Some formal aspects of Field theory and Holography, PCFT
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Fortuity in D1-D5 system

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Introduction

- IIB string theory on $\mathbb{R}^{1,5} \times M$ for $M = T^4$ or $K3$ with coinciding
  - N_1 D1 branes extending along (x^0, x^1)-plane
  - N_5 D5 branes wrapping M and extending along (x^0, x^1)-plane
  - Preserving 8 supercharges

- Near horizon limit: (SUSY $8 \to 16$) [Maldacena]

  IIB on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times M \leftrightarrow$ 2d D1-D5 CFT

  $= \mathrm{Sym}^N M$ with $(N = N_1 N_5)$ deformations

Introduction

- 5D black holes: IIB string theory on $\mathbb{R}^{1,4} \times \mathrm{S}^1 \times M$
  - N_1 D5 branes wrapping $\mathrm{S}^1 \times M$
  - N_1 D1 branes wrapping S^1
  - P units of momentum along S^1
- The microstates of the 5D black holes correspond to states in the Hilbert space of D1-D5 CFT on S^1 with fixed momentum P. [Strominger, Vafa]

$$\frac{A}{4G_N} = S = \log(\#\text{microstates}) = 2\pi\sqrt{\frac{1}{2}N_1 N_5 P}$$

Outlines

- D1-D5 CFT basics — symmetric orbifold and deformation
- Q-cohomology and a classification
- Large N equivalence relation
- Fortuitous states at large N and BPS strings

D1-D5 CFT Basics: SCA

- $M = K3$: D1-D5 CFT has small $\mathcal{N} = 4$ superconformal algebra (SCA).
  - T: stress tensor, $G^{\pm}$, $G'^{\pm}$: spin $\frac{3}{2}$ super-current
  - $K^{\pm}, K^3$: SU(2) spin 1 currents
- $M = T^4$: D1-D5 CFT has contracted large $\mathcal{N} = 4$ SCA
  - $\alpha^i, \bar{\alpha}^i$, $i = 1,2$: four U(1) currents
  - $\psi^{\pm}, \bar{\psi}^{\pm}$: four free fermions

D1-D5 CFT Basics: BPS States

- 8 supercharges: (NS sector)
- left-moving: $G^{\pm}_{-1/2}, G'^{\pm}_{-1/2},$ right-moving: $\tilde{G}^{\pm}_{-1/2}, \tilde{G}'^{\pm}_{-1/2}$
- Pick the right-moving supercharges $Q = \tilde{G}^{+}_{-1/2}, \quad Q' = \tilde{G}'^{+}_{-1/2}$

$$\{Q, Q^{\dagger}\} = 2(\tilde{L}_0 - \tilde{K}^3_0) \equiv \Delta = \{Q', Q'^{\dagger}\}$$

- $\frac{1}{4}$ BPS states: $\tilde{h} = \tilde{j},$ $\frac{1}{2}$ BPS states: $h = j$ and $\tilde{h} = \tilde{j}$

Q-Cohomology

- By Hodge theory argument: Q-cohomology classes $\leftrightarrow \frac{1}{4}$ BPS states
- (refined) elliptic genus = Euler characteristic, and is a deformation invariant.
- Is the Q-cohomology also an invariant?
- At the orbifold point, it is easy to find all BPS states. $\frac{1}{2}$ BPS states are unlifted, while some of the $\frac{1}{4}$ BPS states are lifted by the deformation.

Symmetric Orbifold Point

- We focus on $\mathrm{Sym}^N(T^4)$ theory.
- The Hilbert space is a sum over twisted sectors: $\mathscr{H} = \oplus_p \mathscr{H}_p$, where p is a conjugacy class of S_N.
- Quantize the product theory $(T^4)^N$ on S^1 with twisted boundary condition:

$$X^{[i]}(\sigma + 2\pi) = X^{[g(i)]}(\sigma) \quad \text{for } g \in S_N$$

- The twisted sector space is $\mathscr{H}_p = (\oplus_{g\in p}\mathscr{H}_g)^{S_N}$.

