Fortuity in SYK Models

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Motivation and Backgrounds

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- Recently, a new property called **fortuity** was discovered, particularly for supersymmetric (BPS) black holes.

- Fortuity distinguishes the microstates of **black holes** from those of **horizonless geometries**, by relating BPS states in the dual theories of different N.

Outline

- A Brief Introduction to Fortuity
- Fortuity in $\mathcal{N} = 2$ SYK Model and R-charge Concentration
- R-charge Concentration in $\mathcal{N} = 4$ SYM
- Relation between Fortuity to Chaos, Randomness, and Entanglement

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2. The Hilbert space $\mathscr{H}_N$ of the finite N theory (as a vector space) can be realized as a quotient of the infinite N Hilbert space $\mathscr{H}_\infty$ by certain equivalence relations I_N.

$$\mathscr{H}_N \simeq \mathscr{H}_\infty / I_N \quad \text{with quotient map } \pi : \mathscr{H}_\infty \to \mathscr{H}_N \text{ and } [Q, \pi] = 0$$

I_N: trace relations in $\mathcal{N} = 4$ SYM.

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- Let a Q-cohomology class $[\Psi]$ be represented by a state $|\Psi\rangle \in \mathscr{H}_N$ further lifted to a (non-unique) state $|\tilde{\Psi}\rangle \in \mathscr{H}_\infty$, i.e. $|\Psi\rangle = \pi|\tilde{\Psi}\rangle$

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  2. **Fortuitous cohomology**: $\forall$ lifts $|\tilde{\Psi}\rangle$, $Q|\tilde{\Psi}\rangle \neq 0$

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- Let us add an subscipt N to the quotient map $\pi_N : \mathscr{H}_\infty \to \mathscr{H}_N$.

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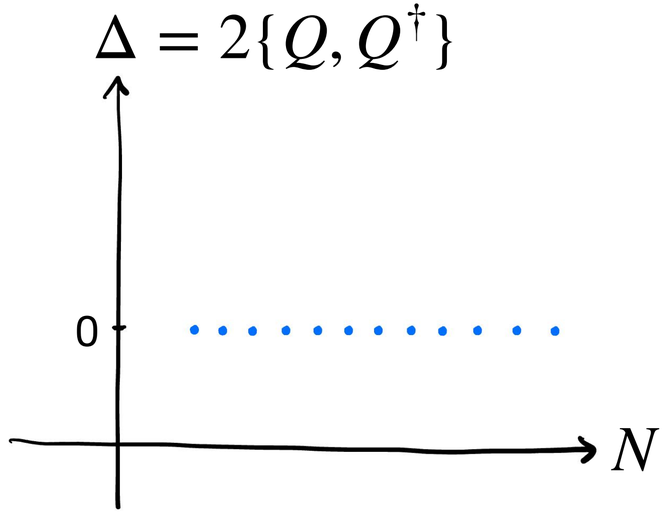
- Monotonous cohomology:

$$Q\pi_{N+i}|\tilde{\Psi}\rangle = \pi_{N+i}Q|\tilde{\Psi}\rangle = \pi_{N+i}(0) = 0$$

We have an infinite N limit of BPS operators.



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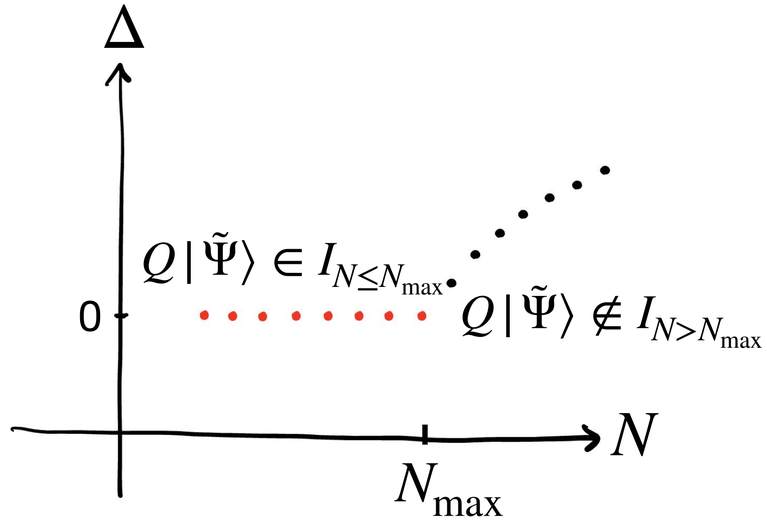


$\Rightarrow \exists N_{\max}$ s.t. $Q|\tilde{\Psi}\rangle \in I_{N+i\le N_{\max}}$ and $Q|\tilde{\Psi}\rangle \notin I_{N+i>N_{\max}}$

No infinite N limit!

