Perturbations of black holes and stars by generalized vector fields

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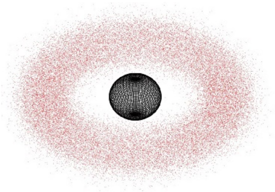
Based on **2104.08049 + to appear** with A. Held and J. Zhang

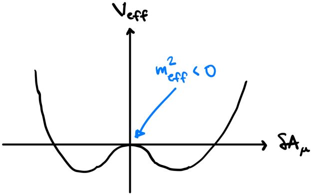
- Main question

  Under what conditions do **black holes** and **stars** in GR grow **massive vector hair**?



- Partial answer

  In **consistent theories**, massive vector hair **cannot** be generated through a tachyonic instability (**vectorization**) in static, spherically symmetric backgrounds



  While tachyonic instabilities are possible, they are necessarily accompanied by **ghost** or **gradient-unstable** modes

Outline

- Introduction
  - Why massive vector fields?
  - Why hairy black holes and stars?
- Generalized Proca theory
  - Review
  - Linearization about GR background
- Dynamics and stability of physical modes
- Applications of stability bounds
  - Black holes
  - Stars

Introduction

Why **massive vector** fields?

- Candidates for dark energy

  De Felice, Heisenberg, Kase, Mukohyama, Tsujikawa, Zhang (2016)

- Candidates for (light) dark matter

  Arkani-Hamed, Weiner (2008)

- Natural alternative to scalar-tensor theories

- Equally (better?) motivated as modifications of gravity

Introduction

Why **hairy** black holes and stars?

- Ideal laboratories to test fundamental physics

  Cardoso, Pani (2019)

- Opportunity to detect new light particles, e.g. dark matter

- Unique observational signatures, e.g. black hole bomb

  Press, Teukolsky (1972)

- Important theoretical objects, e.g. AdS/CFT

  Hartnoll, Herzog, Horowitz (2008)

Generalized Proca theory

Generalized Proca is a **vector-tensor** theory defined by

$$S[g,A]=\int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{\mu^2}{2}A^\mu A_\mu+\sum_{I=2}^{6}\mathcal{L}_I[g,A]\right]$$

$$\begin{aligned}
\mathcal{L}_2 &= G_2(X,\mathcal{F},\mathcal{G})\\
\mathcal{L}_3 &= G_3(X)\nabla_\mu A^\mu\\
\mathcal{L}_4 &= G_4(X)R+G_{4,X}(X)\Big[(\nabla_\mu A^\mu)^2-\nabla_\mu A^\nu\nabla_\nu A^\mu\Big]\\
\mathcal{L}_5 &= G_5(X)G^{\mu\nu}\nabla_\mu A_\nu-\frac{G_{5,X}(X)}{6}\Big[(\nabla_\mu A^\mu)^3\\
&\quad -3\nabla_\rho A^\rho\nabla_\mu A^\nu\nabla_\nu A^\mu+2\nabla_\mu A^\nu\nabla_\nu A^\rho\nabla_\rho A^\mu\Big]\\
\mathcal{L}_6 &= G_6(X)\widetilde{R}^{\mu\nu\rho\sigma}\nabla_\mu A_\nu\nabla_\rho A_\sigma+\frac{G_{6,X}(X)}{2}\widetilde{F}^{\mu\nu}\widetilde{F}^{\rho\sigma}\nabla_\mu A_\rho\nabla_\nu A_\sigma
\end{aligned}$$

Tasinato (2014), Heisenberg (2014)

Generalized Proca theory

$$\mathcal{L}_2 = G_2(X, \mathcal{F}, \mathcal{G})$$
$$\mathcal{L}_3 = G_3(X)\nabla_\mu A^\mu$$
$$\mathcal{L}_4 = G_4(X)R + G_{4,X}(X)\Big[(\nabla_\mu A^\mu)^2 - \nabla_\mu A^\nu \nabla_\nu A^\mu\Big]$$
$$\mathcal{L}_5 = G_5(X)G^{\mu\nu}\nabla_\mu A_\nu - \frac{G_{5,X}(X)}{6}\Big[(\nabla_\mu A^\mu)^3 - 3\nabla_\rho A^\rho \nabla_\mu A^\nu \nabla_\nu A^\mu + 2\nabla_\mu A^\nu \nabla_\nu A^\rho \nabla_\rho A^\mu\Big]$$
$$\mathcal{L}_6 = G_6(X)\widetilde{R}^{\mu\nu\rho\sigma}\nabla_\mu A_\nu \nabla_\rho A_\sigma + \frac{G_{6,X}(X)}{2}\widetilde{F}^{\mu\nu}\widetilde{F}^{\rho\sigma}\nabla_\mu A_\rho \nabla_\nu A_\sigma$$

