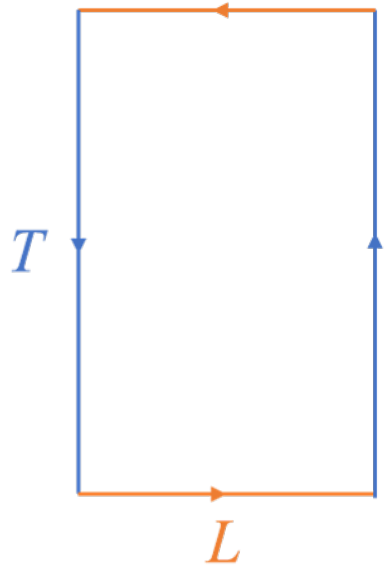
- More physically speaking, a thermodynamic system (e.g., a black hole) always has both thermal fluctuations and quantum fluctuations.
- At extremely low temperature ($T < C^{-1}$), quantum fluctuations cannot be neglected.

Applications

- Quantum corrected holographic strange metal:
  (2410.11487 Liu, **JN**, Pando Zayas)
- Quantum corrected Wilson loop and SYM confinement:
  (2412.11107 Liu, **JN**, Yue, Zheng)
- Quantum corrected Wilson loop and SYM mass gap:
  ('25 Liu, **JN**, Yue, Zheng, Huang)
- Quantum corrected fluid/gravity correspondence:
  ('25 **JN**, Pando Zayas, Yue)
- Quantum corrected quasi-normal modes:
  ('25 Jiang, **JN**, Tian)
- Quantum corrected the ratio of shear viscosity and entropy:
  ('25 Liu, **JN**, Li)

$\langle W \rangle$ and Quark Anti-Quark Potential

- Consider a static heavy quark-antiquark pair with distance L. They form a rectangular temporal Wilson loop:



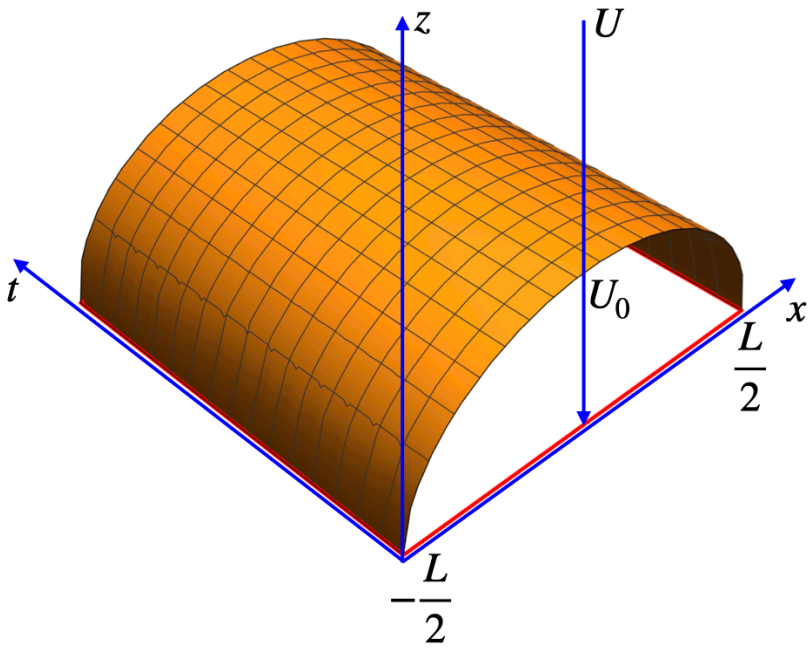
- It is known long ago that a temporal Wilson loop obeys

$$\langle W \rangle \simeq e^{-T \cdot V_{q\bar{q}}(L)} .$$

An area law of $\log\langle W \rangle$ implies a linear potential, i.e., confinement.

