Resurgence of TTbar deformed partition function

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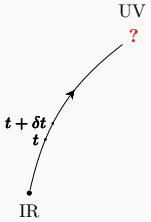
Some Formal Aspects of Field Theories and Holography, USTC, Apr 17-20 2025

Southeast University, Shing-Tung Yau Center

2410.19633: JG, Jiang, Wang
2503.19350: JG, Jiang, Wang

Climbing the RG flow

- Studying RG flow from IR to UV is usually difficult.



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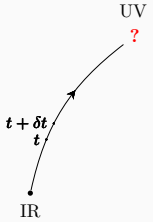

- $\mathrm{T\overline{T}}$ deformation: An irrelevant deformation of 2d QFT that has nice properties. [Smirnov, Zamolodchikov'16; Cavaglia, Negro, Szecsenyi, Tateo'16]

Why $\mathrm{T\overline{T}}$?

- **Solvable**: Many physical observables of the deformed theory are computable from those of the undeformed theory: S-matrix, deformed Lagrangian, finite volume spectrum and torus partition function. [Aharony, Bonelli, Cardy, Cavaglia, Conti, Datta, Doroud, Dubovsky, Flauger, Giveon, Gorbenko, Iannella, Jiang, Kutasov, Mirbabayi, Negro, Smirnov, Szecsenyi, Tateo, Zamolodchikov, Zhu, ...]

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- Preserve **integrability** of IQFT [Smirnov, Zamolodchikov'16] and preserver *uniquely* **modularity** of CFT [Datta, Jiang'18; Aharony, Datta, Giveon, Jiang, Kutasov'18].

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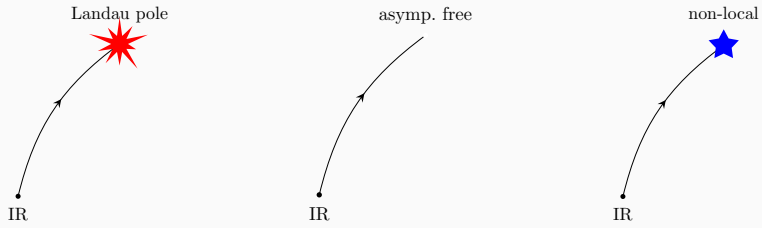
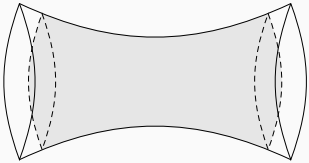
- A third type of UV fixed point: **Non-local** Lagrangian, ... [Cavaglia, Negro, Szecsenyi, Tateo'16]

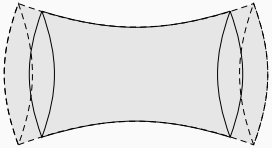


- The bulk *holographic* dual of a $\mathrm{T\overline{T}}$ deformed CFT?

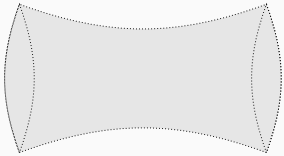


glue-on for $\lambda > 0$

[Apolo,Hao,Lai,Song'23]



cut-off for $\lambda < 0$ [McGough, Mezei, Verlinde'16; Kraus, Liu, Marolf'18]



AdS/flat interpolation for single-trace $\mathrm{T\overline{T}}$ [Giveon, Itzhaki, Kutasov'17]

Some of the current issues

- For $\mathrm{T\overline{T}}$ deformed CFT, torus partition function can only be computed perturbatively

$$Z(\tau, \bar{\tau}; \lambda) = \sum_{k=0}^{\infty} Z_k(\tau, \bar{\tau}) \lambda^k$$

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- Q1: Is this perturbative series *convergent* or *divergent*?
- Q2: Are there *non-perturbative* corrections to this perturbative series?
- We answer both questions [JG, Jiang, Wang'24, '25]
  - by developping an efficient perturbative algorithm, and
  - by using the resurgence theory to extract non-perturbative corrections.

Perturbative coefficients

- For $\mathrm{T\overline{T}}$ deformed CFT, the flow equation dictates that the coefficients Z_k satisfy the recursion relation [Datta, Jiang'18]

$$Z_{k+1} = \frac{1}{2}\left(\frac{2\tau_2}{k+1} D^{(k)} \overline{D}^{(k)} - \frac{k}{2\tau_2}\right) Z_k$$

using the Ramanujan-Serre derivative

$$D = D^{(k)} = \frac{\partial}{\partial \tau} - \frac{\mathrm{i}k}{2\tau_2}, \quad \overline{D} = \overline{D}^{(k)} = \frac{\partial}{\partial \bar{\tau}} + \frac{\mathrm{i}k}{2\tau_2}.$$

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- $D^{(k)}$, $\overline{D}^{(k')}$ act on functions of modular weight (k, k')

$$\Phi_{(k,k')}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = (c\tau+d)^k (c\bar{\tau}+d)^{k'} \Phi_{(k,k')}(\tau,\bar{\tau}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