The Lagrangian includes **5 arbitrary functions** G_I of

$$X = -\frac{1}{2}A^\mu A_\mu\,, \qquad \mathcal{F} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \qquad \mathcal{G} = A^\mu A^\nu F_\mu{}^\rho F_{\nu\rho}$$

and $G_{4,X} \equiv \frac{dG_4}{dX}$, etc.

Generalized Proca theory

$$\mathcal{L}_2 = G_2(X, \mathcal{F}, \mathcal{G})$$
$$\mathcal{L}_3 = G_3(X)\nabla_\mu A^\mu$$
$$\mathcal{L}_4 = G_4(X)R + G_{4,X}(X)\Big[(\nabla_\mu A^\mu)^2 - \nabla_\mu A^\nu \nabla_\nu A^\mu\Big]$$
$$\mathcal{L}_5 = G_5(X)G^{\mu\nu}\nabla_\mu A_\nu - \frac{G_{5,X}(X)}{6}\Big[(\nabla_\mu A^\mu)^3 - 3\nabla_\rho A^\rho \nabla_\mu A^\nu \nabla_\nu A^\mu + 2\nabla_\mu A^\nu \nabla_\nu A^\rho \nabla_\rho A^\mu\Big]$$
$$\mathcal{L}_6 = G_6(X)\widetilde{R}^{\mu\nu\rho\sigma}\nabla_\mu A_\nu \nabla_\rho A_\sigma + \frac{G_{6,X}(X)}{2}\widetilde{F}^{\mu\nu}\widetilde{F}^{\rho\sigma}\nabla_\mu A_\rho \nabla_\nu A_\sigma$$

The Lagrangian includes **non-minimal coupling** terms, in particular with the double dual Riemann tensor

$$\widetilde{R}^{\mu\nu\rho\sigma} = \frac{1}{4}\,\epsilon^{\mu\nu\mu'\nu'}\epsilon^{\rho\sigma\rho'\sigma'}R_{\mu'\nu'\rho'\sigma'}$$

Horndeski (1976)

Generalized Proca theory

$$S[g,A]=\int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\,R-\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu}-\frac{\mu^2}{2}\,A^\mu A_\mu+\sum_{I=2}^{6}\mathcal{L}_I[g,A]\right]$$

Generalized Proca is the unique **consistent** extension of the linear Proca theory in the sense that

- it describes **2+3 degrees of freedom** at the full non-linear level
- among the 4 components of A_μ, the **time component** is non-dynamical

i.e. the contraction

$$n^\mu A_\mu$$

has no time derivatives in the action

$n^\mu \;\rightarrow\;$ normal vector to constant-time hypersurfaces

Generalized Proca theory

More generally we define a **consistent** Proca theory by an action $S[g, A]$ that has a **pair of second-class constraints**

Extensions of Generalized Proca do exist

- Beyond Generalized Proca

  Heisenberg, Kase, Tsujikawa (2016)

- Extended vector-tensor theories

  Kimura, Naruko, Yoshida (2016)

- Proca-Nuevo

  de Rham, Pozsgay (2020)

- Extended Proca-Nuevo

  de Rham, SGS, Heisenberg, Pozsgay (2021)

Generalized Proca theory

- We are interested in the **linearization** of Generalized Proca about the state
$$\langle A_\mu \rangle = 0$$
but with arbitrary background metric