Symmetric Orbifold Point

- Conjugacy class is labeled by cycle shape $p = (1^{N_1}, 2^{N_2}, \cdots)$, $N = N_1 + 2N_2 + \cdots$
- $\mathscr{H}_p \cong \bigotimes_w S^{N_w} \mathscr{H}_{(w)}^{\mathbb{Z}_w}$, and $\mathscr{H}_{(w)}$ is the Hilbert space of T^4 CFT on S^1 with radius w, which is a Fock space with fractional modes $\alpha^i_{n+\frac{m}{w}}$, $\psi^{\pm}_{r+\frac{m}{w}}$, …
- $\mathscr{H}_{(w)}^{\mathbb{Z}_w}$: $\mathbb{Z}_w$ invariant sector, i.e. $\ell = h - \tilde{h} \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$.
- Ground state $|w\rangle$: $h = \tilde{h} = \frac{6}{24}(w - \frac{1}{w})$, $j = \tilde{j} = 0$.

BPS Quartets

- $\frac{1}{2}$ BPS states: $\prod_{0\le r\le w} \psi^+_{-r/w}\tilde{\psi}^+_{-r/w}|w\rangle\,, \quad h=j=\frac{w-1}{2}=\tilde{h}=\tilde{j}$

  - Annihilated by Q, $Q^\dagger$, Q', $Q'^\dagger$ and left-moving supercharges.

  - Carry zero momenta and windings along T^4

- A 4d Clifford algebra commutes with Q and Q'

$$\{\tilde{\psi}^+_{-1/2},\bar{\tilde{\psi}}^-_{1/2}\}=-1=-\{\bar{\tilde{\psi}}^+_{-1/2},\bar{\psi}^-_{1/2}\} \qquad (\{Q,\psi\}\sim\alpha_0=0)$$

BPS Quartets

- $\frac{1}{4}$ BPS states in $\mathscr{H}_{(w)}$:

$$
\begin{aligned}
&|w_{i,++}\rangle && \tilde{h} = \frac{w+1}{2} = \tilde{j} \\
&|w_{i,+-}\rangle, |w_{i,-+}\rangle && \tilde{h} = \frac{w}{2} = \tilde{j} \\
&|w_{i,--}\rangle && \tilde{h} = \frac{w-1}{2} = \tilde{j}
\end{aligned}
$$

- i : arbitrary left-moving mode. $\pm, \pm$: right-moving quartet of the Clifford

- More general $\frac{1}{4}$ BPS states: $|(w_1)_{i_1,\alpha_1\beta_1}, (w_2)_{i_2,\alpha_2\beta_2}, \cdots\rangle$

Deformation

- D1-D5 CFT is given by the deformation $S_{\text{orb}} + g \int d^2z\, \Phi$, where

$$\Phi \leftrightarrow (G^-_{-1/2}\tilde{G}'^-_{-1/2} - G'^-_{-1/2}\tilde{G}^-_{-1/2})\,|\,2_{--,--}\rangle$$

- The holomorphic and anti-holomorphic anomalous dimensions are δh, $\delta\tilde{h}$. They satisfy $\delta h - \delta\tilde{h} = 0$ (no anomalous spin). $\delta\tilde{h}$ can be computed by

$$\delta\tilde{h}\,|\,\Psi\rangle = (\tilde{L}_0 - \tilde{K}^3_0)\,|\,\Psi\rangle = \frac{1}{2}\{Q, Q^\dagger\}\,|\,\Psi\rangle$$

$$c_{ij} = \langle\Psi_i\,|\,Q\int d^2z\Phi\,|\,\Psi_j\rangle \sim \langle\Psi_i\,|\,V(G'^-_{-1/2}\,|\,2_{--.--}\rangle)\,|\,\Psi_j\rangle$$

Joining and Spliting

- $Q = Q^{(0,1)} + Q^{(-1,0)} + Q^{(1,0)} + Q^{(0,-1)}$ with $Q^{(\#\text{cycles},\text{length})}$
  - $Q^{(0,1)}$: Joining a length-w cycle with a trivial cycle $1__,__$
  - $Q^{(-1,0)}$: Joining a length-w_1 and a length-w_2 cycles
  - $Q^{(1,0)}$: Splitting a length-$(w_1 + w_2)$ cycle into a length-w_1 and a length-w_2 cycles
  - $Q^{(0,1)}$: Splitting out a trivial cycle from a length-w cycle
- $Q^{(0,1)}$, $Q^{(1,0)}$, $Q^{(0,1)}$ obey the Leibniz rule.