Examples

- So far, all the studies on the supercharge Q-cohomology are in $\mathcal{N} = 4$ SYM.
- All monotonous cohomologies in $\mathcal{N} = 4$ SYM are known [Chang-Yin'13].
- Fortuitous cohomologies are only known in several infinite towers in the $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$ theories. [Chang-Lin'22, Choi-Choi-Kim-Lee-Lee-Lee-Park 22,23,23]

Conjectures

- **Conjecture** [Chang-Lin'24]: **Monotonous states** ↔ **Horizonless geometries**
  Fortuitous states ↔ **Typical black hole microstates**
- **Evidence/checks**:
  1. Matching monotonous BPS partition functions

$$\lim_{N\to\infty} Z_{\text{monotonous}} = Z_{\text{supergravitons}} \quad \text{and} \quad Z^{\text{half-BPS}}_{\text{monotonous}} = Z_{\text{LLM geometries}}$$

  2. A bound: $\log(\#\,\text{monotonous}) \lesssim N^{\frac{5}{3}} \ll \log(\#\,\text{fortuitous}) \sim N^2$
- **Argument**: Horizonless geometries can be viewed as coherent states of gravitons, and when taking $G_N \to 0$, they disassemble into non-interacting gravitons. This matches with the behavior of monotonous states.

Conjectures

- **Conjecture** [Chen-Lin-Shenker'24]: **Fortuitous** states are **strongly chaotic**, and **monotonous** states are at most **weakly chaotic**.
- **Evidence/checks**:
  - Apply LMRS diagnostic [Lin-Maldacena-Rozenberg-Shan'22] to half-BPS states in D1D5 CFT and 1/4-BPS states in $\mathcal{N} = 4$ SYM.
- **Argument**: The chaos of non-BPS states invades the fortuitous sector.

- The conjectures involving fortuitous states are rather weak. Due to the lack of data, we do not know much about fortuitous states.

- Could we find simpler models to explore more properties of the fortuitous states?

$\mathcal{N} = 2$ SYK model

$\mathcal{N} = 2$ SYK Model

- The supersymmetric $\mathcal{N} = 2$ SYK model is a system of N complex fermions ψ^i with a complex supercharge Q and Hamiltonian H,

$$Q = i \sum_{i,j,k} C_{i,j,k} \psi^i \psi^j \psi^k \,, \qquad H = \{Q, Q^\dagger\} \quad \text{and} \quad Q^2 = 0 = (Q^\dagger)^2.$$

[Fu-Gaiotto-Maldacena-Sachdev'16]

- The couplings $C_{i,j,k}$ are usually taken to be independent, identically distributed Gaussian variables. But for now, we will not consider ensemble average, and focus on a single realization of $C_{i,j,k}$.

Some Remarks on $\mathcal{N}=2$ SYK

- The correlation functions in the $\mathcal{N}=2$ SYK can be computed exactly at large N by solving the Schwinger-Dyson equaitons. The out-of-time-order four-point function exhibits maximal chaos. [Fu-Gaiotto-Maldacena-Sachdev'16, Peng-Spradlin-Volovich'17]

- At low temperatures, the theory is dominated by a $\mathcal{N}=2$ Schwarzian sector, which is dual to a $\mathcal{N}=2$ Jackiw–Teitelboim supergravity in AdS_2.

- The $\mathcal{N}=2$ JT supergravity also describes the near-horizon excitations of BPS black holes in AdS_5. [Boruch-Heydeman-Iliesiu-Turiaci'22]

Q-cohomology in $\mathcal{N}=2$ SYK

- Our previous classification of Q-cohomologies into monotonous and fortuitous uses the infinite N Hilbert space, which is very tricky to define in the $\mathcal{N}=2$ SYK model.

- Therefore, we use an alternative definition that involves lifting monotonous states not directly to infinite N but step by step from N to $N+1$ then to $N+2$, and so on. But it is not illuminating to present here.

- A simple criterion for monotonous: The $\min\left(N_\psi, N-N_\psi\right)$ of the state does not grow with N, where $N_\psi=\sum \psi_i \psi_i^\dagger$ is the fermion number.

Q-cohomology in $\mathcal{N} = 2$ SYK

- By direct computation, [Fu-Gaiotto-Maldacena-Sachdev'16] found that the BPS spectrum has no ensemble fluctuation, and exhibits **R-charge concentration**: all the cohomology classes/BPS states are concentrated near $N_\psi \sim N/2$.