- At **quadratic order** in A_μ only two terms survive
$$\mathcal{L}_4 = G_{4,X}\left[-\frac{1}{2} R A^\mu A_\mu + (\nabla_\mu A^\mu)^2 - \nabla_\mu A^\nu \nabla_\nu A^\mu \right]$$
$$\mathcal{L}_6 = G_6 \, \widetilde{R}^{\mu\nu\rho\sigma} \nabla_\mu A_\nu \nabla_\rho A_\sigma$$
where
$$G_{4,X} \equiv G_{4,X}(X=0) \, , \qquad G_6 \equiv G_6(X=0)$$

Generalized Proca theory

$$\mathcal{L}_4 = G_{4,X}\left[-\frac{1}{2} R A^\mu A_\mu + (\nabla_\mu A^\mu)^2 - \nabla_\mu A^\nu \nabla_\nu A^\mu \right]$$

$$\mathcal{L}_6 = G_6 \widetilde{R}^{\mu\nu\rho\sigma} \nabla_\mu A_\nu \nabla_\rho A_\sigma$$

- $\mathcal{L}_4$ can be integrated by parts

$$\mathcal{L}_4 = G_{4,X} A^\mu A^\nu G_{\mu\nu}$$

- $\mathcal{L}_6$ can be expanded as

$$\mathcal{L}_6 = -\frac{G_6}{4}\left(F^{\mu\nu} F_{\mu\nu} R - 4 F^{\mu\rho} F^\nu{}_\rho R_{\mu\nu} + F^{\mu\nu} F^{\rho\sigma} R_{\mu\nu\rho\sigma} \right)$$

Remark: same operators appear in Drummond-Hathrell effective action of QED

Generalized Proca theory

General quadratic Lagrangian

$$S = \int d^4x\sqrt{-g}\Big[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\mu^2}{2}A^\mu A_\mu + G_{4,X}A^\mu A^\nu G_{\mu\nu}$$
$$- \frac{G_6}{4}\Big(F^{\mu\nu}F_{\mu\nu}R - 4F^{\mu\rho}F^\nu{}_\rho R_{\mu\nu} + F^{\mu\nu}F^{\rho\sigma}R_{\mu\nu\rho\sigma}\Big)\Big]$$

Remarks

- The model has 3 free parameters: $\mu\,,\ G_{4,X}\,,\ G_6$
- All other known Proca theories either have the **same linearization** or else **do not admit** $\langle A_\mu\rangle = 0$
- Regardless of the non-linear completion, this is the **most general theory** with the properties
  - (i) quadratic in A_μ
  - (ii) 3+2 degrees of freedom

Stability analysis

$$S = \int d^4x \sqrt{-g} \Big[\frac{M_{\rm Pl}^2}{2} R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\mu^2}{2} A^\mu A_\mu + G_{4,X} A^\mu A^\nu G_{\mu\nu} - \frac{G_6}{4} \left(F^{\mu\nu} F_{\mu\nu} R - 4 F^{\mu\rho} F^\nu{}_\rho R_{\mu\nu} + F^{\mu\nu} F^{\rho\sigma} R_{\mu\nu\rho\sigma} \right) \Big]$$

- We want to derive the **dispersion relations** for the physical degrees of freedom
- We focus on static and spherically symmetric backgrounds

$$g_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + \frac{dr^2}{g(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

- Vector and metric perturbations do not mix at linear level
- We assume background is stable under metric perturbations

Stability analysis

Proca field is expanded in **vector spherical harmonics**

$$A_\mu = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{J=1}^{4} C_{l,m}^{(J)}(t,r) \left(Z_{l,m}^{(J)} \right)_\mu (\theta, \phi)$$

$$(Z_{l,m}^{(1)})_\mu = \delta_\mu^t Y_{l,m}(\theta,\phi)$$
$$(Z_{l,m}^{(2)})_\mu = \delta_\mu^r Y_{l,m}(\theta,\phi)$$
$$(Z_{l,m}^{(3)})_\mu = \frac{1}{\sqrt{l(l+1)}} \partial_\mu Y_{l,m}(\theta,\phi)$$
$$(Z_{l,m}^{(4)})_\mu = \frac{1}{\sqrt{l(l+1)}} \left[-\csc\theta\, \delta_\mu^\theta \partial_\phi Y_{l,m}(\theta,\phi) + \sin\theta\, \delta_\mu^\phi \partial_\theta Y_{l,m}(\theta,\phi) \right]$$
$$Z_{0,0}^{(3,4)} = 0 \quad (\textbf{monopole})$$