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  and increase k, k' by 2 respectively.
- Recursively, one can show that Z_k is modular of weight (k, k)

Factorization of torus partition function

- The torus pfn of undeformed CFT factorizes to hol. and anti-hol. characters

$$Z_0 = \sum_n c_n(\tau_2)\chi_n(\tau)\chi_n(\bar{\tau}).$$

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- D acts (almost) exclusively on (almost) hol. sector

$$D\eta = \frac{\pi \mathrm{i}}{12}\eta\tilde{E}_2,\ D\tilde{E}_2 = \frac{\pi \mathrm{i}}{6}(\tilde{E}_2^2 - E_4),\ DE_4 = \frac{2\pi \mathrm{i}}{3}(\tilde{E}_2 E_4 - E_6),\ DE_6 = \pi \mathrm{i}(\tilde{E}_2 E_6 - E_4^2),$$

$$D\bar{\eta} = D\bar{E}_4 = D\bar{E}_6 = 0, \quad D\bar{\tilde{E}}_2 = \frac{3\mathrm{i}}{2\pi\tau_2^2}$$

with $\tau = \tau_1 + \mathrm{i}\tau_2$ and $\tilde{E}_2(\tau, \bar{\tau}) = E_2(\tau) - 3/(\pi\tau_2)$.

Factorisation of deformed torus partition function

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- $\chi_{n,\ell} = D^\ell \chi_n$ are almost holomorphic "characters" consisting of $\tilde{E}_2, E_4, E_6, \ldots$

$$\text{Free boson}: \quad \chi_{0,1} = \eta^{-1} \tilde{E}_2, \quad \chi_{0,2} = \eta^{-1}(\tilde{E}_2^2 - 2E_4).$$

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- $c_{n,\ell}$ are universal coefficients that depend only on the modular weight of χ_n.

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- Perturbative $\mathrm{T}\overline{\mathrm{T}}$ series is **divergent** with *unusual* asymptotic behavior

$$Z_n \sim \frac{\Gamma(n+\nu)}{A^{n+\nu}} \exp\left(B\sqrt{\frac{n+\nu}{A}}\right)(b_0 + \mathcal{O}(n^{-1/2})).$$

where

$$A = -2\tau_2, \quad B = \mathrm{i}(1+\tau_2)\sqrt{\frac{2\pi c}{3}}, \quad \nu = \begin{cases} -1, & \text{free boson,} \\ -1/2, & \text{fermion, Lee-Yang,} \end{cases}$$

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- The unusual asymptotic behavior may be due to *non-locality* of the deformed theory.

Asymptotics of $\mathrm{T\overline{T}}$ series: Free boson

Asymptotic plots of

$$s_n = nZ_n/Z_{n+1} \sim A + \mathcal{O}(n^{-1/2})$$
$$s'_n = \sqrt{n}(s_n - A) \sim -B\sqrt{A}/2 + \mathcal{O}(n^{-1/2})$$
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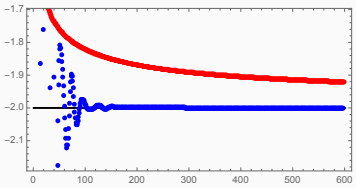
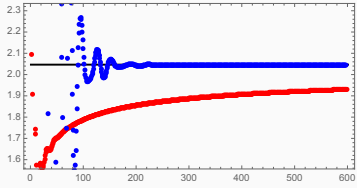
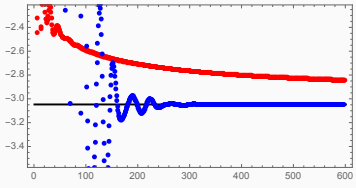
Free Boson ($\tau = \mathrm{i}$)



s_n



s'_n



s''_n

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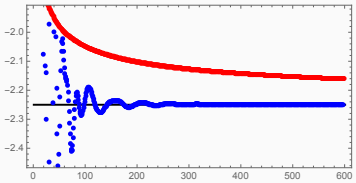
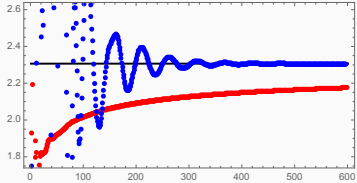
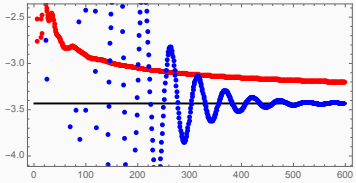
Free boson ($\tau = \mathrm{i}9/8$)



s_n



s'_n



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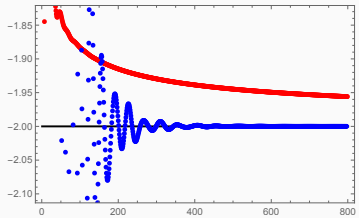