Holographic $\langle W \rangle$

- ('98 Rey, Yee; '98 Maldacena)



$S_{NG} \propto$ (the minimal surface in AdS anchored to the Wilson loop).

$$\langle W \rangle \sim e^{-S_{NG}} \quad \Rightarrow \quad V_{q\bar{q}} = -\frac{4\pi^2 \left(2g_{YM}^2 N\right)^{1/2}}{\Gamma\left(\frac{1}{4}\right)^4 L}$$

- ('98 Witten; '98 Rey, Theisen, Yee; '98 Brandhuber, Itzhaki, Sonnenschein, Yankielowicz; '99 Greensite, Olesen) At finite temperature, holographic Wilson loop $\langle W \rangle$ can produce confining $V_{q\bar{q}}$.

Holographic $\langle W \rangle$ in Phenomenology

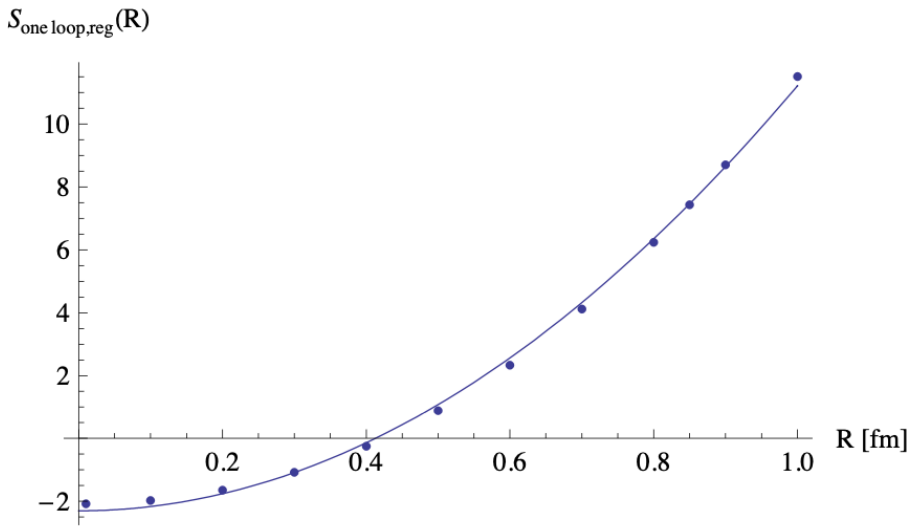
- ('06 Andreev, Zakharov; '09 Pirner, Galow; '09 **JN**, Pirner)

  Phenomenologically, we could modify the the bulk AdS_5 metric at zero temperature by hand:

$$ds^2 = h(z)\frac{L_{AdS}^2}{z^2}\left[-dt^2 + dz^2 + d\vec{x}^2\right]$$

$$h(z) = e^{-\frac{1}{2}cz^2} \quad \text{or} \quad \frac{\log(\epsilon)}{\log\left[(\Lambda z)^2 + \epsilon\right]}.$$

  Numerical result:



Quantum Corrected AdS_2 Metric

- The AdS_2 metric can be rewritten in the light-cone coordinates:

$$ds^2 = -L^2 \frac{4\pi^2}{\beta^2} \frac{du\, dv}{\sin^2 \frac{\pi}{\beta}(u-v)} ,$$

where $u \equiv t + z$, $v \equiv t - z$.

- Let u and v have quantum fluctuations:

$$u \to f(u) , \quad v \to f(v) ,$$

the metric becomes

$$ds^2 = -L^2 \frac{4\pi^2}{\beta^2} \frac{f'(u)\, f'(v)}{\sin^2 \frac{\pi}{\beta}\,[f(u) - f(v)]}\, du\, dv \equiv -4\, L^2 \mathcal{O}(u,\, v)\, du\, dv ,$$

where $\mathcal{O}$ is called a bilocal operator. ('16 Maldacena, Stanford, Yang)

