Eclectic Flavor Groups

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In collaboration with

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Based on **arXiv: 2303.02071 [JHEP 05(2023)144]**

彭桓武高能基础理论研究中心

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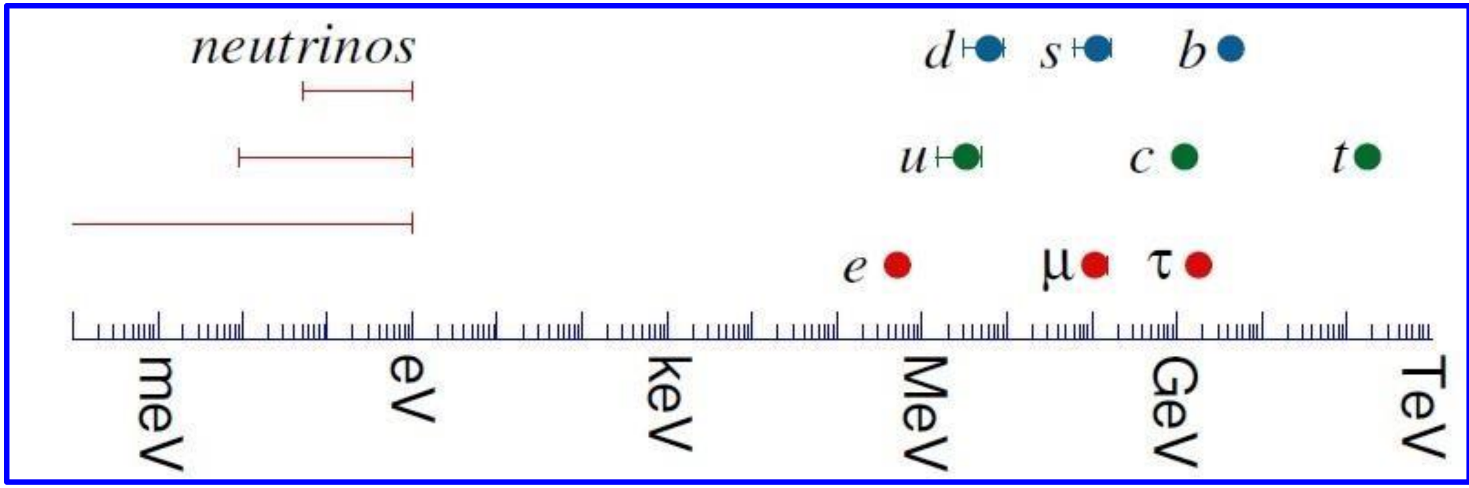
Outline

1. Motivation
2. Eclectic Flavor Groups
3. The EFG $\Omega(1) \cong \Delta(27) \rtimes T'$
4. Effective action invariant under $\Omega(1)$
5. $\Omega(1)$ eclectic lepton model
6. Conclusions

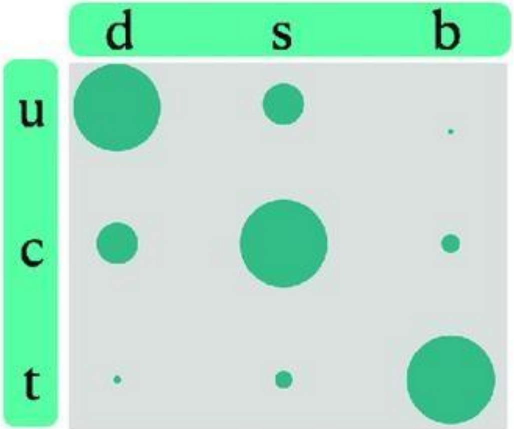
1. Motivation Flavor puzzle

[Z.Z.Xing, 1909.09610]

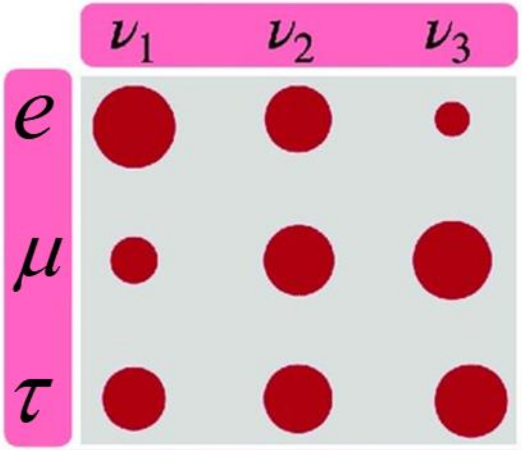
Mass hierarchy



Mixing



Quark CKM mixing matrix



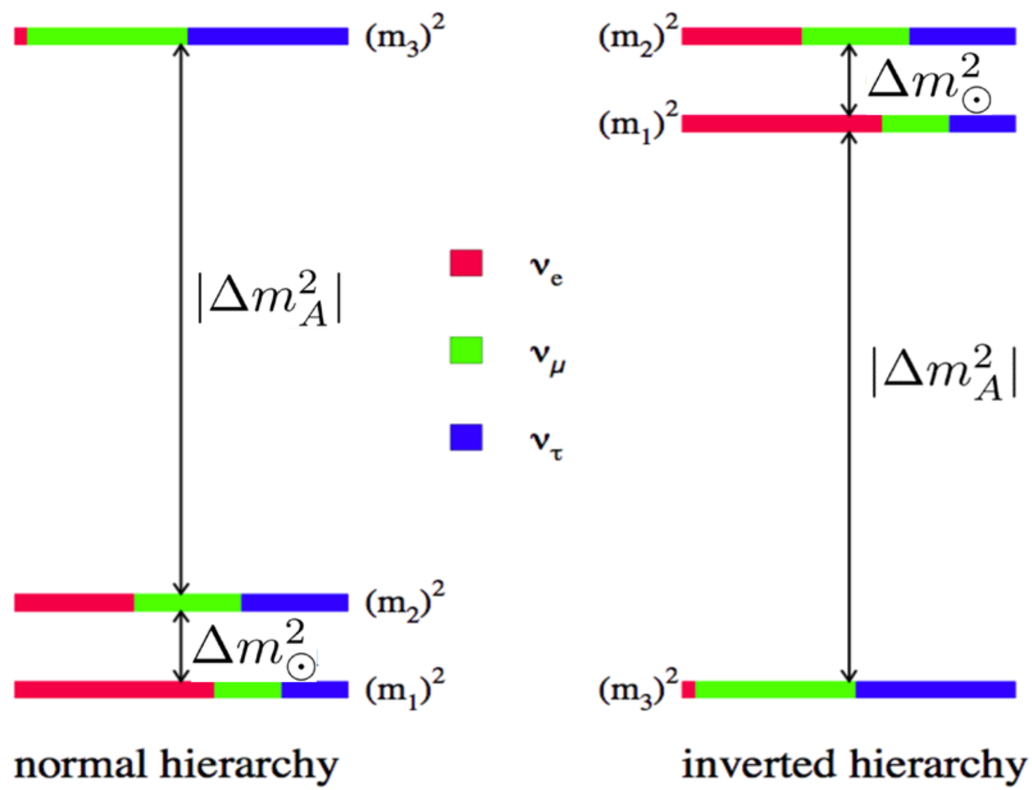
Leptonic PMNS mixing matrix

NO or IO?

3ν flavour paradigm

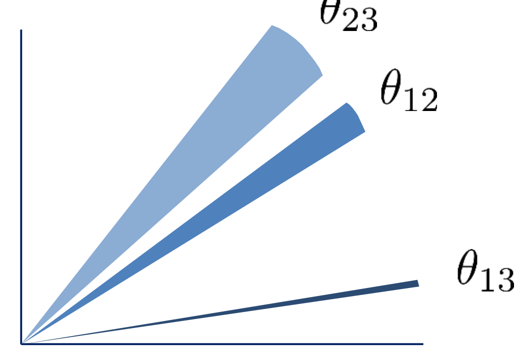
Masses: ordering

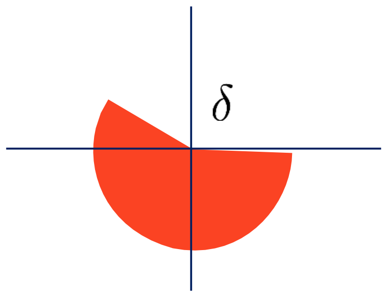
$$\frac{\Delta m^2_\odot}{|\Delta m^2_A|} \sim \frac{1}{30}$$



NuFIT 5.2 (2022)

with SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.4$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \to 0.603$	$0.569^{+0.016}_{-0.021}$	$0.412 \to 0.613$
$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.5$
$\sin^2\theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02052 \to 0.02398$	$0.02223^{+0.00058}_{-0.00058}$	$0.02048 \to 0.02416$
$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$
$\delta_{CP}/^\circ$	232^{+36}_{-26}	$144 \to 350$	276^{+22}_{-29}	$194 \to 344$
$\frac{\Delta m^2_{21}}{10^{-5}\ eV^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ eV^2}$	$+2.507^{+0.026}_{-0.027}$	$+2.427 \to +2.590$	$-2.486^{+0.025}_{-0.028}$	$-2.570 \to -2.406$





Unkonw:

① θ_{23} >45° or θ_{23} <45° ?

② CP : δ_{CP} , α_{21}, α_{31} ?

③neutrino mass ordering and absolute masses

Hyper-K

DEEP UNDERGROUND NEUTRINO EXPERIMENT

Before Daya Bay

[P.F. Harrison,W.G. Scott, Phys. Lett. B 535 (2002) 163-169]

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\theta_{12}^{TB} = 35.26^\circ, \theta_{23}^{TB} = 45^\circ, \theta_{13}^{TB} = 0^\circ$$

[V. D. Barger, S. Pakvasa, T. J. Weiler, K. Whisnant, Phys.Lett.B 437 (1998) 107-116]

$$U_{BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\theta_{12}^{BM} = \theta_{23}^{BM} = 45^\circ, \theta_{13}^{BM} = 0^\circ$$

$$U_{GR} = \begin{pmatrix} -\sqrt{\frac{5+\sqrt{5}}{10}} & \sqrt{\frac{5-\sqrt{5}}{10}} & 0 \\ \frac{1}{\sqrt{5+\sqrt{5}}} & \frac{1}{\sqrt{5-\sqrt{5}}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{5+\sqrt{5}}} & \frac{1}{\sqrt{5-\sqrt{5}}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

[A. Datta, F.-S. Ling, P. Ramond, Nucl.Phys.B 671 (2003) 383-400]

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A 2-3 rotation to TB mixing

[X.G. He, A. Zee, Phys.Rev.D 84 (2011) 053004]

$$U_{PMNS}=U_{TB}R_{23}=\begin{pmatrix}\sqrt{\frac{2}{3}} & \frac{\cos\theta}{\sqrt{3}} & \frac{e^{-i\delta}\sin\theta}{\sqrt{3}}\\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}}+\frac{e^{i\delta}\sin\theta}{\sqrt{2}} & \frac{e^{-i\delta}\sin\theta}{\sqrt{3}}-\frac{\cos\theta}{\sqrt{2}}\\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}}-\frac{e^{i\delta}\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}}+\frac{e^{-i\delta}\sin\theta}{\sqrt{3}}\end{pmatrix}$$

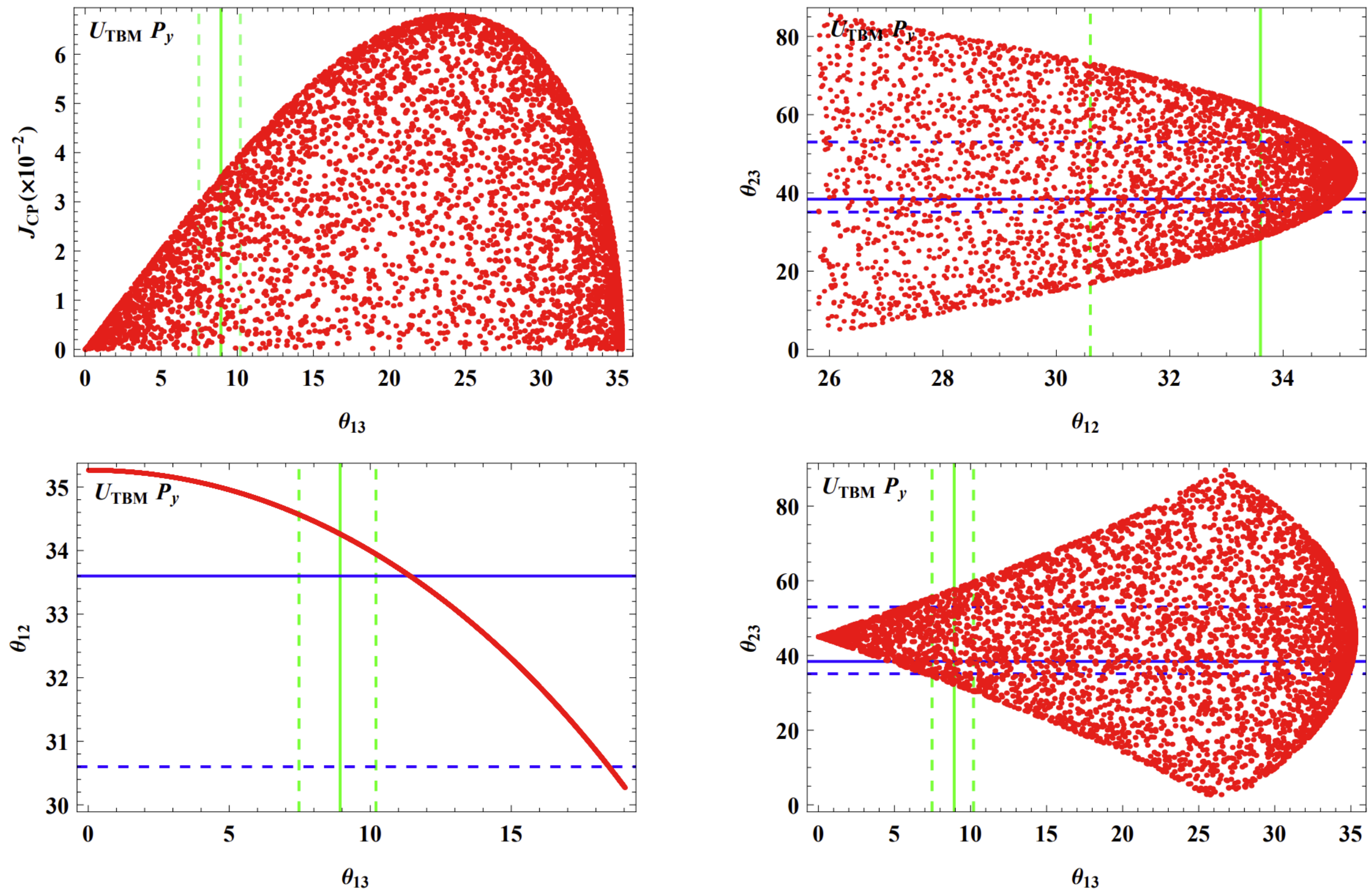
with $$R_{23}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta e^{-i\delta}\\ 0 & -\sin\theta e^{i\delta} & \cos\theta\end{pmatrix}$$

$$\sin^2\theta_{13}=\frac{1}{3}\sin^2\theta,\quad \sin^2\theta_{12}=\frac{\cos^2\theta}{2+\cos^2\theta}$$

$$\sin^2\theta_{23}=\frac{1}{2}-\frac{\sqrt{6}\sin2\theta\cos\delta}{5+\cos2\theta},\quad J_{\rm CP}=-\frac{\sin2\theta\sin\delta}{6\sqrt{6}}.$$

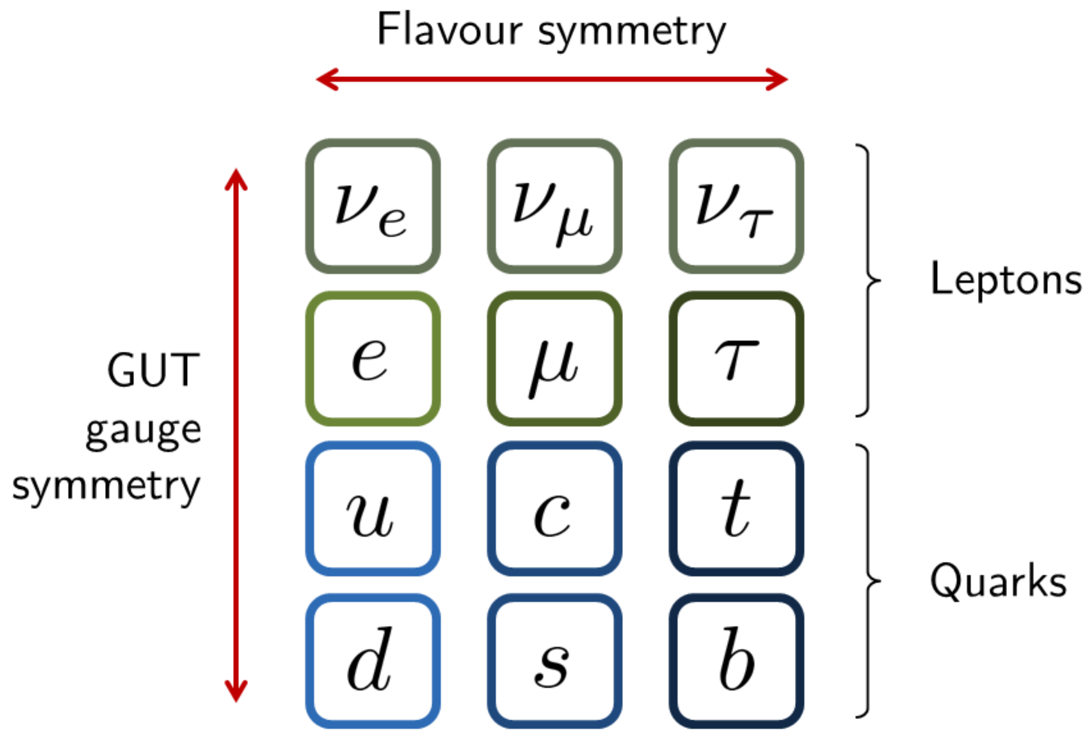
[Bin Wang, Jian Tang, Xue-Qian Li, Phys.Rev.D 88 (2013) 073003]