Infinite N limit

$$Q^{(0,1)} \sim N, \quad Q^{(-1,0)}, Q^{(1,0)} \sim 1, \quad Q^{(0,-1)} \sim N^{-1}$$

- In the infinite N limit, we focus on the cohomology of $Q^{(0,1)}$.
- Since $Q^{(0,1)}$ obeys the Leibniz rule, we focus on single-cycle cohomology.
- The cohomology classes are represented by $|w_{i,\pm\pm}\rangle$ with i corresponding to the $\mathrm{SU}(1,1|2)$ (global part of $\mathcal{N} = 4$ SCA) creation modes.

Infinite N limit

- The $|w_{i,\pm\pm}\rangle$ (with $i \in \mathrm{SU}(1,1|2)$) correspond to single supergraviton, and $|w_{i_1,\pm\pm}, w_{i_2,\pm\pm}, w_{i_3,\pm\pm}, \cdots\rangle$ are multi-graviton states.

- When the number of supergravitons is of order N, they condense to a coherent state of smooth horizonless geometries, the superstrata geometries. [Bena, Giusto, Russo, Shigemori, Warner, …]

From large N to finite N

- Let $\mathscr{H}_N$ be the Hilbert space of the $\mathrm{Sym}^N(T^4)$. There is a projection map (called stringy exclusion principle)

$$\pi_{N',N}: \ \mathscr{H}_{N'} \to \mathscr{H}_N, \quad N' > N$$

  which is defined by projecting the last copy onto the ground state.

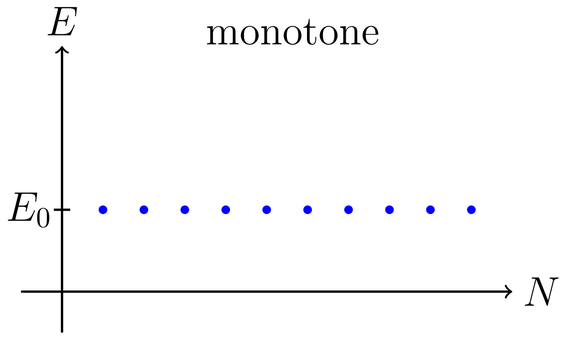
- Consider a nontrivial cohomology class representative $|\Psi\rangle \in \mathscr{H}_N$, which is further lifted to a state:

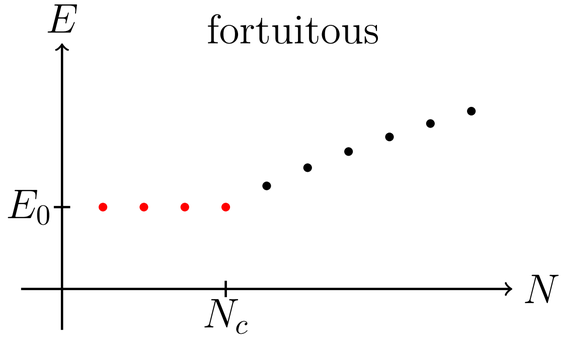
$$|\Psi'\rangle \in \pi^{-1}_{N',N}(|\Psi\rangle) \in \mathscr{H}_{N'}$$

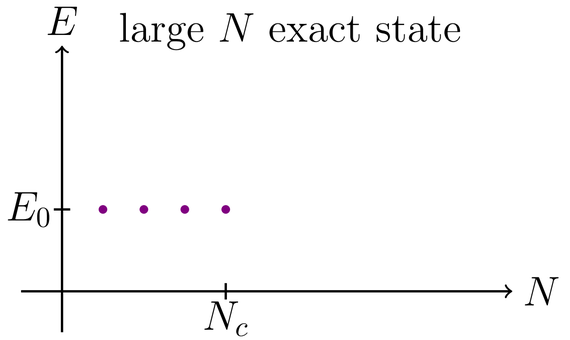
From large N to finite N

- There are three possibilities: $|\Psi'\rangle \in \pi^{-1}_{N',N}(|\Psi\rangle)$