- The quantum-corrected metric is ('18 Blommaert, Mertens, Verschelde)

$$\langle ds^2 \rangle = -4\, L^2 \langle \mathcal{O}(u,\, v) \rangle\, du\, dv = \langle G_{\partial\partial}(2z) \rangle\, (dt^2 + dz^2)$$

with

$$\langle G_{\partial\partial}(2z) \rangle = \frac{1}{2C} \int_0^\infty d\omega \sinh(2\pi\sqrt{2C\omega})\, e^{-\omega z}\, \Gamma(1 \pm i\sqrt{2C\omega})^2 .$$

Quantum Corrected $\langle W \rangle$ and Super-Yang-Mills Confinement

('24 Liu, Yue, **JN**, Zheng)

- Near-extremal AdS_5 Reissner-Nordström black brane:

$$ds^2 = \frac{u^2}{L_{\text{AdS}}^2} f(u)\, dt_E^2 + \frac{L_{\text{AdS}}^2}{u^2} \frac{du^2}{f(u)} + \frac{u^2}{L_{\text{AdS}}^2} d\vec{x}^{\,2}$$

- After some changes of coordinates and introducing quantum fluctuations in near-horizon AdS_2, we obtain

$$ds^2 = \alpha' \left[\frac{U^2}{R^2} \left(f(U)\, h(U)\, dt_E^2 + d\vec{x}^{\,2} \right) + R^2 f^{-1}(U)\, h(U)\, \frac{dU^2}{U^2} \right]$$

with $\quad R^2 \equiv L_{AdS}^2/\alpha\,, \quad$ a constant C with length dimension,

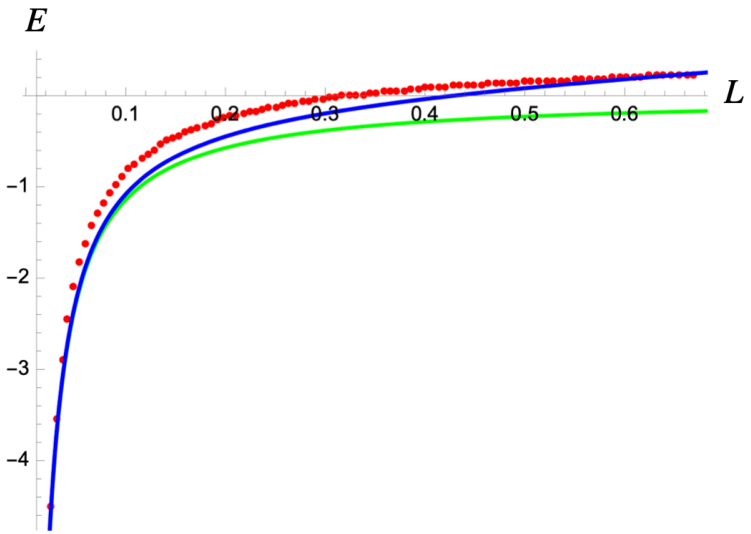
$$h(U) = 1 + \frac{R^4}{96\,\pi^4 C^2 (U - U_T)^2} - \frac{5R^6}{1152\, C^3\, (\pi^6 (U - U_T)^3)} + \frac{35R^8}{18432\,\pi^8 C^4 (U - U_T)^4} + \cdots$$

Quantum Corrected $\langle W \rangle$ and Super-Yang-Mills Confinement

('24 Liu, Yue, **JN**, Zheng)

- Using this quantum-corrected AdS_5 metric, we can compute $\langle W \rangle$ and then extract the quark anti-quark potential:

$$V_{q\bar{q}} = -\frac{4\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}\frac{R^2}{L} + \frac{5\,\Gamma\left(\frac{1}{4}\right)^2}{4608\,\pi^5\,\Gamma\left(\frac{3}{4}\right)^2}\frac{R^2}{C^2}L + \mathcal{O}\left(C^{-3}, \frac{U_T}{U_0}\right)$$