- Hence, **all the BPS states are fortuitous**, as their fermion numbers scale with N in the large N limit.

R-charge concentration

- The R-charge concentration follows from the genericity of the supercharge.
- Let $|\Omega\rangle$ be the vacuum state with $\psi_i^\dagger |\Omega\rangle = 0$ for all i. A state $|\alpha\rangle$ with fermion number p can be viewed as a p-form

$$|\alpha\rangle = \sum_{i_1 \cdots i_p} \alpha_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p} |\Omega\rangle$$

- The supercharge $Q \sim C\psi^3$ acts as a wedge product of a generic 3-form C_3

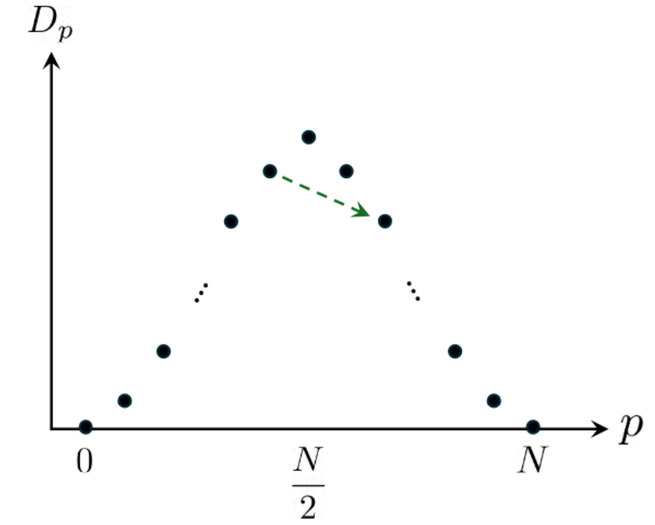
$$Q : \ \alpha_p \mapsto C_3 \wedge \alpha_p$$

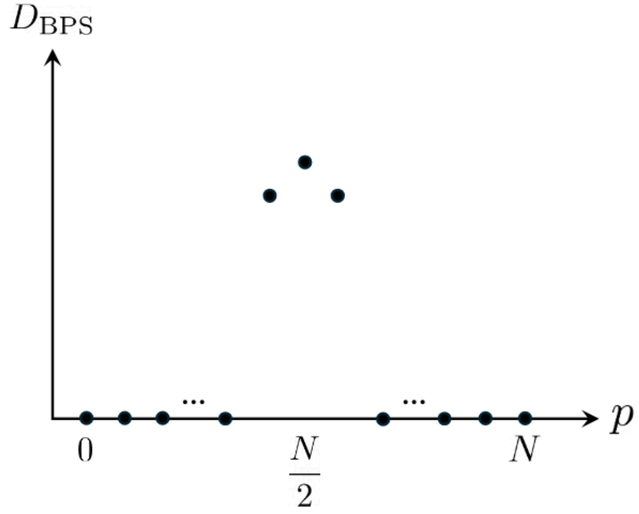
R-charge concentration

- The dimension of the space Ω^p of p-forms is $D_p = \dim \Omega^p = \binom{N}{p}$.

- When $p \leq (N-3)/2$, we have $\dim \Omega^p \leq \dim \Omega^{p+3}$.

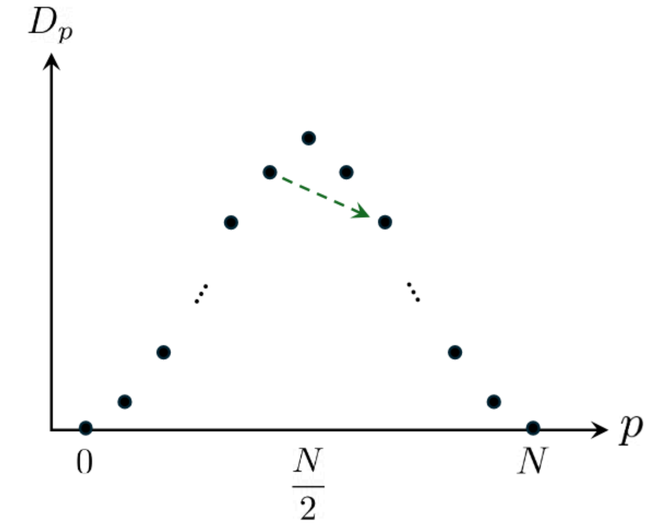
- We observe that as C_3 is generic, the only solution to $C_3 \wedge \alpha_p = 0$ is $\alpha_p = C_3 \wedge \beta_{p-3}$. So, no Q-cohomology and no BPS states.