The correlations corresponding to the mixings

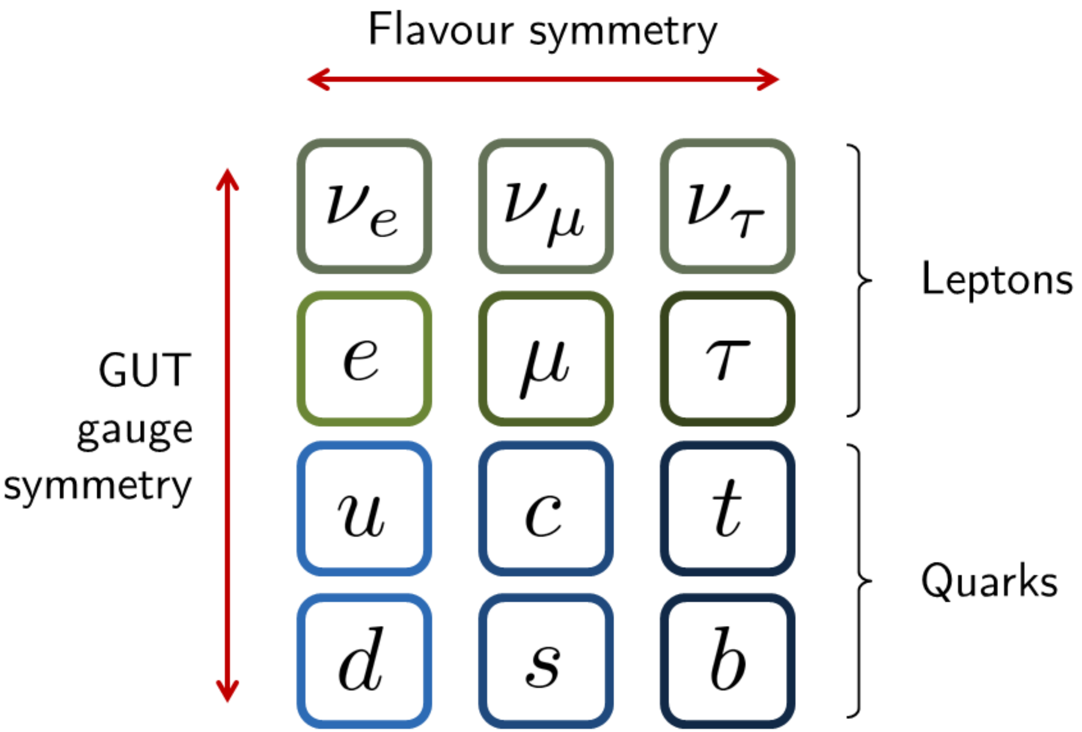


Is there an organizing principle behind the flavor structure of SM?

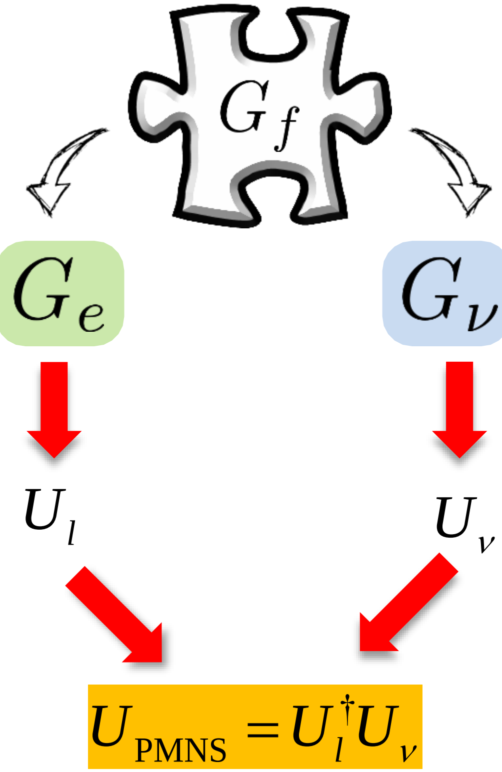
Flavor symmetries



Flavor symmetries



Non Abelian discrete flavor symmetry



constrain mixing and Dirac phase

[Altarelli and Feruglio, Nucl.Phys.B 741 (2006) 215-235;
C.S. Lam, Phys.Lett.B 656 (2007) 193-198;
Feruglio, Romanino, Rev.Mod.Phys.93(2021);
Altarelli and Feruglio, Rev. Mod. Phys. 82, 2701 (2010)…]

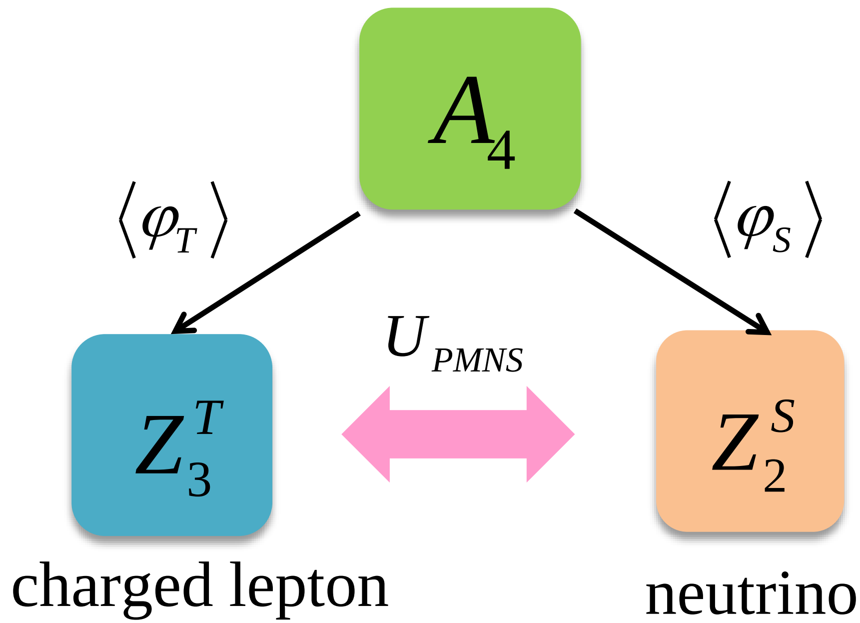
[Altarelli and Feruglio, Nucl.Phys.B 741 (2006) 215-235]

Field	L	e^c	μ^c	τ^c	$h_{u,d}$	φ_T	φ_S	ξ	$\tilde{\xi}$	φ_0^T	φ_0^S	ξ_0
A_4	3	1	$1'$	$1''$	1	3	3	1	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	ω	1	ω	ω
$U(1)_R$	1	1	1	1	0	0	0	0	0	2	2	2

matter fields & Higgs | flavons | driving fields

- Vacuum alignment：

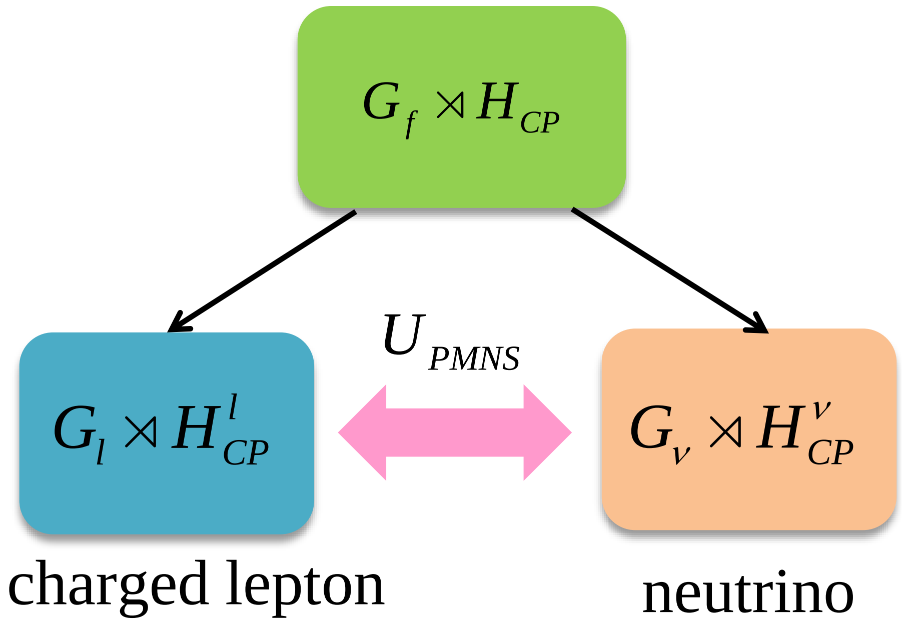
$$\langle\varphi_T\rangle = (v_T, 0, 0)$$
$$\langle\varphi_S\rangle = (v_S, v_S, v_S)$$
$$\langle\xi\rangle = u, \quad \langle\tilde{\xi}\rangle = 0$$



- Mixing matrix：

$$U_{PMNS} = U_{TB}$$

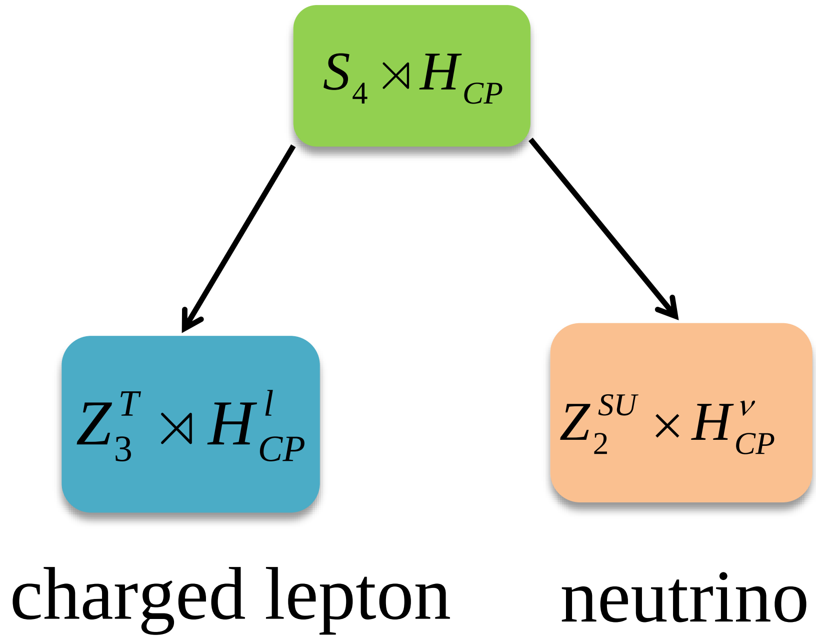
Flavor symmetries + gCP



Constrain mixing angles, Dirac phase and Majorana phase

[Feruglio, Hagedorn, Ziegler , JHEP 07 (2013) 027;
Holthausen, Lindner, Schmidt, JHEP 04 (2013) 122;
Ding, King, Luhn, Stuart, JHEP 05 (2013) 084;
Chen, Fallbacher, Mahanthappa, Ratz, Trautner, Nucl.Phys.B 883 (2014) 267-305…]

[C. C. Li and G. J. Ding, Nucl.Phys.B 881 (2014) 206-232]



[C. C. Li and G. J. Ding, Nucl.Phys.B 881 (2014) 206-232]

- Mixing matrix:

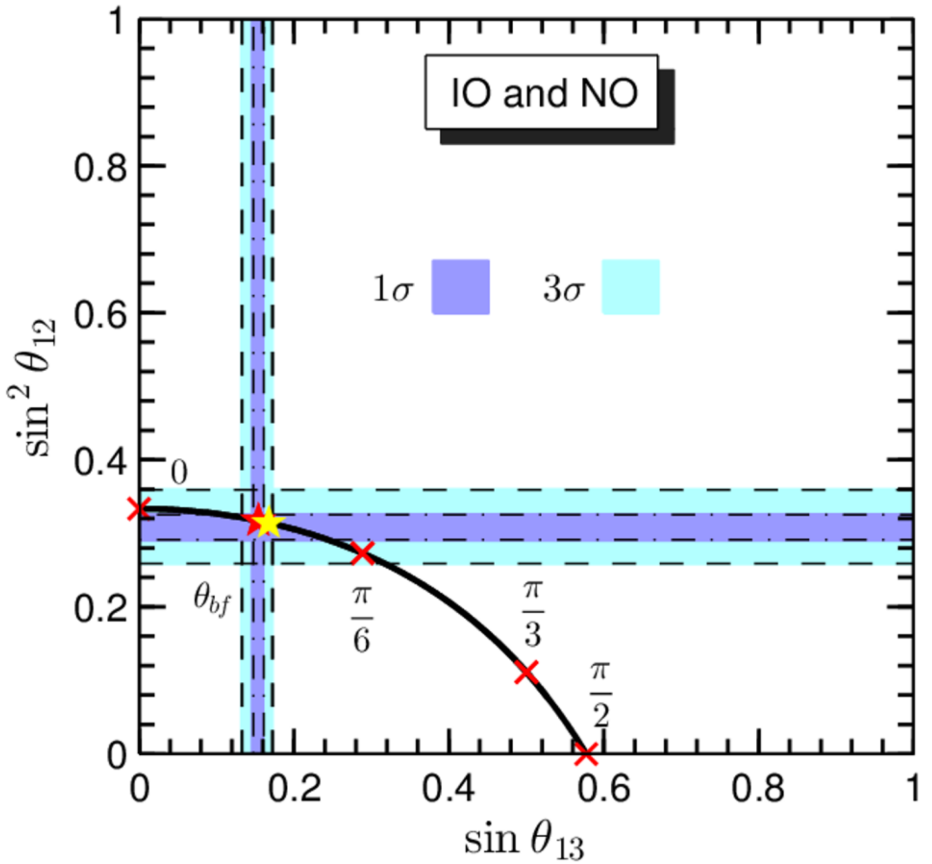
$$U_{PMNS}=\begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{\cos\theta}{\sqrt{3}} & \frac{\sin\theta}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}}+\frac{i\sin\theta}{\sqrt{2}} & -\frac{i\cos\theta}{\sqrt{2}}+\frac{\sin\theta}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{\cos\theta}{\sqrt{3}}-\frac{i\sin\theta}{\sqrt{2}} & \frac{i\cos\theta}{\sqrt{2}}+\frac{\sin\theta}{\sqrt{3}} \end{pmatrix}$$

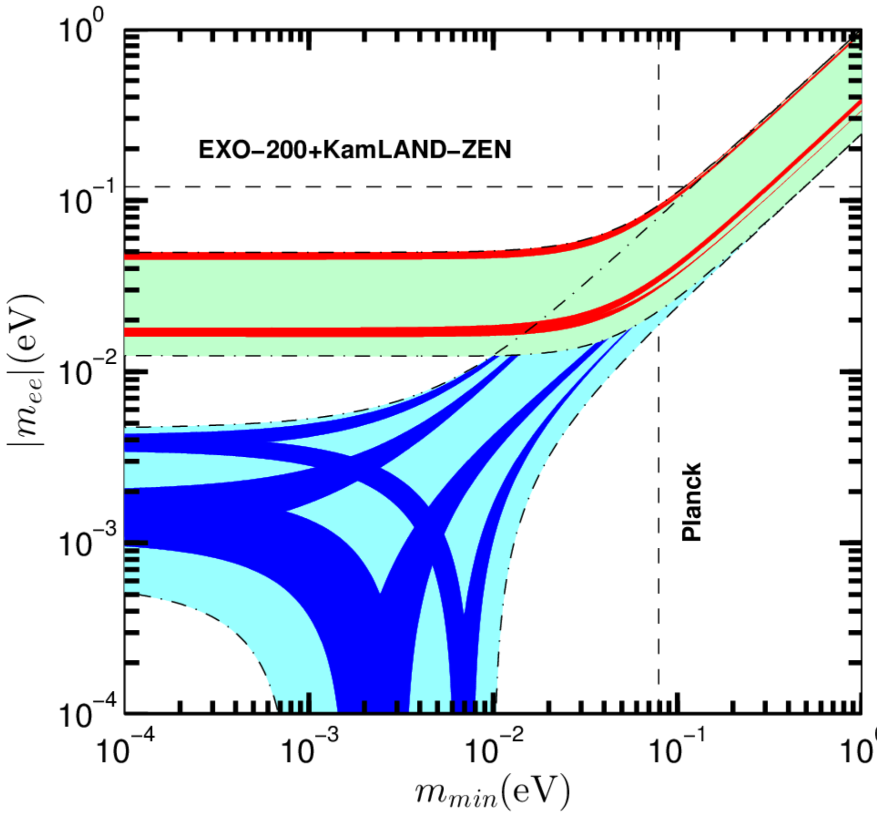
- Mixing parameters:

$$|\sin\delta_{CP}|=1, \quad \sin\alpha_{21}=\sin\alpha_{31}=0, \quad \sin^2\theta_{23}=\frac{1}{2}$$

$$\sin^2\theta_{13}=\frac{1}{3}\sin^2\theta, \quad \sin^2\theta_{12}=\frac{\cos^2\theta}{2+\cos^2\theta}=\frac{1}{3}-\frac{2}{3}\tan^2\theta_{13}$$