Stability analysis

Proca field is expanded in **vector spherical harmonics**

$$A_\mu = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{J=1}^{4} C^{(J)}_{l,m}(t,r) \left(Z^{(J)}_{l,m} \right)_\mu (\theta, \phi)$$

Mode functions $C^{(J)}_{l,m}$ correspond to perturbations with even or odd parity

- $C^{(1,2,3)}_{l,m}$ are **even**, only <u>two</u> combinations are **dynamical** (<u>one</u> for the monopole)
- $C^{(4)}_{l,m}$ is **odd** and it is **dynamical**

Even and odd modes **decouple** at linear order

Stability analysis

Lagrangian for **odd modes**

$$S_{\rm odd} = \frac{1}{2}\int dtdr \sum_{l,m}(-1)^m \left[\frac{\mathcal{H}_1}{f}\,|\dot{C}^{(4)}_{l,m}|^2 - g\mathcal{H}_2\,|C^{(4)\prime}_{l,m}|^2 - \left(\mathcal{N}_m + \frac{l(l+1)}{r^2}\mathcal{N}_j\right)|C^{(4)}_{l,m}|^2\right]$$

$$\left(\dot{F} \equiv \frac{\partial F}{\partial t}\ ,\ F' \equiv \frac{\partial F}{\partial r}\right)$$

Coefficient functions

$$\mathcal{H}_1 = 1 - G_6\,\frac{g'}{r}\ , \qquad \mathcal{H}_2 = 1 - G_6\,\frac{f'g}{fr}$$

$$\mathcal{N}_m = \mu^2 + G_{4,X}\left(R - 2r^2R^{\theta\theta}\right)$$

$$\mathcal{N}_j = 1 + G_6\left(R - 4r^2R^{\theta\theta} + \frac{2(1-g)}{r^2}\right)$$

Stability analysis

- To obtain the **dispersion relations** one assumes localized perturbations, or smooth enough background

$$\left|\frac{f'}{f}\right| \ll k \, , \, \omega \, , \, m_{\rm eff}$$

- One can then perform a standard Fourier transform

$$C \rightarrow \widetilde{C}\, e^{i(kr-\omega t)}$$

- For the **odd modes** the dispersion relation is then

$$\frac{\mathcal{H}_1}{f}\,\omega^2 - g\mathcal{H}_2\,k^2 - \left(\mathcal{N}_m + \frac{l(l+1)}{r^2}\,\mathcal{N}_j\right) = 0$$

Stability analysis

Lagrangian for **even modes**

$$S_{\rm pol} = \frac{1}{2}\int dtdr\, r^2 \sum_{l,m}(-1)^m \Bigg[\frac{g}{f}\,\mathcal{G}_1 \left| \dot{C}^{(2)}_{l,m} - C^{(1)\prime}_{l,m} \right|^2 + \frac{1}{fr^2}\,\mathcal{H}_1 \left| \dot{C}^{(3)}_{l,m} - \sqrt{l(l+1)}\,C^{(1)}_{l,m} \right|^2 - \frac{g}{r^2}\,\mathcal{H}_2 \left| C^{(3)\prime}_{l,m} - \sqrt{l(l+1)}\,C^{(2)}_{l,m} \right|^2 + \frac{1}{f}\,\mathcal{M}_1 |C^{(1)}_{l,m}|^2 - g\,\mathcal{M}_2 |C^{(2)}_{l,m}|^2 - \frac{\mathcal{N}_m}{r^2}\,|C^{(3)}_{l,m}|^2 \Bigg]$$

Coefficient functions $\mathcal{G}_1 = 1 + 2G_6\,\frac{1-g}{r^2}$

$$\mathcal{M}_1 = \mu^2 - 2G_{4,X}\left(\frac{g'}{r} - \frac{1-g}{r^2}\right)\,, \quad \mathcal{M}_2 = \mu^2 - 2G_{4,X}\left(\frac{f'g}{fr} - \frac{1-g}{r^2}\right)$$

while $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{N}_m$ are the **same functions** that appear in the odd sector