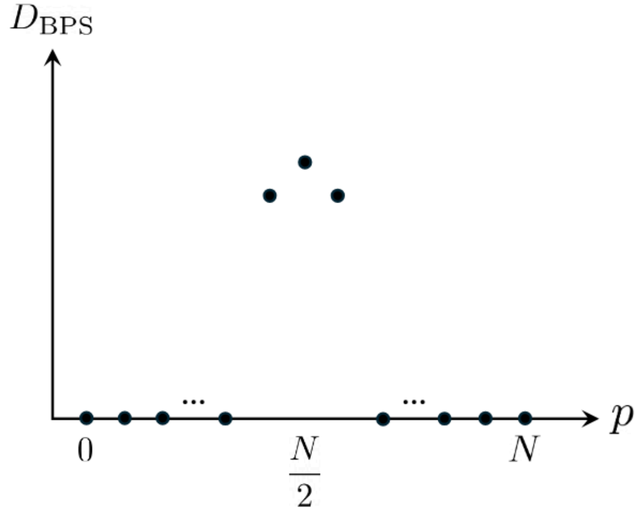




R-charge concentration

- $\psi_i \leftrightarrow \psi_i^\dagger$ symmetry: no BPS state when $p \geq (N+3)/2$.
- All the Q-cohomology/BPS states are concentrated near $p \sim N/2$.
- In fact, there are in total 3 cochain complexes, and the BPS states are concentrated in a single cochain of each cochain complex.





Universality of R-charge concentration

- R-charge concentration is valid for each realization of the SYK.
- R-charge concentration is predicted by Schwarzian [Stanford-Witten], universal sector of BPS black holes.
- Due to $\frac{1}{\beta}$ enhancement, near the BPS states, Schwarzian effects dominate even including stringy corrections, valid in the weak coupling region.

A Failure of Random Matrix Theory

- We may wonder if the R-charge concentration can be captured by the random matrix theory.

- Consider a cochain complex: $\mathscr{H}_0 \overset{Q_1}{\to} \mathscr{H}_1 \overset{Q_2}{\to} \cdots \overset{Q_N}{\to} \mathscr{H}_N$ with random matrices Q_i subject to the nilpotent constraint $Q_{i+1}Q_i = 0$. [Turiaci-Witten'23]

- However, in this case, genericity (maximizing the number of independent variables) of Q_i implies R-charge spreading instead of concentration. [Turiaci-Witten'23] has to impose the BPS spectrum of $\mathcal{N} = 2$ JT by hand.

- The discrepancy may be because the supercharge in $\mathcal{N} = 2$ SYK is q-local.

Supercharge Chaos

- **Supercharge Chaos**: Let Q be a generic q-local complex supercharge acting on a finite L-dimensional Hilbert space, satisfying the N = 2 supersymmetry algebra with an R-symmetry. In an irreducible cochain complex of differential Q with a macroscopic index of order L^{v} (where $0 < v \leq 1$) in the large L limit, the BPS states are sharply concentrated on a single cochain with a specific R-charge. Near the BPS states, the supercharge Q is approximated by the random matrix ensemble of Turiaci-Witten, which is dual to the N = 2 JT gravity.

Supercharge Chaos

- A direct consequence, index counts total number of states.

- The large N scaling of R_c can be determined, more precisely the $r_c = R_c/N^{\#}$ (# depends on the specific model), by a saddle point approximation of the index as

$$I = \mathrm{Tr}\,(-1)^R = \sum_R (-1)^R D_R \approx \int dr\, e^{i\pi r N^{\#} + S(r) N^{\#}} \approx e^{i\pi r_* N^{\#} + S(r_*) N^{\#}},$$

$i\pi + S'(r_*) = 0$. R-charge concentration implies $I = D_{\mathrm{BPS},\, R_c} e^{i\pi r_c N^{\#}}$.

Matching phase gives $r_c = \mathrm{Im}\,\left[ir_* + \pi^{-1} S(r_*)\right]$

Application in $\mathcal{N} = 4$ SYM

A Long-Standing Puzzle

- The IIB supergravity on asymptotic $\mathrm{AdS}_5 \times \mathrm{S}^5$ has five conserved charges J_1, J_2, R_1, R_2, R_3 (for the five orthogonal rotations).