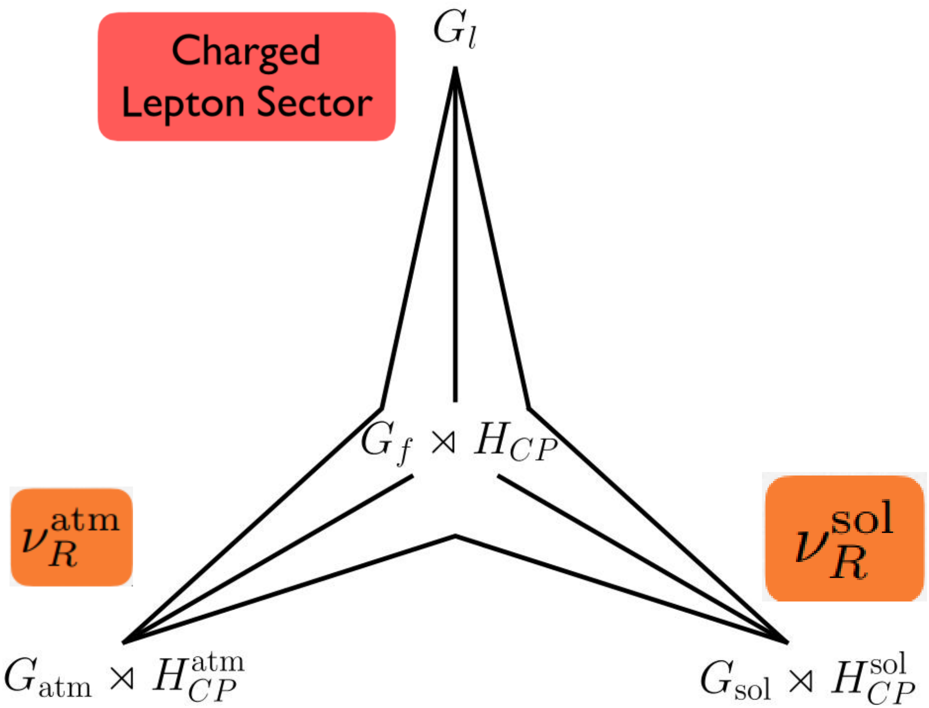
[C. C. Li and G. J. Ding, Nucl.Phys.B 881 (2014) 206-232]

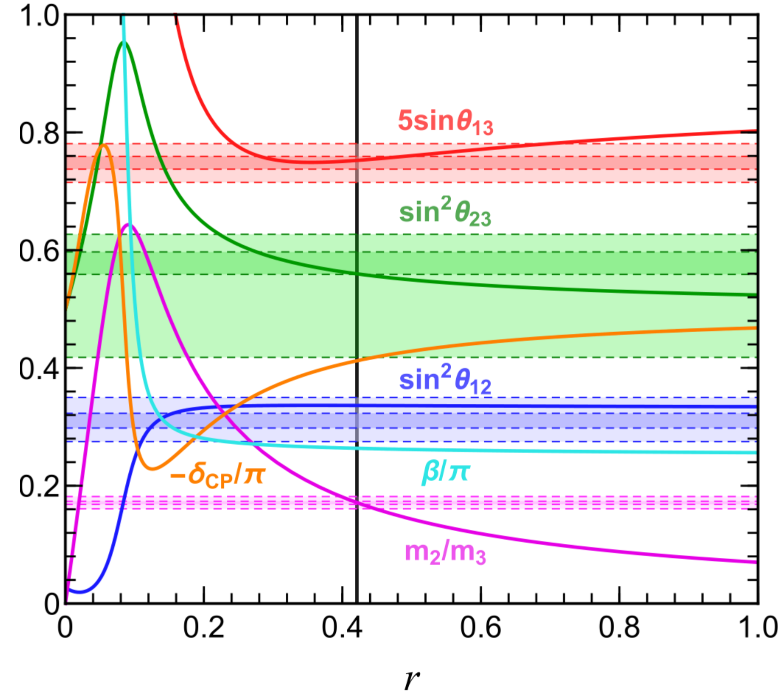




Correlations between $\sin^2\theta_{12}, \sin^2\theta_{23}$ and the effective mass

Tri-Direct CP approach



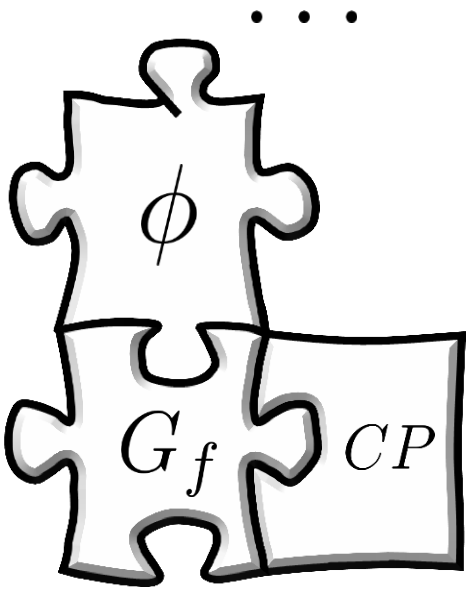


G.J.Ding, S.F.K. and C.C.Li, JHEP 12 (2018) 003

Constrain mixing angles, Dirac phase, Majorana phase and neutrino masses

Problems with the traditional approach

J.T. Penedo
FLASY 2019



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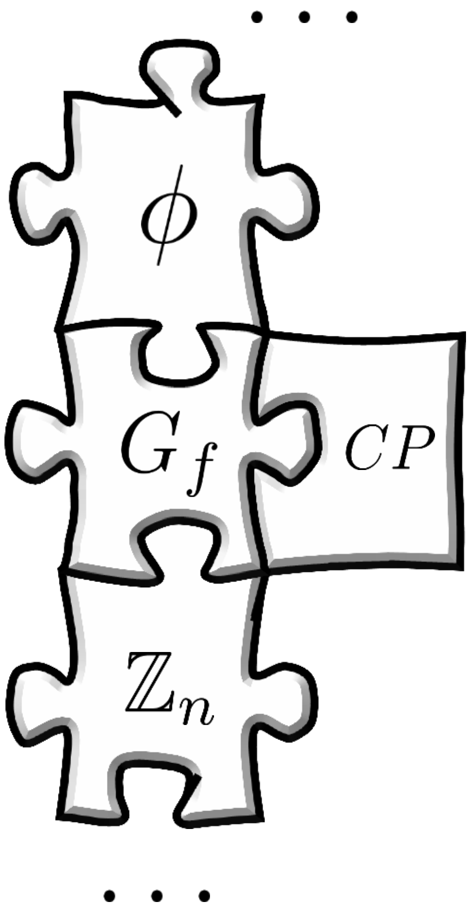

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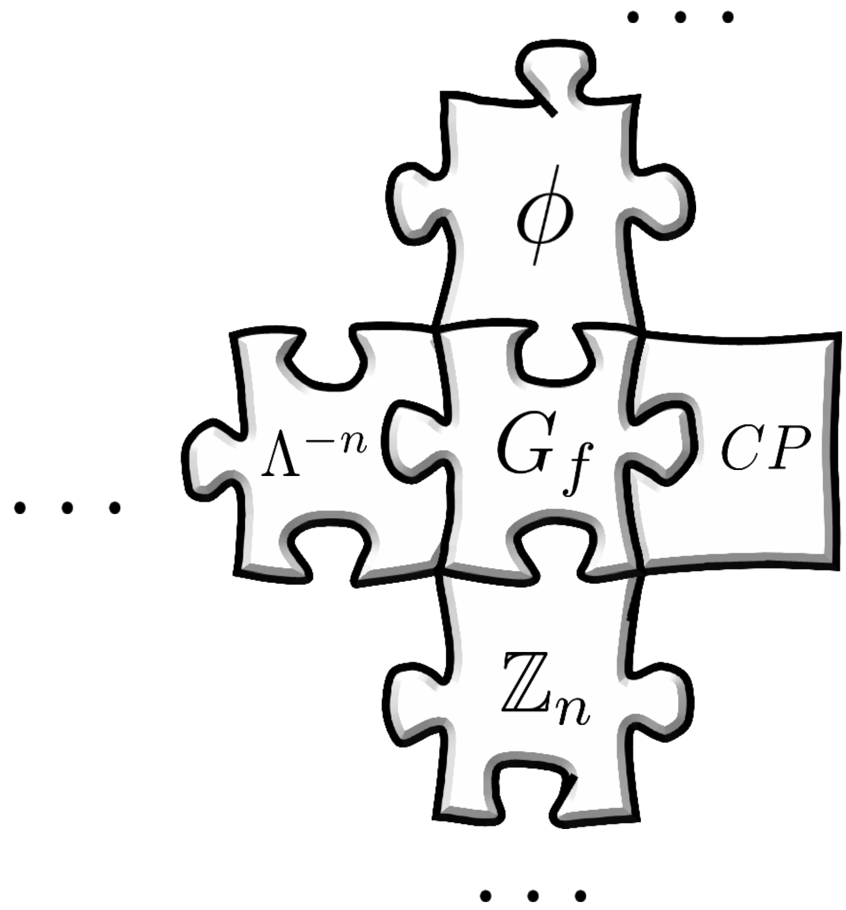

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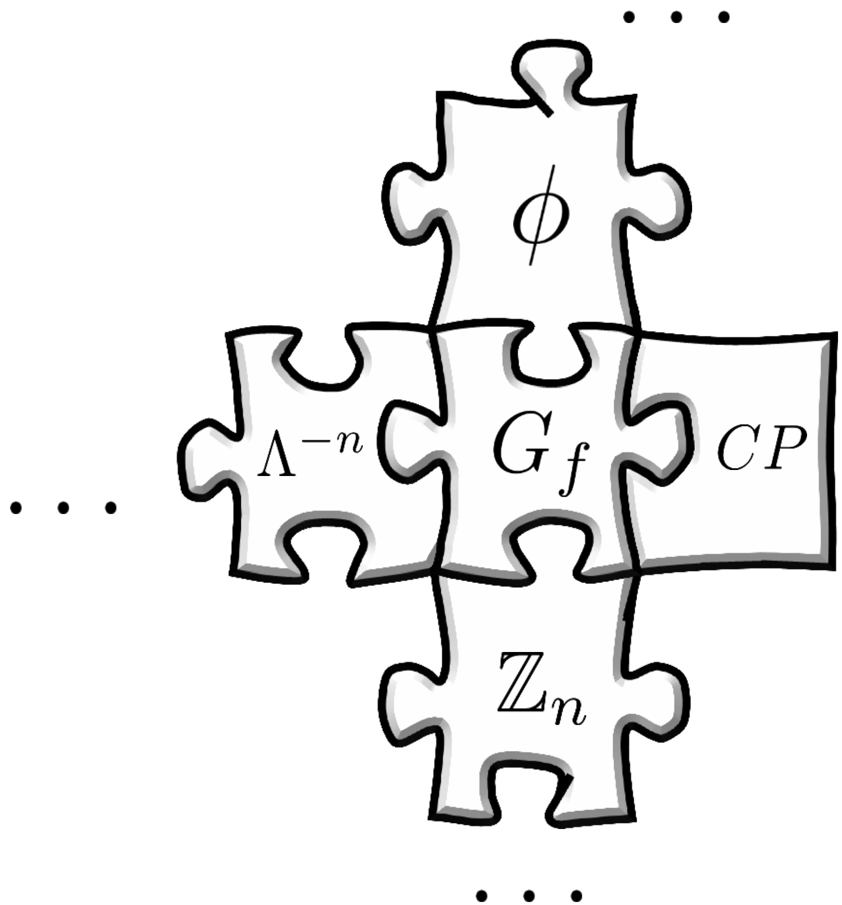

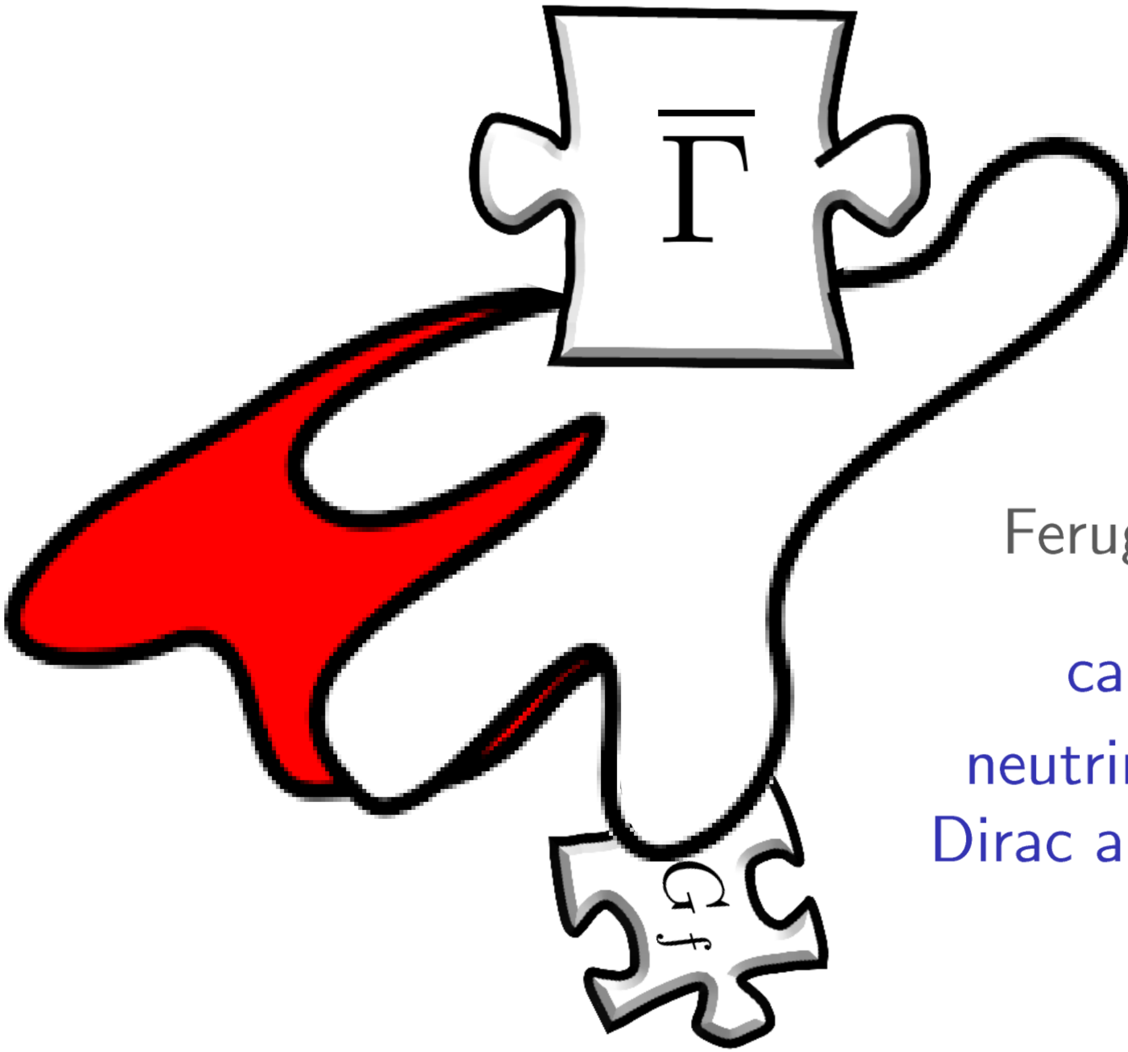

J.T. Penedo
FLASY 2019

Feruglio, 1706.08749

can constrain all:

neutrino masses, mixing,
Dirac and Majorana phases

Modular symmetry

:

$$\Gamma \cong SL(2, \mathbb{Z})$$

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

$$a, b, c, d \in \mathbb{Z} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$$

$$\begin{cases} S:\ \tau \to -1/\tau\,, & S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ T:\ \tau \to \tau + 1\,, & T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{cases}$$

$$S^4 = (ST)^3 = \mathbb{1}_2, \qquad S^2T = TS^2$$

Modular symmetry

Quotient behaves like a flavor group

[Feruglio, 1706.08749…]

$$\underbrace{\Gamma / \pm\Gamma(N)}_{\Gamma_N}$$

$$\underbrace{\Gamma / \Gamma(N)}_{\Gamma'_N}$$

[Liu,Ding,1907.01488]

$$\Gamma(N) = \left\{ \gamma \in SL(2,\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