Green: Coulomb potential

Blue: analytic expansion

Red: numerical result

Quantum Corrected $\langle W \rangle$ and Super-Yang-Mills Confinement

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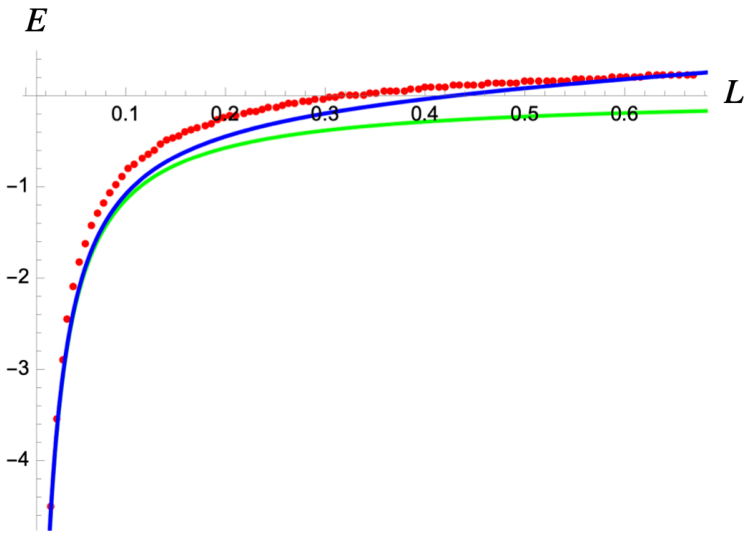

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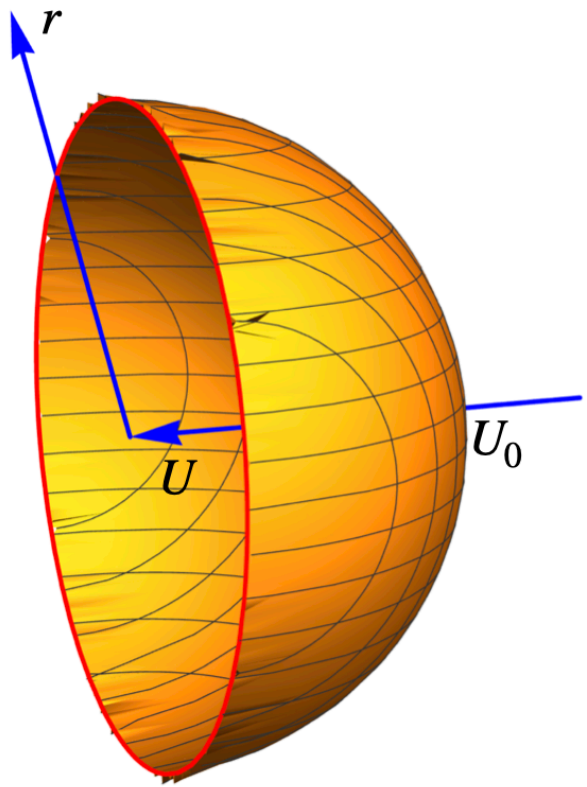
Red: numerical result

- Within AdS/CFT, confinement comes from quantum gravity!

Circular Spatial $\langle W \rangle$ and Super-Yang-Mills Confinement

('24 Liu, Yue, **JN**, Zheng)

- We can also consider a circular spatial $\langle W \rangle$:



Circular Spatial $\langle W \rangle$ and Super-Yang-Mills Confinement

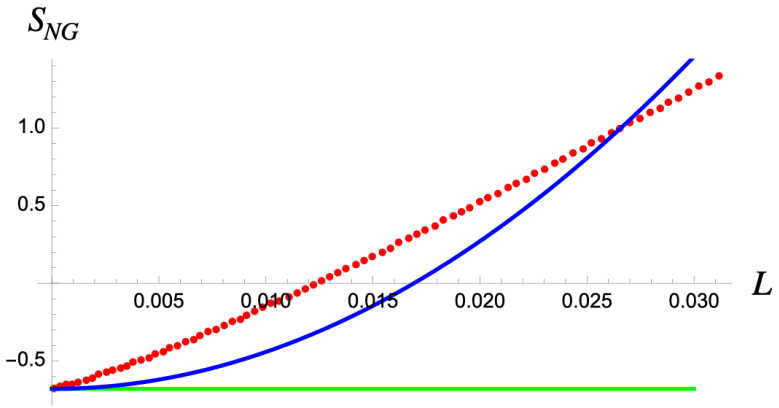
('24 Liu, Yue, **JN**, Zheng)