- It is natural to expect that the BPS black holes in $\mathrm{AdS}_5 \times \mathrm{S}^5$ should carry these five charges independently. However, the known BPS black hole solutions in supergravity all satisfy a nonlinear charge constraint

$$(R_1R_2R_3+\frac{N^2}{2}J_1J_2)-(R_1+R_2+R_3+\frac{N^2}{2})(R_1R_2+R_2R_3+R_3R_1-\frac{N^2}{2}(J_1+J_2))=0$$

- The microstates of the BPS black holes are dual to BPS states in $\mathcal{N}=4$ SYM.

Dressed Black Holes

- [Choi-Jain-Kim-Krishna-Kundu-Lee-Lee-Minwalla-Patel'23, 24] considered dressing the BPS black holes with graviton gas (grey galaxy) or D3 branes. The dressed black holes do not satisfy the charge constraint.

- Boundary Interpretation:
  1. dressing ↔ monotonous states
  2. "pure" fortuitous states ↔ pure BPS black hole
  3. pure fortuitous dressed by monotonous ↔ dressed BPS black hole

- At $N = 2, 3$, the known fortuitous coho's are organized as the pure fortuitous coho's and products of them and monotonous coho's of large enough charges.

[Choi-Choi-Kim-Lee-Lee-Lee-Park 22,23,23]

R-charge Concentration in $\mathcal{N} = 4$ SYM

- The dressings are far from the horizon (with large angular momentum/ charge).
- The near-horizon theory is still the $\mathcal{N} = 2$ Schwarzian theory (the same low energy theory as $\mathcal{N} = 2$ SYK), so we expect R-charge concentration for the pure fortuitous states.
- Assuming R-charge concentration, and using the superconformal index (SCI), we can determine where the pure fortuitous states are concentrated.

SCI: $$I = \mathrm{Tr}\left[(-1)^{2J_2} e^{-\omega_1(J_1-J_2)-\Delta_1(R_1+J_2)-\Delta_2(R_2+J_2)-\Delta_3(R_3+J_2)}\right]$$

SCI: $$I = \mathrm{Tr}\left[(-1)^{2J_2} e^{-\omega_1(J_1-J_2)-\Delta_1(R_1+J_2)-\Delta_2(R_2+J_2)-\Delta_3(R_3+J_2)}\right]$$

The SCI in the large N limit: $$\log(I) = N^2 \frac{\Delta_1\Delta_2\Delta_3}{2\omega_1\omega_2}\Big|_{\Delta_1+\Delta_2+\Delta_3-\omega_1-\omega_2=2\pi i}$$

[Cabo-Bizet, Cassani, Martelli, Murthy'18, Choi, Kim, Kim, Nahmgoong'18, Benini, Milan'18]

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Obtain the index with fixed $J_1 - J_2$ and $R_a + J_2$ by inverse Laplace transform:
($J_i = N^2 j_i$, $R_a = N^2 r_a$)

$$S(j_1 - j_2, r_a + j_2) = \log\left[\mathrm{Tr}_{J_1-J_2, R_a+J_2}(-1)^{2J_2}\right] \approx \mathrm{Ext}\left[\frac{\Delta_1\Delta_2\Delta_3}{2\omega_1(\sum\Delta_a - \omega_1 - 2\pi i)} + \omega_1(j_1 - j_2) + \sum \Delta_a(r_a + j_2)\right]$$

saddle point approx

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R-charge concentration: $2\pi j_{2,c} = \mathrm{Im}\, S(j_1 - j_2, r_a + j_2)$

This reproduces the non-linear charge constraint of the BPS BH solutions.

Notice: R-charge concentration means microscopically BPS BH is concentrated

Summary

- We studied supercharge Q-cohomology in $\mathcal{N} = 2$ SYK models.
- Fortuitous states are concentrated at a single R-charge in the cochain complex.
- The R-charge concentration is closely related to the genericity of the supercharge.
- Supercharge Chaos Conjecture.
- The supercharge in $\mathcal{N} = 4$ SYM is not generic, but we think that it is generic in the sector of pure fortuitous states. The R-charge concentration reproduces microscopically the non-linear charge relation of BPS black holes in supergravity.

Outlook

- Understand more about the boundary duals of the dressed black holes and the dressing mechanism.
- Understand the interactions between the dressing at far-horizon and the near-horizon $\mathcal{N} = 2$ Schwarzian modes.
- Finding Schwarzian in the weakly coupled region.
- This may still require us to find more fortuitous states in $\mathcal{N} = 4$ SYM.
- More interplays between Quantum Chaos and SUSY?

Thank you