Modular symmetry

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[Feruglio, 1706.08749…]

$$\underbrace{\Gamma/_{\pm\Gamma(N)}}_{\Gamma_N}$$

$$\underbrace{\Gamma/_{\Gamma(N)}}_{\Gamma'_N}$$

[Liu,Ding,1907.01488]

$$S^{N_s} = (ST)^3 = T^N = 1, \qquad S^2T = TS^2$$

$$N_s = 2 \quad \Rightarrow \quad \Gamma_N, \qquad\qquad N_s = 4 \quad \Rightarrow \quad \Gamma'_N$$

Modular invariant theory [Ferrara et al, 1989; Feruglio, 1706.08749]

- **$\mathcal{N}$=1 global supersymmetry theory with modular symmetry:**

$$\mathcal{S} = \int d^4 x d^2\theta d^2\bar{\theta} \mathcal{K}(\psi_I, \bar{\psi}_I; \tau, \bar{\tau}) + \int d^4 x d^2\theta \mathcal{W}(\psi_I, \tau) + \mathrm{h.c.}$$

- Minimal Kahler potential :

$$\mathcal{K} = -h \ln(-i\tau + i\bar{\tau}) + \sum_n (-i\tau + i\bar{\tau})^{-k_n} |\psi_n|^2$$

- Superpotential :

$$\mathcal{W} = \sum_n Y_{I_1 I_2 \ldots I_n}(\tau) \psi_{I_1} \psi_{I_2} \ldots \psi_{I_n}$$

- **Modular invariance requires Yukawa couplings are Modular Forms!**

$$\psi_{I_i} \xrightarrow{\gamma} (c\tau + d)^{-k_{\psi_{I_i}}} \rho_{I_i}(\gamma) \psi_{I_i}$$

$$Y_{I_1 I_2 \ldots I_n}(\tau) \to Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$

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weight

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weight

Irreps of finite modular groups

Modular invariant theory [Ferrara et al, 1989; Feruglio, 1706.08749]

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weight

Irreps of finite modular groups

$$\begin{cases} k_Y = k_{I_1} + k_{I_2} + \cdots + k_{I_n} \\ \rho_Y \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_n} \supset 1 \end{cases}$$

Finite modular groups

	Γ_N	Γ'_N	
[Kobayashi et al, 1803.10391…]	$\Gamma_2 \simeq S_3$	$\Gamma'_2 \simeq S_3$	[Kobayashi et al, 1803.10391…]
[Feruglio, 1706.08749…]	$\Gamma_3 \simeq A_4$	$\Gamma'_3 \simeq T'$	[Liu,Ding,1907.01488…]
[Novichkov et al, 1811.04933…]	$\Gamma_4 \simeq S_4$	$\Gamma'_4 \simeq S'_4$	[Liu,Yao,Ding,2006.10722]
[Novichkov et al, 11812.02158…]	$\Gamma_5 \simeq A_5$	$\Gamma'_5 \simeq A'_5$	[Yao,Liu,Ding,2011.03501…]
	$\Gamma_7 \simeq \Sigma(168)$	$\Gamma'_6 \cong S_3 \times T'$	

[Ding, King, Li, Zhou, 2004.12662(JHEP)]

[Li, Liu, Ding, 2108.02181(JHEP)]

❑ Remarks:

- Freedom of model building in this bottom-up approach: k_I, ρ_I
- For a given k_Y, ρ_Y , the modular forms space is finite-dimensional

  ➡ Only finite possible Yukawa couplings! Highly predictive !

N	$\dim\mathcal{M}_k(\Gamma(N))$	Γ_N (Γ'_N)	Modular forms multiplets			
			$k=1$	$k=2$	$k=3$	$k\geq 4$
2	$k/2+1$ ($k\in$ even)	S_3 (S_3)	$-$	$Y^{(2)}_{\mathbf{2}}$	$-$	...
3	$k+1$	A_4 (T')	$Y^{(1)}_{\mathbf{2}}$	$Y^{(2)}_{\mathbf{3}}$	$Y^{(3)}_{\mathbf{2}}$, $Y^{(3)}_{\mathbf{2}''}$	...
4	$2k+1$	S_4 (S'_4)	$Y^{(1)}_{\hat{\mathbf{3}}'}$	$Y^{(2)}_{\mathbf{2}}$, $Y^{(2)}_{\mathbf{3}}$	$Y^{(3)}_{\hat{\mathbf{1}}'}$, $Y^{(3)}_{\hat{\mathbf{3}}}$, $Y^{(3)}_{\hat{\mathbf{3}}'}$	...
5	$5k+1$	A_5 (A'_5)	$Y^{(1)}_{\mathbf{6}}$	$Y^{(2)}_{\mathbf{3}}$, $Y^{(2)}_{\mathbf{3}'}$, $Y^{(2)}_{\mathbf{5}}$	$Y^{(3)}_{\mathbf{4}'}$, $Y^{(3)}_{\mathbf{6}I}$, $Y^{(3)}_{\mathbf{6}II}$	...

[Kobayashi, Tanaka, and Tatsuishi 2018; Feruglio 2017; Penedo and Petcov 2019; Novichkov et al. 2019; Ding, King, and Liu 2019b; Liu and Ding 2019; Liu, Yao, and Ding 2021; Novichkov, Penedo, and Petcov 2021; Wang, Yu, and Zhou 2021; Yao, Liu, and Ding 2021]

❑ Drawback: The Kahler potential is not under control!

[Chen, Ramos-Sanchez, Ratz 1909.06910]

$$\mathcal{K} \supset \sum_{\Phi_n} \sum_{k\geq 1} (-i\tau + i\bar{\tau})^{-k+k_n} \sum_a \kappa_a^{(k)} \left[Y^{(k)}(\tau) \otimes \bar{Y}^{(k)}(\tau) \otimes \psi_n \otimes \bar{\psi}_n \right]_{1,a}$$

The Kahler potential under control

- In traditional flavor symmetry the Kahler potential is under control [Chen M.C. et al, 1208.2947,…]

$$\mathcal{K} \supset (\varphi L\bar{L})_{1_0} / \Lambda \quad \text{and/or} \quad \left(\varphi\bar{\varphi}L\bar{L}\right)_{1_0} / \Lambda^2 .$$

- Combine the advantages of both approaches traditional flavor symmetry and modular symmetry

$$G_{\text{flavor}} = G_{\text{traditional}} \times G_{\text{modular}}$$

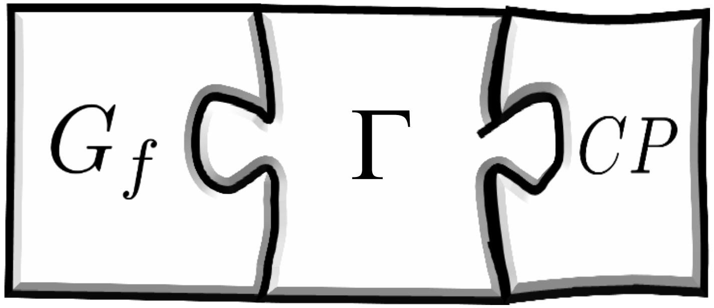
[Chen M.C. et al, 2108.02240]

$$G_{\text{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}}$$

[A. Baur et al, 1901.03251;
[A. Baur et al, 1908.00805]

Eclectic flavor groups

How does this work?



[Hans Peter Nilles et al, 2001.01736;
Hans Peter Nilles et al, 2004.05200]

2. Eclectic Flavor Groups

- Transformation properties

TFS:

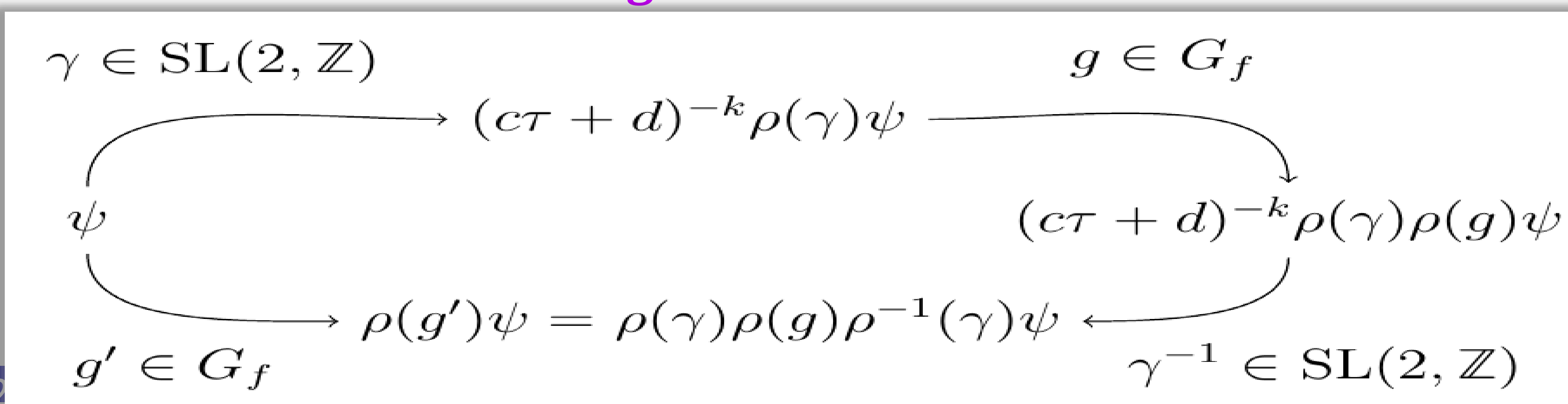
$$\psi \xrightarrow{g} \rho(g)\psi, \qquad g \in G_f \,,$$

MS:

$$\tau \xrightarrow{\gamma} \gamma\tau \equiv \frac{a\tau+b}{c\tau+d},$$

$$\psi \xrightarrow{\gamma} (c\tau+d)^{-k_\psi}\rho(\gamma)\psi, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,,$$

Combining MS with TFS



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Combining MS with TFS

$\gamma \in \mathrm{SL}(2,\mathbb{Z})$ $\qquad$ $g \in G_f$

$$\psi \longrightarrow (c\tau+d)^{-k}\rho(\gamma)\psi \longrightarrow (c\tau+d)^{-k}\rho(\gamma)\rho(g)\psi$$

$$\rho(\gamma)\rho(g)\rho^{-1}(\gamma) = \rho(g'), \qquad g, g' \in G_f, \quad \gamma \in \Gamma.$$

$$\psi \longrightarrow \rho(g')\psi = \rho(\gamma)\rho(g)\rho^{-1}(\gamma)\psi \longleftarrow (c\tau+d)^{-k}\rho(\gamma)\rho(g)\psi$$

$g' \in G_f$ $\qquad$ $\gamma^{-1} \in \mathrm{SL}(2,\mathbb{Z})$

- The consistency condition

$$\rho(\gamma)\rho(g)\rho^{-1}(\gamma) = \rho(u_\gamma(g))\,, \qquad \forall g \in G_f\,,$$

$\rho(\gamma)$ ↓

$$u_\gamma : G_f \to G_f$$

↓

$$G_{ecl} \cong G_f \rtimes \Gamma'_N\ (G_f \rtimes \Gamma_N)\,.$$

Finite modular group must be subgroup of the outer automorphism group of G_f.

- It is sufficient to only discuss:

$$\rho(S)\,\rho(g)\,\rho^{-1}(S) = \rho(u_S(g)),$$
$$\rho(T)\,\rho(g)\,\rho^{-1}(T) \;=\; \rho(u_T(g))\,.$$

- The finite modular group:

$$\Gamma_N = \left\{S, T \middle| S^2 = (ST)^3 = T^N = 1\right\},$$

$$\Gamma'_N = \left\{S, T \middle| S^4 = (ST)^3 = T^N = 1,\ S^2T = TS^2\right\}, \qquad N \leq 5.$$

$$(u_S)^{N_s} = (u_T)^N = (u_S \circ u_T)^3 = 1, \qquad (u_S)^2 \circ u_T = u_T \circ (u_S)^2 \ ,$$

$$N_s = 2 \quad \Rightarrow \quad \Gamma_N, \qquad\qquad N_s = 4 \quad \Rightarrow \quad \Gamma'_N$$

[Hans Peter Nilles et al, 2001.01736]

EFG and GCP

- The consistency between the modular symmetry and gCP

$$\rho(K_*)\rho^*(S)\rho^{-1}(K_*) = \rho^{-1}(S), \qquad \rho(K_*)\rho^*(T)\rho^{-1}(K_*) = \rho^{-1}(T)\,.$$

[Novichkov et al, 1905.11970;
Ding et al, 2102.06716]

- The consistency between the traditional symmetry and gCP

$$\rho(K_*)\rho^*(g)\rho^{-1}(K_*) = \rho(u_{K_*}(g))\,, \qquad \forall g \in G_f\,,$$

[Feruglio, Hagedorn, Ziegler , JHEP 07 (2013) 027;
Holthausen, Lindner, Schmidt, JHEP 04 (2013) 122;
Ding, King, Luhn, Stuart, JHEP 05 (2013) 084;
Chen, Fallbacher, Mahanthappa, Ratz, Trautner, Nucl.Phys.B 883 (2014) 267-305…]

- The automorphisms of G_f satisfy

$$(u_S)^{N_s} = (u_T)^N = (u_S \circ u_T)^3 = 1, \qquad (u_S)^2 \circ u_T = u_T \circ (u_S)^2\,,$$
$$(u_{K_*})^2 = 1, \qquad u_{K_*} \circ u_S \circ u_{K_*} = u_S^{-1}, \qquad u_{K_*} \circ u_T \circ u_{K_*} = u_T^{-1}\,,$$

[Hans Peter Nilles et al, 2001.01736]

- **Advantages of EFG:**
  - There are very few candidates for self-consistently EFG groups G_{ecl} when u_γ nontrival: $G_f = \mathbb{Z}_3 \times \mathbb{Z}_3,\ \Delta(27),\ \Delta(54)\ \ldots$
  - The superpotential in EFG models is highly constrained because it satisfies both traditional flavor invariance and modular invariance.
  - Kahler potential is under control due to the traditional flavor symmetry!
  - EFG has a natural UV completion——Heterotic string on orbifold

There is currently no bottom-up EFG model

We choose $\mathbf{G_{ecl} = \Omega(1) \cong \Delta(27) \rtimes T'}$

3. The EFG $\Omega(1) \cong \Delta(27) \rtimes T'$

- The multiplication rules of Δ(27)

$$A^3 = B^3 = (AB)^3 = (AB^2)^3 = 1\,.$$

our working basis

$\omega = e^{2\pi i/3}$

$$\mathbf{1}_{r,s} \;:\quad \rho_{\mathbf{1}_{r,s}}(A) = \omega^r, \qquad \rho_{\mathbf{1}_{r,s}}(B) = \omega^s\,, \quad \text{with} \quad r,s = 0,1,2\,,$$

$$\mathbf{3} \;:\quad \rho_{\mathbf{3}}(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho_{\mathbf{3}}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

$$\bar{\mathbf{3}} \;:\quad \rho_{\bar{\mathbf{3}}}(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho_{\bar{\mathbf{3}}}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

- The outer automorphism group of $\Delta(27)$

$$\mathrm{Out}\left(\Delta(27)\right) \;\cong\; \mathrm{Aut}\left(\Delta(27)\right)/\mathrm{Inn}\left(\Delta(27)\right) \;\cong\; GL(2,3)\,,$$

- The three outer automorphisms u_S , u_T and u_{K*}

$$\begin{aligned} u_S(A) &= B^2A, & u_S(B) &= B^2A^2, \\ u_T(A) &= BA, & u_T(B) &= B\,, \\ u_{K_*}(A) &= A^2B, & u_{K_*}(B) &= A^2BA\,. \end{aligned}$$

[Hans Peter Nilles et al, 2001.01736]

$$(u_S)^4 = (u_T)^3 = (u_S \circ u_T)^3 = 1, \qquad (u_S)^2 \circ u_T = u_T \circ (u_S)^2$$

$$u_S,\ u_T \quad \Rightarrow \quad \Gamma_3' \simeq T'$$

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$$u_S,\ u_T \quad \Rightarrow \quad \Gamma'_3 \cong T'$$

no GCP

EFG: $\Omega(1) \cong \Delta(27) \rtimes T'$.

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no GCP

EFG: $\Omega(1) \cong \Delta(27) \rtimes T'$.