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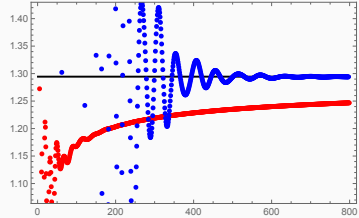

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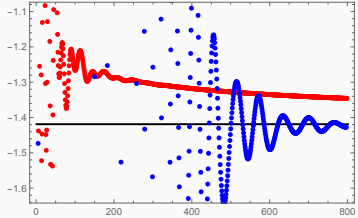

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Resurgence in a nutshell

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- **Resurgent relation**: Possible non-perturbative contribution of the type

$$\varphi^{(\text{np})}(z) \sim \mathrm{e}^{-A/z} z^{-\nu} \sum_n b_n z^n$$

which controls large order behavior of a_n by

$$a_n \sim \frac{\Gamma(n+\nu)}{A^{n+\nu}} \left(b_0 + \frac{b_1 A}{n+\nu-1} + \dots \right).$$

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$$\phi = \int \mathrm{e}^{-S(x)/z} \mathrm{d}x, \quad \text{with } \partial_x S(x_*) = 0.$$

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  - If $A > 0$, the np contribution may exist.
  - If $A < 0$, the np contribution usually do not exist, but may appear after $z < 0$.

Resurgent relation

- In the case of $\mathrm{T}\overline{\mathrm{T}}$ deformed CFT, the *unusual* large order behavior of pert. coefficients

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- More details are revealed in the Thursday Seminar next week!

Saddle point analysis

- $\mathrm{T}\overline{\mathrm{T}}$ deformed partition function has the integral form [Dukovsky, Gorbenko, Hernandez-Chifflet,'18; Hashimoto, Kutasov'19]

$$Z(\tau,\bar{\tau};\lambda)=\frac{\tau_2}{\pi\lambda}\int_{\mathcal{H}^+}\frac{\mathrm{d}^2\zeta}{(\zeta_2)^2}\mathrm{e}^{-\frac{|\zeta-\tau|^2}{\lambda\tau_2}}Z_{\mathsf{CFT}}(\zeta,\bar{\zeta}).$$

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- Saddle point equations

$$\partial_{\zeta_{1,2}}\left(V_0(\zeta^*)+\lambda V_1(\zeta^*)\right)=0$$

with

$$V_0=|\zeta-\tau|^2/\tau_2,\quad V_1=2\log\zeta_2-\log Z_{\mathrm{CFT}}(\zeta,\bar{\beta})$$

Type I and type II saddle points

- *Type I saddle points*: V_1 finite so that λV_1 is negligible in limit $\lambda \to 0$

$$\text{Trivial saddle}: \quad (\zeta_1, \zeta_2)^* = (\tau_1, \tau_2) \;\Rightarrow\; \varphi^{(\text{pert})}.$$

$$\text{Non-trivial saddle}: \quad (\zeta_1, \zeta_2)^* = (\tau_1, -\tau_2) \;\Rightarrow\; \varphi^{(\text{np}),I} \sim \mathrm{e}^{+4\tau_2/\lambda}(\ldots).$$

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- *Type II saddle points*: V_1 singular so that λV_1 has non-zero contribution at limit $\lambda \to 0$

$$\text{Non-trivial saddle}: \quad \lim_{\lambda\to 0}(\zeta, \bar{\zeta})^* = (0, \mathrm{i}\infty)$$

which leads to $\varphi^{(\text{np}),II} = \varphi^{(\text{np})} \sim \mathrm{e}^{-A/\lambda - B/\sqrt{\lambda}}\lambda^{-\nu}$ with the correct A, B, ν

$$A = -2\tau_2, \;\; B = \mathrm{i}(1+\tau_2)\sqrt{2\pi c/3}.$$

Stokes phenomenon

- **Stokes phenomenon**: Both $\varphi^{(\mathrm{np}),I}$ and $\varphi^{(\mathrm{np}),II}$

$$\varphi^{(\mathrm{np}),I} \sim \mathrm{e}^{4\tau_2/\lambda}, \quad \varphi^{(\mathrm{np}),II} \sim \mathrm{e}^{2\tau_2/\lambda - \mathrm{i}(1+\tau_2)\sqrt{2\pi c/(3\lambda)}}$$

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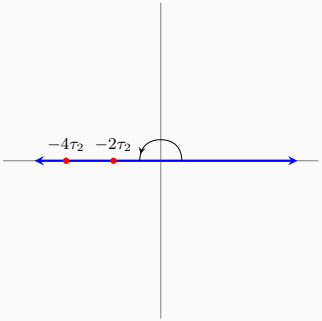
- But they begin to contribute when $\lambda < 0$ due to the Stokes phenomeon.



Conclusion and outlook

- We focus on torus partition function of $\mathrm{T}\overline{\mathrm{T}}$ deformed CFT

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- In the future, we wish to understand
  - implication of non-perturbative contributions for holography, and
  - wiggles of asymptotics of Z_n before settling down.

Thank you for your attention!