Stability analysis

Lagrangian for **even monopole mode**

$$S_{\mathrm{even}}^{(l=0)} = \frac{1}{2}\int dtdr\, r^2\left[\frac{g}{f}\,\mathcal{G}_1\left|\dot{C}_{0,0}^{(2)} - C_{0,0}^{(1)\prime}\right|^2 + \frac{1}{f}\,\mathcal{M}_1|C_{0,0}^{(1)}|^2 - g\,\mathcal{M}_2|C_{0,0}^{(2)}|^2\right]$$

Trick is to integrate out non-dynamical mode by introducing an **additional field**

$$B_{0,0} = a_0\left(\dot{C}_{0,0}^{(2)} - C_{0,0}^{(1)\prime}\right) \qquad a_0 \equiv \sqrt{\frac{g|\mathcal{G}_1|}{f}}$$

This relation is enforced as an equation of motion with the **auxiliary action**

$$S_{\mathrm{even}}^{(l=0)} = \frac{1}{2}\int dtdr\, r^2\left\{-\sigma_0|B_{0,0}|^2 + \sigma_0 a_0\left[B_{0,0}^*\left(\dot{C}_{0,0}^{(2)} - C_{0,0}^{(1)\prime}\right) + \mathrm{c.c.}\right] + \frac{1}{f}\,\mathcal{M}_1|C_{0,0}^{(1)}|^2 - g\mathcal{M}_2|C_{0,0}^{(2)}|^2\right\} \qquad \sigma_0 \equiv \mathrm{sign}(\mathcal{G}_1)$$

Stability analysis

$$S_{\rm even}^{(l=0)} = \frac{1}{2}\int dtdr\, r^2 \Big\{ -\sigma_0 |B_{0,0}|^2 + \sigma_0 a_0 \left[B_{0,0}^* \left(\dot{C}_{0,0}^{(2)} - C_{0,0}^{(1)\prime} \right) + \text{c.c.} \right]$$
$$+ \frac{1}{f}\mathcal{M}_1 |C_{0,0}^{(1)}|^2 - g\mathcal{M}_2 |C_{0,0}^{(2)}|^2 \Big\}$$

- Integrating out $B_{0,0}$ gives back the original action
- But now we can also integrate out $C_{0,0}^{(1)}$ and $C_{0,0}^{(2)}$ because their EoM are algebraic

$$C_{0,0}^{(1)} = -\frac{\sigma_0}{r^2}\frac{f}{\mathcal{M}_1}\left(r^2 a_0 B_{0,0}\right)' \,, \qquad C_{0,0}^{(2)} = -\sigma_0 \frac{a_0}{g\mathcal{M}_2}\dot{B}_{0,0}$$

Stability analysis

The result is

$$S^{(l=0)}_{\text{even}} = \frac{1}{2}\int dtdr\, r^2 \left[\frac{|\mathcal{G}_1|}{f\mathcal{M}_2}\, |\dot{B}_{0,0}|^2 - \frac{g|\mathcal{G}_1|}{\mathcal{M}_1} \left| B'_{0,0} + \frac{(r^2 a_0)'}{r^2 a_0}\, B_{0,0} \right|^2 - \sigma_0 |B_{0,0}|^2 \right]$$

with $\quad a_0 \equiv \sqrt{\frac{g|\mathcal{G}_1|}{f}}\,,\ \sigma_0 \equiv \mathrm{sign}(\mathcal{G}_1)$

Remarks

- As expected, there is a single dynamical **monopole** mode
- The naive "mass" coefficients control the **kinetic** and **gradient** terms of the dynamical mode
- This has no analog for scalars

Stability analysis

The same procedure works for the $l \geq 1$ **even modes**

- We introduce two **additional fields**

$$B_{l,m} = a_0 \left(\dot{C}^{(2)}_{l,m} - C^{(1)\prime}_{l,m} \right) \ , \quad C_{l,m} = C^{(3)}_{l,m}$$