GCP

EFG: [1296, 2891]

- Solving the following consistency conditions

$$\rho_{\boldsymbol{r}}(S)\,\rho_{\boldsymbol{r}}(A)\,\rho_{\boldsymbol{r}}^{-1}(S) \;=\; \rho_{\boldsymbol{r}}(B^2 A), \qquad \rho_{\boldsymbol{r}}(S)\,\rho_{\boldsymbol{r}}(B)\,\rho_{\boldsymbol{r}}^{-1}(S) \;=\; \rho_{\boldsymbol{r}}(B^2 A^2)\,,$$

$$\rho_{\boldsymbol{r}}(T)\,\rho_{\boldsymbol{r}}(A)\,\rho_{\boldsymbol{r}}^{-1}(T) \;=\; \rho_{\boldsymbol{r}}(BA), \qquad \rho_{\boldsymbol{r}}(T)\,\rho_{\boldsymbol{r}}(B)\,\rho_{\boldsymbol{r}}^{-1}(T) \;=\; \rho_{\boldsymbol{r}}(B)\,,$$

	E	A^2B^2	A^2B	A	AB^2	AB	A^2	B^2	B	BAB^2A^2	ABA^2B^2
	$1C_1$	$3C_3^{(1)}$	$3C_3^{(2)}$	$3C_3^{(3)}$	$3C_3^{(4)}$	$3C_3^{(5)}$	$3C_3^{(6)}$	$3C_3^{(7)}$	$3C_3^{(8)}$	$1C_3^{(1)}$	$1C_3^{(2)}$
$\mathbf{1_{0,0}}$	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1_{0,1}}$	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω	1	1
$\mathbf{1_{0,2}}$	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2	1	1
$\mathbf{1_{1,0}}$	1	ω^2	ω^2	ω	ω	ω	ω^2	1	1	1	1
$\mathbf{1_{1,1}}$	1	ω	1	ω	1	ω^2	ω^2	ω^2	ω	1	1
$\mathbf{1_{1,2}}$	1	1	ω	ω	ω^2	1	ω^2	ω	ω^2	1	1
$\mathbf{1_{2,0}}$	1	ω	ω	ω^2	ω^2	ω^2	ω	1	1	1	1
$\mathbf{1_{2,1}}$	1	1	ω^2	ω^2	ω	1	ω	ω^2	ω	1	1
$\mathbf{1_{2,2}}$	1	ω^2	1	ω^2	1	ω	ω	ω	ω^2	1	1
$\mathbf{3}$	3	0	0	0	0	0	0	0	0	3ω	$3\omega^2$
$\bar{\mathbf{3}}$	3	0	0	0	0	0	0	0	0	$3\omega^2$	3ω

- u_S and u_T act on irreducible representations as

$$u_S: \quad \mathbf{1}_{0,0} \to \mathbf{1}_{0,0}, \quad \mathbf{3} \to \mathbf{3}, \quad \bar{\mathbf{3}} \to \bar{\mathbf{3}},$$
$$\mathbf{1}_{0,1} \to \mathbf{1}_{2,2} \to \mathbf{1}_{0,2} \to \mathbf{1}_{1,1} \to \mathbf{1}_{0,1},$$
$$\mathbf{1}_{1,0} \to \mathbf{1}_{1,2} \to \mathbf{1}_{2,0} \to \mathbf{1}_{2,1} \to \mathbf{1}_{1,0}.$$

$$u_T: \quad \mathbf{1}_{0,0} \to \mathbf{1}_{0,0}, \quad \mathbf{3} \to \mathbf{3}, \quad \bar{\mathbf{3}} \to \bar{\mathbf{3}},$$
$$\mathbf{1}_{0,1} \to \mathbf{1}_{1,1} \to \mathbf{1}_{2,1} \to \mathbf{1}_{0,1},$$
$$\mathbf{1}_{0,2} \to \mathbf{1}_{2,2} \to \mathbf{1}_{1,2} \to \mathbf{1}_{0,2},$$
$$\mathbf{1}_{1,0} \to \mathbf{1}_{1,0}, \quad \mathbf{1}_{2,0} \to \mathbf{1}_{2,0}.$$

$\mathbf{1}_{0,0}$, $\mathbf{3}$, $\bar{\mathbf{3}}$ and $\mathbf{8}$

$$\mathbf{8} = \mathbf{1}_{0,1} \oplus \mathbf{1}_{0,2} \oplus \mathbf{1}_{1,0} \oplus \mathbf{1}_{1,1} \oplus \mathbf{1}_{1,2} \oplus \mathbf{1}_{2,0} \oplus \mathbf{1}_{2,1} \oplus \mathbf{1}_{2,2}.$$

$$r = \mathbf{1}_{0,0} : \quad \rho_{\mathbf{1}^k}(S) = 1\,, \qquad \rho_{\mathbf{1}^k}(T) = \omega^k\,,$$

$$r = \mathbf{3} : \quad \rho_{\mathbf{3}_k}(S) = \frac{i}{\sqrt{3}} \begin{pmatrix} \omega^2 & \omega & \omega \\ \omega & \omega^2 & \omega \\ \omega^2 & \omega^2 & 1 \end{pmatrix}, \qquad \rho_{\mathbf{3}_k}(T) = \omega^k \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rightarrow \quad \mathbf{3}_k = \mathbf{1}^{[k+1]} \oplus \mathbf{2}^{[k+2]}\,.$$

$$r = \bar{\mathbf{3}} : \quad \rightarrow \quad \bar{\mathbf{3}}_k = \mathbf{1}^{[2-k]} \oplus \mathbf{2}^{[1-k]}\,.$$

$r = 8:$

$$\rho_{8_k}(S) = \begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{pmatrix}, \rho_{8_k}(T) = \omega^k \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \end{pmatrix},$$

$$\mathbf{8}_k = \mathbf{1}^k \oplus \mathbf{2}^{[k+1]} \oplus \mathbf{2}^{[k+2]} \oplus \mathbf{3},$$

4. Effective action invariant under Ω(1)

Superpotential

Level 3 modular form multiplets

$$Y_{\mathbf{1}}^{(k_Y)}(\tau)=Y_1,\quad Y_{\mathbf{1}'}^{(k_Y)}(\tau)=Y_2,\quad Y_{\mathbf{1}''}^{(k_Y)}(\tau)=Y_3,$$

$$Y_{\mathbf{2}}^{(k_Y)}(\tau)=\begin{pmatrix}Y_4\\Y_5\end{pmatrix},\quad Y_{\mathbf{2}'}^{(k_Y)}(\tau)=\begin{pmatrix}Y_6\\Y_7\end{pmatrix},\quad Y_{\mathbf{2}''}^{(k_Y)}(\tau)=\begin{pmatrix}Y_8\\Y_9\end{pmatrix},\quad Y_{\mathbf{3}}^{(k_Y)}(\tau)=\begin{pmatrix}Y_{10}\\Y_{11}\\Y_{12}\end{pmatrix}.$$

■ The assignment

$$\psi,\psi^c\sim(\mathbf{3},\mathbf{3}_\mathbf{0}),\qquad \Phi\equiv(\phi_1,\phi_2,\phi_3)^T\sim(\mathbf{3},\mathbf{3}_\mathbf{0}).$$

Superpotential:

$$\mathcal{W}_D=\frac{1}{\Lambda}\sum_{\mathbf{r},s}c_{\mathbf{r},s}\left(Y_{\mathbf{r}}^{(k_Y)}\Phi\psi^c\psi\right)_{(\mathbf{1}_{\mathbf{0},\mathbf{0}},\mathbf{1}),s}H_{u/d},$$

■ Invariance under the action of $\Delta(27)$

$$\mathcal{W}_D = \frac{1}{\Lambda}\sum_{\mathbf{r}}\left(Y_{\mathbf{r}}^{(k_Y)}\left[c_{\mathbf{r},1}\mathcal{I}_1 + c_{\mathbf{r},2}\mathcal{I}_2 + c_{\mathbf{r},3}\mathcal{I}_3\right]\right)_{\mathbf{1}} H_{u,d}\,.$$

$$\mathcal{I}_1 = \left(\left(\psi^c\psi\right)_{\bar{\mathbf{3}}_{\mathbf{S},1}}\Phi\right)_{\mathbf{1}_{\mathbf{0},\mathbf{0}}}\,,\quad \mathcal{I}_2 = \left(\left(\psi^c\psi\right)_{\bar{\mathbf{3}}_{\mathbf{S},2}}\Phi\right)_{\mathbf{1}_{\mathbf{0},\mathbf{0}}}\,,\quad \mathcal{I}_3 = \left(\left(\psi^c\psi\right)_{\bar{\mathbf{3}}_{\mathbf{A}}}\Phi\right)_{\mathbf{1}_{\mathbf{0},\mathbf{0}}}\,.$$

■ Invariance under the action of Δ(27)

$$\mathcal{W}_D = \frac{1}{\Lambda}\sum_{\mathbf{r}}\left(Y_{\mathbf{r}}^{(k_Y)}\left[c_{\mathbf{r},1}\mathcal{I}_1 + c_{\mathbf{r},2}\mathcal{I}_2 + c_{\mathbf{r},3}\mathcal{I}_3\right]\right)_{\mathbf{1}} H_{u,d}\,.$$

$$\mathcal{I}_1 = \left(\left(\psi^c\psi\right)_{\bar{\mathbf{3}}_{\mathrm{S},1}}\Phi\right)_{\mathbf{1}_{0,0}},\quad \mathcal{I}_2 = \left(\left(\psi^c\psi\right)_{\bar{\mathbf{3}}_{\mathrm{S},2}}\Phi\right)_{\mathbf{1}_{0,0}},\quad \mathcal{I}_3 = \left(\left(\psi^c\psi\right)_{\bar{\mathbf{3}}_{\mathrm{A}}}\Phi\right)_{\mathbf{1}_{0,0}}\,.$$

$$\begin{pmatrix}\mathcal{I}_2\\ -\sqrt{2}\,\omega^2\mathcal{I}_1\end{pmatrix}\sim\mathbf{2}',\qquad \mathcal{I}_3\sim\mathbf{1}''\,.$$

$$Y_{\mathbf{1}'}^{(k_Y)} = Y_2,\qquad Y_{\mathbf{2}''}^{(k_Y)} = \left(Y_8, Y_9\right)^T$$

- Invariance under the action of Δ(27)

$$\mathcal{W}_D = \frac{1}{\Lambda}\sum_{\mathbf{r}}\left(Y_{\mathbf{r}}^{(k_Y)}\left[c_{\mathbf{r},1}\mathcal{I}_1 + c_{\mathbf{r},2}\mathcal{I}_2 + c_{\mathbf{r},3}\mathcal{I}_3\right]\right)_{\mathbf{1}} H_{u,d}\,.$$

$$\mathcal{I}_1 = \left(\left(\psi^c\psi\right)_{\bar{\mathbf{3}}_{\mathbf{S},1}}\Phi\right)_{\mathbf{1}_{\mathbf{0},\mathbf{0}}}\,,\quad \mathcal{I}_2 = \left(\left(\psi^c\psi\right)_{\bar{\mathbf{3}}_{\mathbf{S},2}}\Phi\right)_{\mathbf{1}_{\mathbf{0},\mathbf{0}}}\,,\quad \mathcal{I}_3 = \left(\left(\psi^c\psi\right)_{\bar{\mathbf{3}}_{\mathbf{A}}}\Phi\right)_{\mathbf{1}_{\mathbf{0},\mathbf{0}}}\,.$$

$$\begin{pmatrix} \mathcal{I}_2 \\ -\sqrt{2}\,\omega^2\mathcal{I}_1 \end{pmatrix} \sim \mathbf{2}',\qquad \mathcal{I}_3 \sim \mathbf{1}''.$$

$$Y_{\mathbf{1}'}^{(k_Y)} = Y_2\,,\qquad Y_{\mathbf{2}''}^{(k_Y)} = \left(Y_8, Y_9\right)^T$$

$$\mathcal{W}_D = \frac{1}{\Lambda}\left[i\omega\alpha_1 Y_2\mathcal{I}_3 + \sqrt{2}\,\alpha_2 Y_8\mathcal{I}_1 + \omega\alpha_2 Y_9\mathcal{I}_2\right]H_{u,d}\,,$$

- The mass matrix of the fermion ψ

$$M_\psi = \frac{v_{u,d}}{\Lambda}\left[i\omega\alpha_1 Y_2 \begin{pmatrix} 0 & \phi_3 & -\phi_2 \\ -\phi_3 & 0 & \phi_1 \\ \phi_2 & -\phi_1 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} \sqrt{2}\phi_1 Y_8 & \omega\phi_3 Y_9 & \omega\phi_2 Y_9 \\ \omega\phi_3 Y_9 & \sqrt{2}\phi_2 Y_8 & \omega\phi_1 Y_9 \\ \omega\phi_2 Y_9 & \omega\phi_1 Y_9 & \sqrt{2}\phi_3 Y_8 \end{pmatrix}\right],$$

GCP invariance

$$\alpha_1 = \alpha_1^*, \qquad \alpha_2 = \alpha_2^*.$$

$\mathcal{W}$	$(\Delta(27), T')$	$Y_{\boldsymbol{r}}^{(k_Y)}$	M_ψ
$\mathcal{W}_{D1}$	$\psi \sim (\mathbf{3}, \mathbf{3_0})$, $\Phi \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$ $(\psi_1^c, \psi_2^c, \psi_3^c) \sim (\mathbf{1_{(0,0)}}, \mathbf{1^{[3-k]}})$	$Y_{\mathbf{1^k}}^{(k_Y)}$	$\frac{v_{u,d} Y_k}{\Lambda} \begin{pmatrix} \alpha_1\phi_1 & \alpha_1\phi_2 & \alpha_1\phi_3 \\ \alpha_2\phi_1 & \alpha_2\phi_2 & \alpha_2\phi_3 \\ \alpha_3\phi_1 & \alpha_3\phi_2 & \alpha_3\phi_3 \end{pmatrix}$
$\mathcal{W}_{D2}$	$\psi \sim (\mathbf{3}, \mathbf{3_0})$, $\psi_m^c \sim (\mathbf{1_{(0,0)}}, \mathbf{1^{m-1}})$, $\Phi \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_{\boldsymbol{k}})$	$Y_{\mathbf{1}}^{(k_Y)}$ $Y_{\mathbf{1'}}^{(k_Y)}$ $Y_{\mathbf{1''}}^{(k_Y)}$	$\frac{v_{u,d}}{\Lambda} P^k \begin{pmatrix} \alpha_1 Y_1\phi_1 & \alpha_1 Y_1\phi_2 & \alpha_1 Y_1\phi_3 \\ \alpha_2 Y_3\phi_1 & \alpha_2 Y_3\phi_2 & \alpha_2 Y_3\phi_3 \\ \alpha_3 Y_2\phi_1 & \alpha_3 Y_2\phi_2 & \alpha_3 Y_2\phi_3 \end{pmatrix}$
$\mathcal{W}_{D3}$	$\psi,\ \psi^c \sim (\mathbf{3}, \mathbf{3_0})$, $\Phi \sim (\mathbf{3}, \mathbf{3_0})$	$Y_{\mathbf{1'}}^{(k_Y)}$, $Y_{\mathbf{2''}}^{(k_Y)}$	M_ψ in Eq. (4.8)
$\mathcal{W}_{D4}$	$\psi,\ \psi^c \sim (\mathbf{3}, \mathbf{3_0})$, $\Phi \sim (\mathbf{3}, \mathbf{3_1})$	$Y_{\mathbf{1}}^{(k_Y)}$, $Y_{\mathbf{2'}}^{(k_Y)}$	$M_\psi\ (Y_2 \to Y_1, Y_8 \to Y_6, Y_9 \to Y_7)$
$\mathcal{W}_{D5}$	$\psi,\ \psi^c \sim (\mathbf{3}, \mathbf{3_0})$, $\Phi \sim (\mathbf{3}, \mathbf{3_2})$	$Y_{\mathbf{1''}}^{(k_Y)}$, $Y_{\mathbf{2}}^{(k_Y)}$	$M_\psi\ (Y_2 \to Y_3, Y_8 \to Y_4, Y_9 \to Y_5)$
$\mathcal{W}_{D6}$	$\psi,\ \psi^c \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$, $\Phi \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$	$Y_{\mathbf{1''}}^{(k_Y)}$, $Y_{\mathbf{2'}}^{(k_Y)}$	$M_\psi\ (Y_2 \to \omega Y_3, Y_8 \to Y_7, Y_9 \to -\omega Y_6)$
$\mathcal{W}_{D7}$	$\psi,\ \psi^c \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$, $\Phi \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{1})$	$Y_{\mathbf{1}}^{(k_Y)}$, $Y_{\mathbf{2''}}^{(k_Y)}$	$M_\psi\ (Y_2 \to \omega Y_1, Y_8 \to Y_9, Y_9 \to -\omega Y_8)$
$\mathcal{W}_{D8}$	$\psi,\ \psi^c \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$, $\Phi \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{2})$	$Y_{\mathbf{1'}}^{(k_Y)}$, $Y_{\mathbf{2}}^{(k_Y)}$	$M_\psi\ (Y_2 \to \omega Y_2, Y_8 \to Y_5, Y_9 \to -\omega Y_4)$
$\mathcal{W}_{D9}$	$\psi \sim (\mathbf{3}, \mathbf{3_0})$, $\psi^c \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$, $\Phi \sim (\mathbf{8}, \mathbf{8_0})$	$Y_{\mathbf{1}}^{(k_Y)}$, $Y_{\mathbf{3}}^{(k_Y)}$ $Y_{\mathbf{2'}}^{(k_Y)}$, $Y_{\mathbf{2''}}^{(k_Y)}$	M'_ψ in Eq. (4.13) for even k_Y M''_ψ in Eq. (4.14) for odd k_Y
$\mathcal{W}_{D10}$	$\psi \sim (\mathbf{3}, \mathbf{3_0})$, $\psi^c \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$, $\Phi \sim (\mathbf{8}, \mathbf{8_1})$	$Y_{\mathbf{1''}}^{(k_Y)}$, $Y_{\mathbf{3}}^{(k_Y)}$ $Y_{\mathbf{2}}^{(k_Y)}$, $Y_{\mathbf{2'}}^{(k_Y)}$	$M'_\psi \left(Y_1 \to Y_3, Y_{\mathbf{3}}^{(k_Y)} \to P Y_{\mathbf{3}}^{(k_Y)}\right)$ $M''_\psi \left(Y_{\mathbf{2'}}^{(k_Y)} \to Y_{\mathbf{2}}^{(k_Y)}, Y_{\mathbf{2''}}^{(k_Y)} \to Y_{\mathbf{2'}}^{(k_Y)}\right)$
$\mathcal{W}_{D11}$	$\psi \sim (\mathbf{3}, \mathbf{3_0})$, $\psi^c \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$, $\Phi \sim (\mathbf{8}, \mathbf{8_2})$	$Y_{\mathbf{1'}}^{(k_Y)}$, $Y_{\mathbf{3}}^{(k_Y)}$ $Y_{\mathbf{2}}^{(k_Y)}$, $Y_{\mathbf{2''}}^{(k_Y)}$	$M'_\psi \left(Y_1 \to Y_2, Y_{\mathbf{3}}^{(k_Y)} \to P^2 Y_{\mathbf{3}}^{(k_Y)}\right)$ $M''_\psi \left(Y_{\mathbf{2'}}^{(k_Y)} \to \omega Y_{\mathbf{2''}}^{(k_Y)}, Y_{\mathbf{2''}}^{(k_Y)} \to \omega^2 Y_{\mathbf{2}}^{(k_Y)}\right)$