- From $\langle W \rangle$, we can extract the quark anti-quark potential:

$$S_{\text{NG}} = \frac{\pi R^2}{12}\left(-{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{3}{4};1,\frac{7}{4};1\right)-\frac{24\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}\right)$$

$$+\frac{BR^2\,\Gamma\left(\frac{1}{4}\right)^4}{2\pi^3}\frac{L^2}{C^2}+\mathcal{O}\left(C^{-3},\,U_T/U_0\right),$$ which obeys the area law!

- Numerical results:



Green: constant (classical)

Blue: analytic expansion

Red: numerical result

How to Understand This Result?

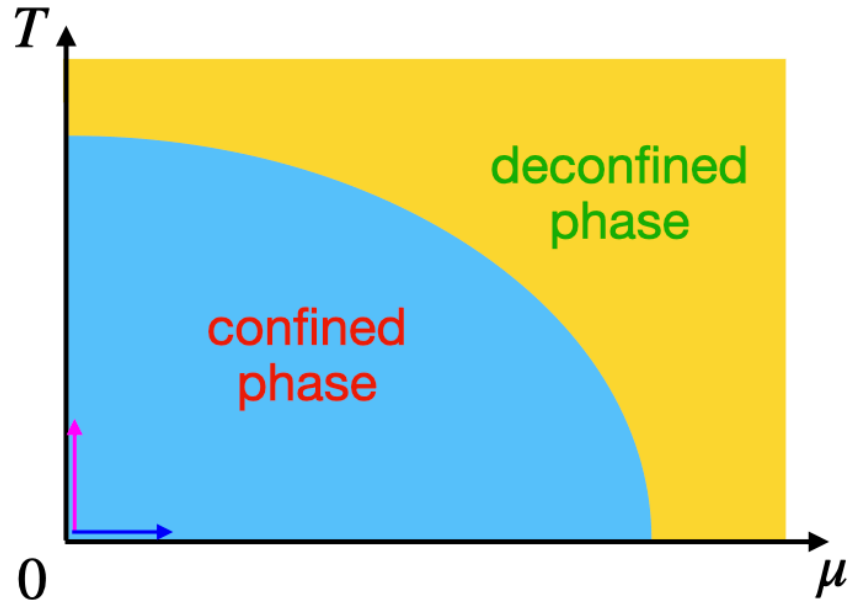
- ('98 Witten)

  At finite temperature, the confinement can be obtained, i.e., the conformal symmetry is broken.

- ('24 Liu, Yue, **JN**, Zheng)

  Our works corresponds to finite chemical potential at zero temperature.

- SYM phase diagram: ('06 Yamada, Yaffe)



Summary and Prospect

A few applications of near-horizon AdS_2 quantum gravity include:

- Resolution of the Hawking radiation paradox ('20 Iliesiu, Turiaci)
- Holographic strange metal, low-T fluid/gravity correspondence, $\cdots$
- Holographically, super-Yang-Mills confinement originates from

  quantum gravity fluctuations in the near-horizon region.

In the near future:

- Can quantum gravity fluctuations generate mass gap of SYM?
  More precisely, we expect $\langle W(L)\, W(0) \rangle \sim \exp(-ML)$.
  ('25 Liu, Yue, **JN**, Zheng, Huang)

  Similar to the confinement, the previous computation of $\langle WW \rangle$ at $T = 0$ on classical global AdS_5 shows only a polynomial decay.
  ('98 Berenstein, Corrado, Fischler, Maldacena; '99 Sonnenschein, Loewy)

Thank you!