  1. $|\Psi'\rangle$ is Q-closed but not Q-closed
  2. $|\Psi'\rangle$ is not Q-closed
  3. $|\Psi'\rangle$ is Q-exact







Large N Exact States

- If $[\pi_{N',N}, Q] = 0$, then there exists an induced map from Q-cohomology at rank N' to rank N. So $|\Psi'\rangle$ cannot be Q-exact.
- The large N exact states give equivalence relations between monotone states.
- Consider two monotone states $|\Psi_1\rangle, |\Psi_2\rangle \in \mathscr{H}_N$ that represent distinct cohomology classes, i.e. $|\Psi_1\rangle - |\Psi_2\rangle$ is not Q-exact.
- However, it is possible that the lift $|\Psi_1'\rangle - |\Psi_2'\rangle$ is Q-exact.

Large N Equivalence

- The large N equivalence relation is important in understanding the matching between the boundary and bulk spectrum at large N.

- In D1-D5 CFT, it seems there are extra BPS states at large N, for example

$$L_{-2}\,|\,w_{i_1,\pm\pm}, w_{i_2,\pm\pm}, \cdots\rangle\ , \quad (i_1, i_2, \cdots \in \mathrm{SU}(1,1\,|\,2))$$

  which is interpreted as boundary graviton dressing of $|\,w_{i_1,\pm\pm}, w_{i_2,\pm\pm}, \cdots\rangle$.

- However, $|\,w_{i_1,\pm\pm}, w_{i_2,\pm\pm}, \cdots\rangle$ already give the complete bulk spectrum.
  $\Rightarrow$ Large N equivalence: $L_{-2}\,|\,w_{i_1,\pm\pm}, w_{i_2,\pm\pm}, \cdots\rangle \sim |\,w_{i_1,\pm\pm}, w_{i_2,\pm\pm}, \cdots\rangle$

Fortuitous States at Large N

- In theories like $\mathcal{N} = 4$ SYM, it is very hard to find examples of fortuitous states at large N.

- However, surprisingly, such examples can be easily obtained in D1-D5 CFT by considering composite states

$$u^i v^j \,|\, (w_1)_{i,\alpha\beta}, (w_2)_{j,\gamma\delta}\rangle \quad (i,j \text{ not necessary in } \mathrm{SU}(1,1\,|\,2))$$

- Using the Clifford algebra, we can project the state such that the joining is forbidden:

$$Q: \;\; P^{\alpha\beta\gamma\delta} u^i v^j \,|\, (w_1)_{i,\alpha\beta}, (w_2)_{j,\gamma\delta}\rangle \not\to |\,(w_1 + w_2)\rangle$$

Fortuitous States at Large N

- Q obeys Leibniz rule when acting on $P^{\alpha\beta\gamma\delta}u^i v^j \,|\, (w_1)_{i,\alpha\beta}, (w_2)_{j,\gamma\delta}\rangle$.
- If $u^i \,|\, (w_1)_{i,\alpha\beta}\rangle$ and $v^j \,|\, (w_2)_{j,\gamma\delta}\rangle$ are BPS states, then the composite state is also BPS.
- Assume $w_1 \gg w_2$ and $u^i \,|\, (w_1)_{i,\alpha\beta}\rangle$ is monotone that corresponds to a superstrata geometry, and $v^j \,|\, (w_2)_{j,\gamma\delta}\rangle$ is fortuitous.
- $v^j \,|\, 1^{w_1}, (w_2)_j\rangle \longleftrightarrow$ massive string state in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times T^4$

  $\Rightarrow u^i v^j \,|\, (w_1)_i, (w_2)_j\rangle \longleftrightarrow$ BPS stringy excitation on superstrata geometry

Conclusion

- Q-cohomology in D1-D5 CFT has provided many insights into the bulk theory.

- The examples of large N fortuitous states are non-generic, but at least tell us what the bulk dual of fortuitous states may look like.

- Constructing more examples of fortuitous states in D1-D5 CFT and other models may shed light on the question: What is a black hole microstate?