■ Leading order

Kahler potential

$$\mathcal{K}_{\rm LO} = \sum_{n,\boldsymbol{r_1},\boldsymbol{r_2},s} (-i\tau + i\bar{\tau})^{-k_\psi+n} \left(Y^{(n)\dagger}_{\boldsymbol{r_1}} Y^{(n)}_{\boldsymbol{r_2}} \psi^\dagger \psi \right)_{(\mathbf{1_{0,0}},\mathbf{1}),s}$$

$\Delta(27)$

$$\mathcal{K}_{\rm LO} = \sum_{n,\boldsymbol{r_1},\boldsymbol{r_2}} (-i\tau + i\bar{\tau})^{-k_\psi+n} \left(Y^{(n)\dagger}_{\boldsymbol{r_1}} Y^{(n)}_{\boldsymbol{r_2}} \right)_{(\mathbf{1_{0,0}},\mathbf{1})} \left(\psi^\dagger \psi \right)_{(\mathbf{1_{0,0}},\mathbf{1})} .$$

with

$$\left(\psi^\dagger \psi \right)_{\mathbf{1_{0,0}}} = \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 + \psi_3^\dagger \psi_3 = \left(\psi^\dagger \psi \right)_{(\mathbf{1_{0,0}},\mathbf{1})} ,$$

$$\mathcal{K}_{\rm LO} = \left(\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 + \psi_3^\dagger \psi_3 \right) \sum_{n,\boldsymbol{r}} (-i\tau + i\bar{\tau})^{-k_\psi+n} \left(Y^{(n)\dagger}_{\boldsymbol{r}} Y^{(n)}_{\boldsymbol{r}} \right)_{(\mathbf{1_{0,0}},\mathbf{1})} .$$

without off-diagonal element of the Kahler metric

■ Next-to-leading order :

$$\mathcal{K}_{\rm NLO} = \frac{1}{\Lambda}\sum_{m,n,\boldsymbol{r_1},\boldsymbol{r_2},s}(-i\tau+i\bar{\tau})^{-k_\psi+m}\left(Y^{(m)\dagger}_{\boldsymbol{r_1}}Y^{(n)}_{\boldsymbol{r_2}}\psi^\dagger\psi\Phi\right)_{(\boldsymbol{1_0},\boldsymbol{0},\boldsymbol{1}),s}+\text{h.c.}\,,$$

$\Phi \sim (\mathbf{8}, \mathbf{8}_{\mathbf{k}})$ → off-diagonal elements of the Kahler metric

■ Next-to-next-to-leading order :

$$\mathcal{K}_{\rm NNLO} = \frac{1}{\Lambda^2}\sum_{m,n,\boldsymbol{r_1},\boldsymbol{r_2},s}(-i\tau+i\bar{\tau})^{-k_\psi-k_\Theta+m}\left(Y^{(m)\dagger}_{\boldsymbol{r_1}}Y^{(n)}_{\boldsymbol{r_2}}\psi^\dagger\psi\Theta^\dagger\Phi\right)_{(\boldsymbol{1_0},\boldsymbol{0},\boldsymbol{1}),s}+\text{h.c.}\,.$$

off-diagonal elements of the Kahler metric

5. Ω(1) eclectic lepton model

Fields	L	E^c	H_u	H_d	ϕ	φ	χ	ξ	$Y_{\boldsymbol{r}}^{(k_Y)}$
$\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$	$(\mathbf{2},-\frac{1}{2})$	$(\mathbf{1},1)$	$(\mathbf{2},\frac{1}{2})$	$(\mathbf{2},-\frac{1}{2})$	$(\mathbf{1},0)$	$(\mathbf{1},0)$	$(\mathbf{1},0)$	$(\mathbf{1},0)$	$(\mathbf{1},0)$
$\Delta(27)$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{0,0}$	$\mathbf{1}_{0,0}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{0,0}$	$\mathbf{1}_{0,0}$
$\Gamma'_3\cong T'$	$\mathbf{3}_0$	$\mathbf{3}_0$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}_1$	$\mathbf{3}_0$	$\mathbf{3}_1$	$\mathbf{1}$	$\boldsymbol{r}$
modular weight	0	0	0	0	5	5	7	-1	k_Y
Z_2	1	-1	1	1	-1	1	1	1	1
Z_3	ω	ω^2	1	1	1	ω	ω	1	1

Kahler potential：

$\propto 1$

$$\mathcal{K} = \mathcal{K}_{\mathrm{LO}}+\mathcal{K}_{\mathrm{NLO}}+\mathcal{K}_{\mathrm{NNLO}}+\ldots$$

$$\sum_{m,n,\boldsymbol{r_1},\boldsymbol{r_2},s}\frac{1}{\Lambda^2}(-i\tau+i\bar{\tau})^{-k_\psi-k_\Theta+m}\left(Y_{\boldsymbol{r_1}}^{(m)\dagger}Y_{\boldsymbol{r_2}}^{(n)}\psi^\dagger\psi\Theta^\dagger\Phi\right)_{(\mathbf{1}_{0,0},\mathbf{1}),s}+\text{h.c.}$$

Correction ~ $\frac{\langle\Phi\rangle^2}{\Lambda^2}$

Ignore!

5. Ω(1) eclectic lepton model

Fields	L	E^c	H_u	H_d	ϕ	φ	χ	ξ	$Y_{\boldsymbol{r}}^{(k_Y)}$
$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 1)$	$(\mathbf{2}, \frac{1}{2})$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$
$\Delta(27)$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{0,0}$	$\mathbf{1}_{0,0}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{0,0}$	$\mathbf{1}_{0,0}$
$\Gamma'_3 \cong T'$	$\mathbf{3}_0$	$\mathbf{3}_0$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}_1$	$\mathbf{3}_0$	$\mathbf{3}_1$	$\mathbf{1}$	$\boldsymbol{r}$
modular weight	0	0	0	0	5	5	7	-1	k_Y
Z_2	1	-1	1	1	-1	1	1	1	1
Z_3	ω	ω^2	1	1	1	ω	ω	1	1

Superpotential:

$$\mathcal{W} = \frac{\alpha}{\Lambda}\left(E^c L \phi Y_{\mathbf{2'}}^{(5)}\right)_{(\mathbf{1}_{0,0},\mathbf{1})} H_d + \frac{\beta}{\Lambda^2}\left(E^c L \xi \phi Y_{\mathbf{1}}^{(4)}\right)_{(\mathbf{1}_{0,0},\mathbf{1})} H_d + \frac{g_1}{2\Lambda^2}\left(LL\varphi Y_{\mathbf{2''}}^{(5)}\right)_{(\mathbf{1}_{0,0},\mathbf{1})} H_u H_u + \frac{g_2}{2\Lambda^2}\left(LL\chi Y_{\mathbf{2'}}^{(7)}\right)_{(\mathbf{1}_{0,0},\mathbf{1})} H_u H_u \,.$$

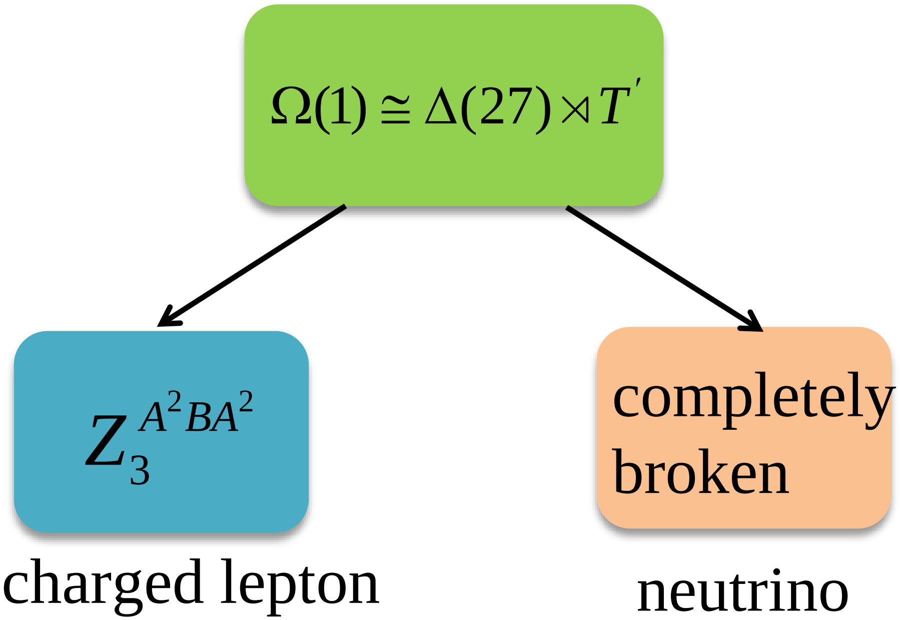
Symmetry breaking

VEVs of flavons

$$\langle\phi\rangle = (\omega^2, 1, 1)^T v_\phi : \qquad \Omega(1) \xrightarrow{\langle\phi\rangle} Z_3^{A^2BA^2} ,$$

$$\langle\varphi\rangle = (0, 0, 1)^T v_\varphi : \qquad \Omega(1) \xrightarrow{\langle\varphi\rangle} Z_3^{ABA^2} ,$$

$$\langle\chi\rangle = (1, 0, 0)^T v_\chi : \qquad \Omega(1) \xrightarrow{\langle\chi\rangle} Z_3^{B} .$$



■ Lepton mass matrices:

$$m_l = \frac{\alpha v_\phi v_d}{\Lambda}\begin{pmatrix} \sqrt{2}\omega^2 Y^{(5)}_{\mathbf{2'},1} & \omega Y^{(5)}_{\mathbf{2'},2} & \omega Y^{(5)}_{\mathbf{2'},2} \\ \omega Y^{(5)}_{\mathbf{2'},2} & \sqrt{2} Y^{(5)}_{\mathbf{2'},1} & Y^{(5)}_{\mathbf{2'},2} \\ \omega Y^{(5)}_{\mathbf{2'},2} & Y^{(5)}_{\mathbf{2'},2} & \sqrt{2} Y^{(5)}_{\mathbf{2'},1} \end{pmatrix} + \frac{i\beta Y^{(4)}_{\mathbf{1}} v_\xi v_\phi v_d}{\Lambda^2}\begin{pmatrix} 0 & \omega & -\omega \\ -\omega & 0 & 1 \\ \omega & -1 & 0 \end{pmatrix},$$

$$m_\nu = \frac{g_1 v_\varphi v_u^2}{\Lambda^2}\begin{pmatrix} 0 & \omega Y^{(5)}_{\mathbf{2''},2} & 0 \\ \omega Y^{(5)}_{\mathbf{2''},2} & 0 & 0 \\ 0 & 0 & \sqrt{2} Y^{(5)}_{\mathbf{2''},1} \end{pmatrix} + \frac{g_2 v_\chi v_u^2}{\Lambda^2}\begin{pmatrix} \sqrt{2} Y^{(7)}_{\mathbf{2'},1} & 0 & 0 \\ 0 & 0 & \omega Y^{(7)}_{\mathbf{2'},2} \\ 0 & \omega Y^{(7)}_{\mathbf{2'},2} & 0 \end{pmatrix},$$

$$U_l^\dagger m_l^\dagger m_l U_l = \mathrm{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right),$$ ➡ $$U_l = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 \end{pmatrix}.$$

■ Lepton mass matrices：

$$m_l = \frac{\alpha v_\phi v_d}{\Lambda}\begin{pmatrix} \sqrt{2}\omega^2 Y^{(5)}_{\mathbf{2'},1} & \omega Y^{(5)}_{\mathbf{2'},2} & \omega Y^{(5)}_{\mathbf{2'},2} \\ \omega Y^{(5)}_{\mathbf{2'},2} & \sqrt{2} Y^{(5)}_{\mathbf{2'},1} & Y^{(5)}_{\mathbf{2'},2} \\ \omega Y^{(5)}_{\mathbf{2'},2} & Y^{(5)}_{\mathbf{2'},2} & \sqrt{2} Y^{(5)}_{\mathbf{2'},1} \end{pmatrix} + \frac{i\beta Y^{(4)}_{\mathbf{1}} v_\xi v_\phi v_d}{\Lambda^2}\begin{pmatrix} 0 & \omega & -\omega \\ -\omega & 0 & 1 \\ \omega & -1 & 0 \end{pmatrix},$$

$$m_\nu = \frac{g_1 v_\varphi v_u^2}{\Lambda^2}\begin{pmatrix} 0 & \omega Y^{(5)}_{\mathbf{2''},2} & 0 \\ \omega Y^{(5)}_{\mathbf{2''},2} & 0 & 0 \\ 0 & 0 & \sqrt{2} Y^{(5)}_{\mathbf{2''},1} \end{pmatrix} + \frac{g_2 v_\chi v_u^2}{\Lambda^2}\begin{pmatrix} \sqrt{2} Y^{(7)}_{\mathbf{2'},1} & 0 & 0 \\ 0 & 0 & \omega Y^{(7)}_{\mathbf{2'},2} \\ 0 & \omega Y^{(7)}_{\mathbf{2'},2} & 0 \end{pmatrix},$$

■ The charged lepton masses：

$$m_e = \left|\sqrt{2} Y^{(5)}_{\mathbf{2'},1} - Y^{(5)}_{\mathbf{2'},2} - \frac{\sqrt{6}\beta v_\xi Y^{(4)}_{\mathbf{1}}}{\alpha\Lambda}\right| \frac{\alpha v_\phi v_d}{\Lambda},$$

$$m_\mu = \left|\sqrt{2} Y^{(5)}_{\mathbf{2'},1} - Y^{(5)}_{\mathbf{2'},2} + \frac{\sqrt{6}\beta v_\xi Y^{(4)}_{\mathbf{1}}}{\alpha\Lambda}\right| \frac{\alpha v_\phi v_d}{\Lambda},$$

$$m_\tau = \left|\sqrt{2} Y^{(5)}_{\mathbf{2'},1} + 2 Y^{(5)}_{\mathbf{2'},2}\right| \frac{\alpha v_\phi v_d}{\Lambda}.$$

■ Best fit values of the free parameters

$$\Re\langle\tau\rangle = 0.00177, \quad \Im\langle\tau\rangle = 1.120,$$

$$|\beta v_\xi/(\alpha\Lambda)| = 0.0480, \quad \arg(\beta v_\xi/(\alpha\Lambda)) = 1.04\pi,$$

$$|g_2 v_\chi/(g_1 v_\varphi)| = 0.9787, \quad \arg(g_2 v_\chi/(g_1 v_\varphi)) = 1.005\pi,$$

$$\alpha v_\xi v_\phi v_d/\Lambda^2 = 262.5\,\mathrm{MeV}, \quad g_1 v_\varphi v_u^2/\Lambda^2 = 5.401\,\mathrm{meV}.$$

■ The predictions for various observable quantities

$$\sin^2\theta_{13} = 0.02251, \quad \sin^2\theta_{12} = 0.3284, \quad \sin^2\theta_{23} = 0.4954, \quad \delta_{CP} = 1.434\pi,$$

$$\alpha_{21} = 0.961\pi, \quad \alpha_{31} = 0.926\pi, \quad m_1 = 15.13\,\mathrm{meV}, \quad m_2 = 17.40\,\mathrm{meV},$$

$$m_3 = 52.31\,\mathrm{meV}, \quad \sum_{i=1}^{3} m_i = 84.84\,\mathrm{meV}, \quad m_{\beta\beta} = 5.619\,\mathrm{meV},$$

$$m_e = 0.511\,\mathrm{MeV}, \quad m_\mu = 106.5\,\mathrm{MeV}, \quad m_\tau = 1.803\,\mathrm{GeV}.$$

In the charged lepton diagonal basis :

$$m'_\nu = U_l^T m_\nu U_l$$

$$= \frac{g_1 v_\varphi v_u^2}{\Lambda^2}\begin{pmatrix} \sqrt{2}Y^{(5)}_{\mathbf{2''},1}+2Y^{(5)}_{\mathbf{2''},2} & \omega\left(\sqrt{2}Y^{(5)}_{\mathbf{2''},1}-Y^{(5)}_{\mathbf{2''},2}\right) & \omega^2\left(\sqrt{2}Y^{(5)}_{\mathbf{2''},1}-Y^{(5)}_{\mathbf{2''},2}\right) \\ \omega\left(\sqrt{2}Y^{(5)}_{\mathbf{2''},1}-Y^{(5)}_{\mathbf{2''},2}\right) & \omega^2\left(\sqrt{2}Y^{(5)}_{\mathbf{2''},1}+2Y^{(5)}_{\mathbf{2''},2}\right) & \sqrt{2}Y^{(5)}_{\mathbf{2''},1}-Y^{(5)}_{\mathbf{2''},2} \\ \omega^2\left(\sqrt{2}Y^{(5)}_{\mathbf{2''},1}-Y^{(5)}_{\mathbf{2''},2}\right) & \sqrt{2}Y^{(5)}_{\mathbf{2''},1}-Y^{(5)}_{\mathbf{2''},2} & \omega\left(\sqrt{2}Y^{(5)}_{\mathbf{2''},1}+2Y^{(5)}_{\mathbf{2''},2}\right) \end{pmatrix}$$

$$+\frac{g_2 v_\chi v_u^2}{\Lambda^2}\begin{pmatrix} \sqrt{2}Y^{(7)}_{\mathbf{2'},1}+2Y^{(7)}_{\mathbf{2'},2} & \omega^2\left(\sqrt{2}Y^{(7)}_{\mathbf{2'},1}-Y^{(7)}_{\mathbf{2'},2}\right) & \omega\left(\sqrt{2}Y^{(7)}_{\mathbf{2'},1}-Y^{(7)}_{\mathbf{2'},2}\right) \\ \omega^2\left(\sqrt{2}Y^{(7)}_{\mathbf{2'},1}-Y^{(7)}_{\mathbf{2'},2}\right) & \omega\left(\sqrt{2}Y^{(7)}_{\mathbf{2'},1}+2Y^{(7)}_{\mathbf{2'},2}\right) & \sqrt{2}Y^{(7)}_{\mathbf{2'},1}-Y^{(7)}_{\mathbf{2'},2} \\ \omega\left(\sqrt{2}Y^{(7)}_{\mathbf{2'},1}-Y^{(7)}_{\mathbf{2'},2}\right) & \sqrt{2}Y^{(7)}_{\mathbf{2'},1}-Y^{(7)}_{\mathbf{2'},2} & \omega^2\left(\sqrt{2}Y^{(7)}_{\mathbf{2'},1}+2Y^{(7)}_{\mathbf{2'},2}\right) \end{pmatrix}.$$

GCP + $\Re\tau = 0$:

$$P^T_{\nu\tau} m'_\nu P_{\nu\tau} = (m'_\nu)^*, \qquad P_{\nu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

μ—τ reflection symmetry

[Xing, Zhao, 1512.04207]

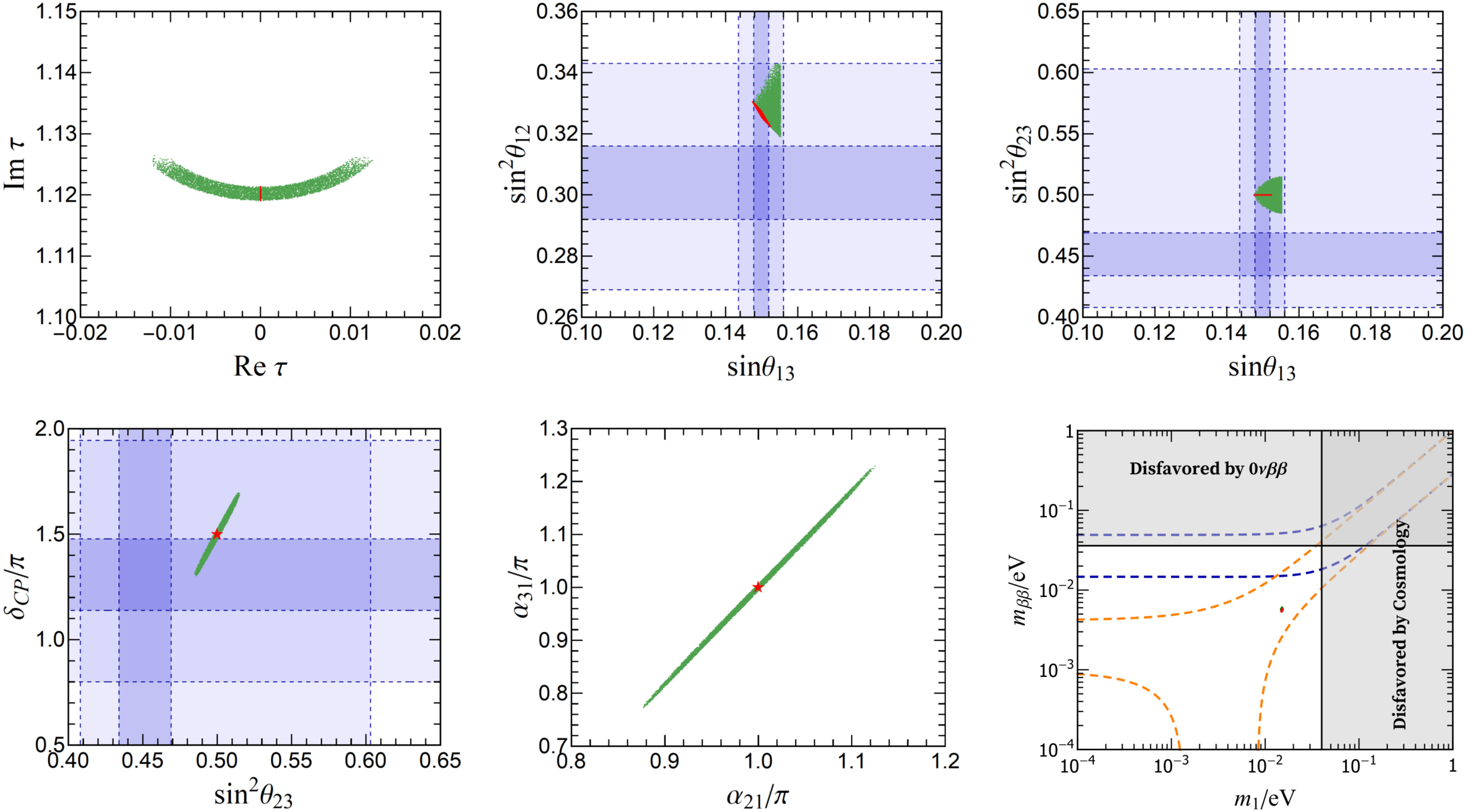
$$\theta_{23} = \pi/4, \quad \delta_{CP} = \pm\pi/2, \quad \alpha_{21}, \alpha_{31} = 0, \pi$$

- Best fit values of the free parameters

$$\langle\tau\rangle = 1.120i, \qquad \beta v_\xi/(\alpha\Lambda) = -0.0484, \qquad g_2 v_\chi/(g_1 v_\varphi) = -0.981,$$
$$\alpha v_d v_\xi v_\phi/\Lambda^2 = 263.0\,\mathrm{MeV}, \qquad g_1 v_u^2 v_\varphi/\Lambda^2 = 5.409\,\mathrm{meV}.$$

- The predictions for various observable quantities

$$\sin^2\theta_{13} = 0.02238, \quad \sin^2\theta_{12} = 0.3266, \quad \sin^2\theta_{23} = 0.5, \quad \delta_{CP} = 1.5\pi,$$
$$\alpha_{21} = \pi, \quad \alpha_{31} = \pi, \quad m_1 = 15.18\,\mathrm{meV}, \quad m_2 = 17.44\,\mathrm{meV},$$
$$m_3 = 52.43\,\mathrm{meV}, \quad \sum_i m_i = 85.05\,\mathrm{meV}, \quad m_{\beta\beta} = 5.595\,\mathrm{meV},$$
$$m_e = 0.511\,\mathrm{MeV}, \quad m_\mu = 106.5\,\mathrm{MeV}, \quad m_\tau = 1.807\,\mathrm{GeV}.$$



The correlations among mixing parameters

5. Conclusions

- **A comprehensive analysis of the superpotential and Kahler potential of models based on the EFG Ω(1)**

- **Kahler potential are suppressed by powers of** $\langle \Phi \rangle^2 / \Lambda^2$

- **Two concrete lepton models invariant under the EFG Ω(1)**

- μ - τ **reflection symmetry for neutrino mass matrices is preserved.**

谢谢！

Backup

flavor group $\mathcal{G}_{\text{fl}}$	GAP ID	$\text{Aut}(\mathcal{G}_{\text{fl}})$	finite modular groups		eclectic flavor group
Q_8	[8, 4]	S_4	without $\mathcal{CP}$	S_3	$\text{GL}(2,3)$
			with $\mathcal{CP}$	–	–
$\mathbb{Z}_3 \times \mathbb{Z}_3$	[9, 2]	$\text{GL}(2,3)$	without $\mathcal{CP}$	S_3	$\Delta(54)$
			with $\mathcal{CP}$	$S_3 \times \mathbb{Z}_2$	[108, 17]
A_4	[12, 3]	S_4	without $\mathcal{CP}$	S_3 S_4	S_4 S_4
			with $\mathcal{CP}$	–	–
T'	[24, 3]	S_4	without $\mathcal{CP}$	S_3	$\text{GL}(2,3)$
			with $\mathcal{CP}$	–	–
$\Delta(27)$	[27, 3]	[432, 734]	without $\mathcal{CP}$	S_3 T'	$\Delta(54)$ $\Omega(1)$
			with $\mathcal{CP}$	$S_3 \times \mathbb{Z}_2$ $\text{GL}(2,3)$	[108, 17] [1296, 2891]
$\Delta(54)$	[54, 8]	[432, 734]	without $\mathcal{CP}$	T'	$\Omega(1)$
			with $\mathcal{CP}$	$\text{GL}(2,3)$	[1296, 2891]

flavor group	finite modular group	EFG	GAP ID
Q_8	S_3	$GL(2,3)$	[48, 29]
$Z_3 \times Z_3$	S_3	$\Delta(54)$	[54, 8]
A_4	S_3	$(Z_3 \times A_4) \rtimes Z_2$	[72, 43]
	S_4	$(A_4 \times A_4) \rtimes Z_2$	[288, 1026]
T'	S_3	$(Z_3 \times SL(2,3)) \rtimes Z_2$	[144, 125]
$\Delta(27)$	S_3	$(\Delta(27) \times Z_3) \rtimes Z_2$	[162, 46]
	T'	$(\Delta(27) \rtimes Q_8) \rtimes Z_3 \cong \Omega(1)$	[648, 533]
$\Delta(54)$	T'	$((\Delta(27) \rtimes Q_8) \rtimes Z_3) \times Z_2 \cong \Omega(1) \times Z_2$	[1296, 2895]

Modular invariant theory

[Ferrara et al, 1989; Feruglio, 1706.08749]

- $N = 1$ global supersymmetry
  - The action：

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\mathcal{K}(\Phi_I, \bar{\Phi}_I; \tau, \bar{\tau}) + \int d^4x d^2\theta \mathcal{W}(\Phi_I, \tau) + \text{h.c.}$$

  - Kahler potential:

$$\mathcal{K} = -h\ln(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\Phi_I|^2$$

  - The superpotential:

$$\mathcal{W} = \sum_n Y_{I_1 I_2 \ldots I_n}(\tau)\Phi_{I_1}\Phi_{I_2}\ldots\Phi_{I_n}$$

The Kahler potential is not under control

[Chen M.C. et al, 1909.06910,…]

- Transformation properties：

$$\tau \xrightarrow{\gamma} \gamma\tau = \frac{a\tau+b}{c\tau+d},$$

$$Y^{(k)}(\tau) \to Y^{(k)}(\gamma\tau) = (c\tau+d)^k \rho(\gamma) Y^{(k)}(\tau),$$

$$\Phi \xrightarrow{\gamma} \gamma(c\tau+d)^{-k_\Phi} \rho(\gamma)\Phi, \quad \text{where} \;\; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

⬇

$$(-i\tau+i\bar{\tau})^k \xrightarrow{\gamma} \big((c\tau+d)(c\bar{\tau}+d)\big)^{-k} (-i\tau+i\bar{\tau})^k,$$

$$\left(Y^{(k)}\overline{Y}^{(k)}\right)_1 \xrightarrow{\gamma} (c\tau+d)^k (c\bar{\tau}+d)^k \left(Y^{(k)}\overline{Y}^{(k)}\right)_1,$$

$$\left(\bar{\Phi}\Phi\right)_1 \xrightarrow{\gamma} (c\tau+d)^{-k_\Phi} (c\bar{\tau}+d)^{-k_\Phi} \left(\bar{\Phi}\Phi\right)_1.$$

$$\mathbf{1}^a\otimes\mathbf{1}^b=\mathbf{1}^{[a+b]},\quad \mathbf{1}^a\otimes\mathbf{2}^b=\mathbf{2}^{[a+b]},\quad \mathbf{1}^a\otimes\mathbf{3}=\mathbf{3},\quad \mathbf{2}^a\otimes\mathbf{2}^b=\mathbf{1}^{[a+b]}\oplus\mathbf{3},$$
$$\mathbf{2}^a\otimes\mathbf{3}=\mathbf{2}\oplus\mathbf{2}'\oplus\mathbf{2}'',\quad \mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}''\oplus\mathbf{3}_{\boldsymbol{S}}\oplus\mathbf{3}_{\boldsymbol{A}}\,, \tag{B.3}$$

$$\mathbf{1}^a:\ \rho_{\mathbf{1}^a}(S)=1,\qquad \rho_{\mathbf{1}^a}(T)=\omega^a\,,$$

$$\mathbf{2}^a:\ \rho_{\mathbf{2}^a}(S)=\frac{i}{\sqrt{3}}\begin{pmatrix}1 & \sqrt{2}\\ \sqrt{2} & -1\end{pmatrix},\qquad \rho_{\mathbf{2}^a}(T)=\omega^{a+1}\begin{pmatrix}1 & 0\\ 0 & \omega\end{pmatrix}$$

$$\mathbf{3}:\quad \rho_{\mathbf{3}}(S)=\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix},\qquad \rho_{\mathbf{3}}(T)=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix},$$

weight	Modular form $Y_{\boldsymbol{r}}^{(k)}$	weight	Modular form $Y_{\boldsymbol{r}}^{(k)}$
$k=1$	$Y_{\mathbf{2}''}^{(1)}$	$k=2$	$Y_{\mathbf{3}}^{(2)}$
$k=3$	$Y_{\mathbf{2}'}^{(3)},\ Y_{\mathbf{2}''}^{(3)}$	$k=4$	$Y_{\mathbf{1}}^{(4)},\ Y_{\mathbf{1}'}^{(4)},\ Y_{\mathbf{3}}^{(4)}$
$k=5$	$Y_{\mathbf{2}}^{(5)},\ Y_{\mathbf{2}'}^{(5)},\ Y_{\mathbf{2}''}^{(5)}$	$k=6$	$Y_{\mathbf{1}}^{(6)},\ Y_{\mathbf{3}i}^{(6)},\ Y_{\mathbf{3}ii}^{(6)}$
$k=7$	$Y_{\mathbf{2}}^{(7)},\ Y_{\mathbf{2}'}^{(7)},\ Y_{\mathbf{2}''i}^{(7)},\ Y_{\mathbf{2}''ii}^{(7)}$	$k=8$	$Y_{\mathbf{1}}^{(8)},\ Y_{\mathbf{1}'}^{(8)},\ Y_{\mathbf{1}''}^{(8)},\ Y_{\mathbf{3}i}^{(8)},\ Y_{\mathbf{3}ii}^{(8)}$
$k=9$	$Y_{\mathbf{2}}^{(9)},\ Y_{\mathbf{2}'i}^{(9)},\ Y_{\mathbf{2}'ii}^{(9)},\ Y_{\mathbf{2}''i}^{(9)},\ Y_{\mathbf{2}''ii}^{(9)}$	$k=10$	$Y_{\mathbf{1}}^{(10)},\ Y_{\mathbf{1}'}^{(10)},\ Y_{\mathbf{3}i}^{(10)},\ Y_{\mathbf{3}ii}^{(10)},\ Y_{\mathbf{3}iii}^{(10)}$

- As a consequence

$$\mathcal{K} \supset \sum_{\Phi_n} \sum_{k\geq 0} (-i\tau + i\bar{\tau})^{-k+k_\Phi} \sum_a \kappa_a^{(k)} \left[Y^{(k)}(T) \otimes \bar{Y}^{(k)} \otimes \Phi(T) \otimes \bar{\Phi} \right]_{1,a}.$$

- The full Kahler potential includes additional terms, such as

$$\mathcal{K} = \alpha_0 (-i\tau + i\bar{\tau})^{-k_L} \left(\bar{L} L \right)_1 + \sum_{k=1} \alpha_k (-i\tau + i\bar{\tau})^{-k_L + k_Y} \left(Y \bar{Y} L \bar{L} \right)_{1,k} + \ldots$$

- The additional terms will modify the Kahler metric

$$K_L^{i\bar{j}} = \frac{\partial^2 K}{\partial L_i \, \partial \bar{L}_{\bar{j}}}$$

- This metric has to be diagonalized,

$$K_L = U_L^\dagger D^2 U_L$$

where U_L is unitary and D is diagonal and positive.

Therefore, the canonically normalized fields are

$$\hat{L} = D U_L L \quad \text{or equivalently} \quad L = U_L^{\dagger} D^{-1} \hat{L} \; .$$

After adding the $\alpha_{i>0}$ contributions and transforming the fields back to canonical normalization, we need to diagonalize

$$U_\nu^T m_\nu U_\nu = \mathrm{diag}(m_1, m_2, m_3) \quad \text{and} \quad U_e^{\dagger} Y_e Y_e^{\dagger} U_e = \mathrm{diag}(y_e^2, y_\mu^2, y_\tau^2).$$

$$\hat{U}_\nu^T D^{-1} U_L^{*} m_\nu U_L^{?} D^{-1} \hat{U}_\nu = \mathrm{diag}(m_1, m_2, m_3) \; ,$$
$$\hat{U}_e^{\dagger} D^{-1} U_L^{*} Y_e Y_e^{?} U_L^{T} D^{-1} \hat{U}_e = \mathrm{diag}(y_e^2, y_\mu^2, y_\tau^2).$$

We see that if D is proportional to the unit matrix, there would be no effect.
However, D is generically not proportional to the unit matrix.

$\mathcal{W}$	$(\Delta(27), T')$	$Y_{\boldsymbol{r}}^{(k_Y)}$	M_{ψ^c}
$\mathcal{W}_{M1}$	$\psi^c \sim (\mathbf{3}, \mathbf{3_0})$, $\Phi \sim (\mathbf{3}, \mathbf{3_0})$	$Y_{\mathbf{2''}}^{(k_Y)}$	M_{ψ^c} in Eq. (4.22)
$\mathcal{W}_{M2}$	$\psi^c \sim (\mathbf{3}, \mathbf{3_0})$, $\Phi \sim (\mathbf{3}, \mathbf{3_1})$	$Y_{\mathbf{2'}}^{(k_Y)}$	$M_{\psi^c}\,(Y_8 \to Y_6, Y_9 \to Y_7)$
$\mathcal{W}_{M3}$	$\psi^c \sim (\mathbf{3}, \mathbf{3_0})$, $\Phi \sim (\mathbf{3}, \mathbf{3_2})$	$Y_{\mathbf{2}}^{(k_Y)}$	$M_{\psi^c}\,(Y_8 \to Y_4, Y_9 \to Y_5)$
$\mathcal{W}_{M4}$	$\psi^c \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$, $\Phi \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$	$Y_{\mathbf{2'}}^{(k_Y)}$	$M_{\psi^c}\,(Y_8 \to Y_7, Y_9 \to -\omega Y_6)$
$\mathcal{W}_{M5}$	$\psi^c \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$, $\Phi \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{1})$	$Y_{\mathbf{2''}}^{(k_Y)}$	$M_{\psi^c}\,(Y_8 \to Y_9, Y_9 \to -\omega Y_8)$
$\mathcal{W}_{M6}$	$\psi^c \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{0})$, $\Phi \sim (\bar{\mathbf{3}}, \bar{\mathbf{3}}_\mathbf{2})$	$Y_{\mathbf{2}}^{(k_Y)}$	$M_{\psi^c}\,(Y_8 \to Y_5, Y_9 \to -\omega Y_4)$

$$M_{\psi^c} = \alpha \begin{pmatrix} \sqrt{2}\,Y_8\phi_1 & \omega Y_9\phi_3 & \omega Y_9\phi_2 \\ \omega Y_9\phi_3 & \sqrt{2}\,Y_8\phi_2 & \omega Y_9\phi_1 \\ \omega Y_9\phi_2 & \omega Y_9\phi_1 & \sqrt{2}\,Y_8\phi_3 \end{pmatrix} . \tag{4.22}$$

Z_3^g	Res. CP	$\langle\phi_{\mathbf{3}}\rangle/v_\phi$	$\langle\phi_{\bar{\mathbf{3}}}\rangle/v_\phi$	$\langle\phi_{\mathbf{8}}\rangle/v_\phi$
Z_3^A	—	$(1,1,1)^T$	$(1,1,1)^T$	$v_{8_1}=(1,x,0,0,0,0,0,0)^T$
Z_3^B	A	$(1,0,0)^T$	$(1,0,0)^T$	$v_{8_2}=(0,0,\omega^2,0,0,-\omega x,0,0)^T$
	AB	$(\omega^2,0,0)^T$	$(\omega,0,0)^T$	v_{8_2}
	BAB^2	$(\omega,0,0)^T$	$(\omega^2,0,0)^T$	v_{8_2}
Z_3^{AB}	BAB^2A^2	$(1,\omega^2,1)^T$	$(1,\omega,1)^T$	$v_{8_3}=(0,0,0,0,1-ix,0,1+ix,0)^T$
	ABA^2B^2	$\omega(1,\omega^2,1)^T$	$\omega^2(1,\omega,1)^T$	v_{8_3}
	AB	$\omega^2(1,\omega^2,1)^T$	$\omega(1,\omega,1)^T$	v_{8_3}
Z_3^{BA}	BAB^2A^2	$\omega(1,1,\omega^2)^T$	$\omega^2(1,1,\omega)^T$	v_{8_3}
	ABA^2B^2	$\omega^2(1,1,\omega^2)^T$	$\omega(1,1,\omega)^T$	v_{8_3}
	BA	$(1,1,\omega^2)^T$	$(1,1,\omega)^T$	v_{8_3}
Z_3^{ABA}	—	$(1,\omega^2,\omega^2)^T$	$(1,\omega,\omega)^T$	$v_{8_4}=(0,0,0,1,0,0,0,x)^T$
$Z_3^{A^2B}$	—	$(1,1,\omega)^T$	$(1,1,\omega^2)^T$	v_{8_4}
$Z_3^{BA^2}$	—	$(1,\omega,1)^T$	$(1,\omega^2,1)^T$	v_{8_4}
$Z_3^{BAB^2}$	—	$(1,\omega,\omega^2)^T$	$(1,\omega^2,\omega)^T$	v_{8_1}
$Z_3^{B^2AB}$	—	$(1,\omega^2,\omega)^T$	$(1,\omega,\omega^2)^T$	v_{8_1}
$Z_3^{A^2BA}$	A^2	$(0,1,0)^T$	$(0,1,0)^T$	$v_{8_2}^*$
	B^2A^2B	$\omega(0,1,0)^T$	$\omega^2(0,1,0)^T$	$v_{8_2}^*$
	BA^2B^2	$\omega^2(0,1,0)^T$	$\omega(0,1,0)^T$	$v_{8_2}^*$
$Z_3^{ABA^2}$	BAB^2A^2	$(0,0,1)^T$	$(0,0,1)^T$	$v_{8_5}=(0,0,1,0,0,x,0,0)^T$
	ABA^2B^2	$\omega(0,0,1)^T$	$\omega^2(0,0,1)^T$	v_{8_5}
	AB^2A^2	$\omega^2(0,0,1)^T$	$\omega(0,0,1)^T$	v_{8_5}
$Z_3^{A^2BA^2}$	BAB^2A^2	$(\omega^2,1,1)^T$	$(\omega,1,1)^T$	v_{8_3}
	ABA^2B^2	$\omega(\omega^2,1,1)^T$	$\omega^2(\omega,1,1)^T$	v_{8_3}
	A^2BA^2	$\omega^2(\omega^2,1,1)^T$	$\omega(\omega,1,1)^T$	v_{8_3}
$Z_3^{BAB^2A^2}$	—	—	—	general 8-D vector

$\mathcal{N}$=1 SUSY modular invariant theories

- known since late 1980s

S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B **225** (1989) 363.

S. Ferrara, .D. Lust and S. Theisen, Phys. Lett. B **233** (1989) 147.

focus on Yukawa interactions and $\mathcal{N}$=1 global SUSY

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\; K(\Phi,\bar{\Phi}) + \int d^4x d^2\theta\; w(\Phi) + h.c.$$

$$\Phi = (\tau, \varphi)$$

Kahler potential, kinetic terms

superpotential, holomorphic function of Φ
Yukawa interactions

S invariant if

$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi,\bar{\Phi}) \to K(\Phi,\bar{\Phi}) + f(\Phi) + f(\bar{\Phi}) \end{cases}$$

- invariance of the Kahler potential easy to achieve. For example:

$$K(\Phi,\bar{\Phi}) = -h\log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

minimal K

- extension to $\mathcal{N}$=1 SUGRA straightforward: ask invariance of $G=K+\log|w|^2$

Few facts about (level-N) Modular Forms

- transformation property under the modular group

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

unitary representation of the finite modular group $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$

- q-expansion

$f(\tau + N) = f(\tau)$ ⇒ $f(\tau) = \sum_{i=0}^{\infty} a_n q_N^n \qquad q_N = e^{\frac{i2\pi\tau}{N}}$

- $k < 0$ ⇒ $f(\tau) = 0$

  $k = 0$ ⇒ $f(\tau) =$ constant

  $k > 0$ (even integer) ⇒ $f(\tau) \in \mathcal{M}_k(\Gamma(N))$ finite-dimensional linear space

- ring of modular forms generated by few elements

$$\mathcal{M}(\Gamma(N)) = \bigoplus_{k=0}^{\infty} \mathcal{M}_{2k}(\Gamma(N))$$

an explicit example in a moment

Kinetic Term

Kinetic term of the modulus τ $$\frac{|\partial_\mu \tau|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$

Modular transformation $$\tau' = \frac{a\tau + b}{c\tau + d}, ad - bc = 1$$

- numerator

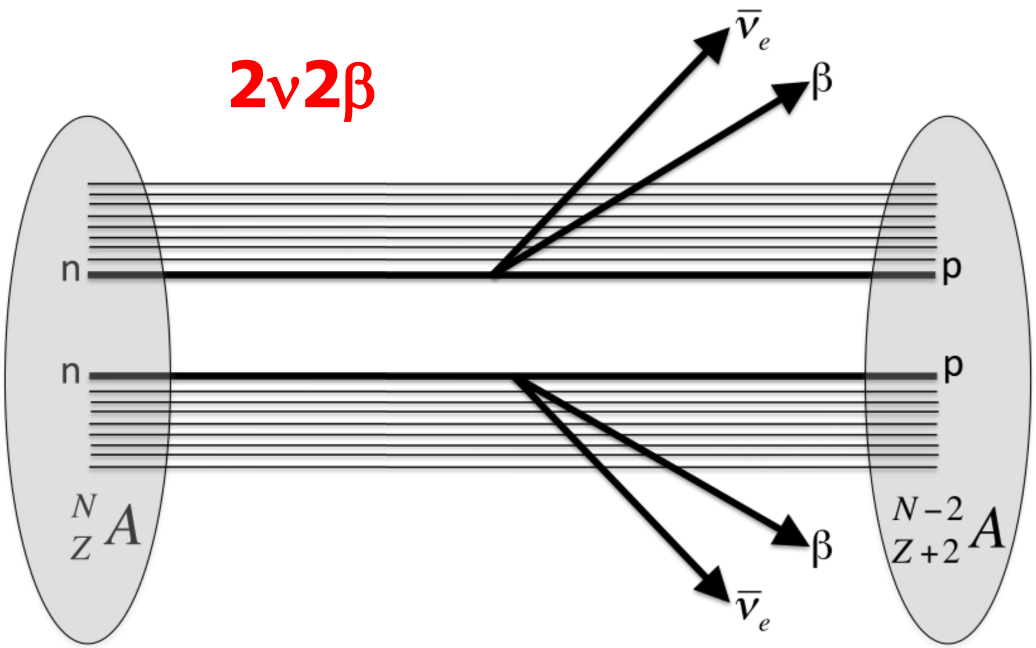
$$\partial_\mu \tau' = \frac{(a\partial_\mu \tau)(c\tau + d) - (a\tau + b)(c\partial_\mu \tau)}{(c\tau + d)^2} = \frac{(ad - bc)\partial_\mu \tau}{(c\tau + d)^2} = \frac{\partial_\mu \tau}{(c\tau + d)^2}$$

- denominator

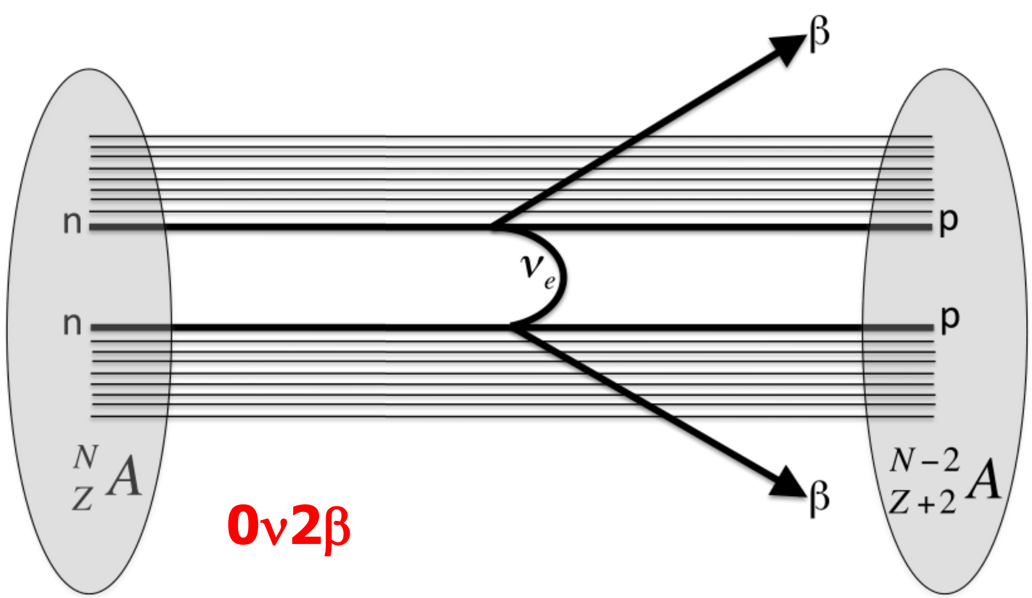
$$\tau' - \bar{\tau}' = \frac{(a\tau + b)(c\bar{\tau} + d) - (a\bar{\tau} + b)(c\tau + d)}{|c\tau + d|^2} = \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\tau - \bar{\tau}}{|c\tau + d|^2}$$

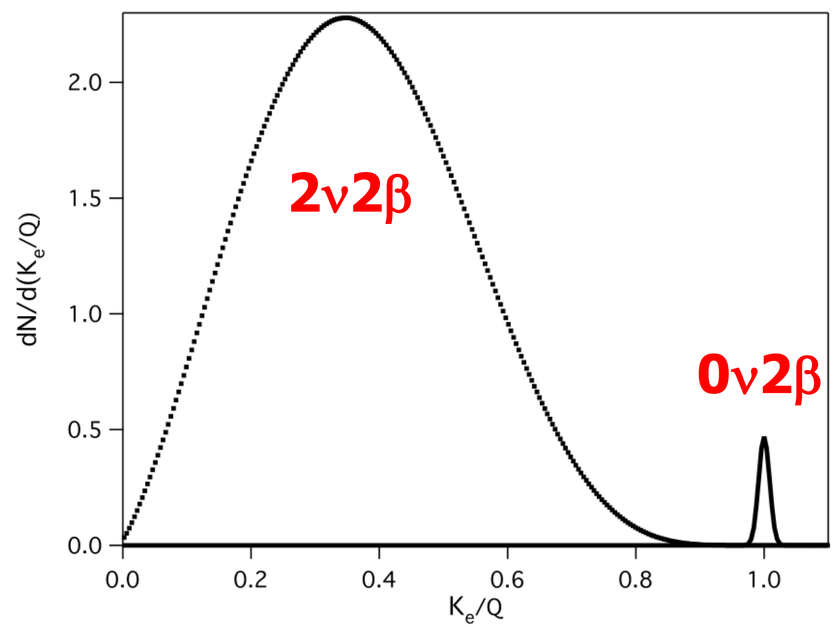
$\Rightarrow$ $$\frac{|\partial_\mu \tau'|^2}{\langle -i\tau' + i\bar{\tau}'\rangle^2} = \frac{|\partial_\mu \tau|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$ Modular invariant

If this is the case, …



$$^N_Z A \rightarrow ^{N-2}_{Z+2} A + 2\beta^- + 2\bar{\nu}_e$$





$$^N_Z A \rightarrow ^{N-2}_{Z+2} A + 2\beta^-$$

1939: 0ν2β decays

A 0ν2β decay can happen if massive ν's have the Majorana nature (Wendell Furry 1939)

$$T^{0\nu}_{1/2} = \left(G^{0\nu}\right)^{-1} \left|M^{0\nu}\right|^{-2} \left|\langle m\rangle_{ee}\right|^{-2}$$