- Then integrate out $C^{(1)}_{l,m}$ and $C^{(2)}_{l,m}$

$$C^{(1)}_{l,m} = \frac{f}{\left(\mathcal{M}_1 + \mathcal{H}_1 \frac{l(l+1)}{r^2} \right)} \left[- \frac{\sigma_0}{r^2} \left(r^2 a_0 B_{l,m} \right)' + \frac{\mathcal{H}_1 \sqrt{l(l+1)}}{f r^2} \dot{C}_{l,m} \right]$$

$$C^{(2)}_{l,m} = \frac{1}{g \left(\mathcal{M}_2 + \mathcal{H}_2 \frac{l(l+1)}{r^2} \right)} \left[- \sigma_0 a_0 \dot{B}_{l,m} + \frac{g \mathcal{H}_2 \sqrt{l(l+1)}}{r^2} C'_{l,m} \right]$$

Stability analysis

- Final action for the dynamical modes

$$S_{\text{even}}^{(l>0)} = \frac{1}{2}\int dtdr \sum_{l,m}(-1)^m \Bigg[\text{(diagonal terms)}$$
$$- \frac{\sigma_0 a_0 \mathcal{H}_2 \sqrt{l(l+1)}}{\left(\mathcal{M}_2 + \mathcal{H}_2 \frac{l(l+1)}{r^2}\right)} \left(\dot{B}^*_{l,m} C'_{l,m} + \text{c.c.}\right)$$
$$+ \frac{\sigma_0 \mathcal{H}_1 \sqrt{l(l+1)}}{r^2\left(\mathcal{M}_1 + \mathcal{H}_1 \frac{l(l+1)}{r^2}\right)} \left((r^2 a_0 B^*_{l,m})' \dot{C}_{l,m} + \text{c.c.}\right)\Bigg]$$

- In general the Lagrangian cannot be diagonalized via a (local) field redefinition

- Dispersion relations are in general non-linear

Stability analysis

Summary of **dispersion relations**

- Odd modes ($l \geq 1$)

$$\frac{\mathcal{H}_1}{f}\,\omega^2 - g\mathcal{H}_2\,k^2 - \left(\mathcal{N}_m + \frac{l(l+1)}{r^2}\,\mathcal{N}_j\right) = 0$$

- Even monopole mode

$$\frac{|\mathcal{G}_1|}{f\mathcal{M}_2}\,\omega^2 - \frac{g|\mathcal{G}_1|}{\mathcal{M}_1}\,k^2 - \mathrm{sign}(\mathcal{G}_1) = 0$$

- Even higher multipole modes

$$\det\begin{pmatrix}\mathcal{P}_{BB} & \mathcal{P}_{BC}\\ \mathcal{P}_{BC} & \mathcal{P}_{CC}\end{pmatrix} = 0$$

Stability analysis

Stability conditions

- Absence of **ghost** and **radial gradient** instabilities

$$\mathcal{H}_1 > 0\,, \quad \mathcal{H}_2 > 0\,, \quad \mathcal{M}_1 > 0\,, \quad \mathcal{M}_2 > 0$$

- Absence of **angular gradient** instabilities

$$\mathcal{N}_j > 0\,, \quad \mathcal{N}_m > 0\,, \quad \mathcal{G}_1 > 0$$

Corollary

- Effective masses of all modes are positive definite if kinetic and gradient terms are healthy
- **Tachyonic** instabilities cannot arise

Stability analysis

No-go theorem* for vectorization

Static spherically symmetric GR backgrounds cannot spontaneously grow vector hair through a tachyonic instability

SGS, Held, Zhang (2021); Silva, Coates, Ramazanoglu, Sotiriou (2021)

*Potential loophole:

- The analysis was based on dispersion relations of **localized** perturbations
- We cannot discard a tachyonic destabilization of **global solutions**
- We have checked that Schwarzschild black holes are stable (more on this later)

Applications

Schwarzschild black hole

Beltran-Jimenez, Durrer, Heisenberg, Thorsrud (2013)

- Metric

$$f = g = 1 - \frac{r_s}{r} \qquad\qquad r_s = 2GM$$

- Coefficient functions

$$\mathcal{H}_1 = \mathcal{H}_2 = 1 - \frac{G_6 r_s}{r^3} \quad , \qquad \mathcal{N}_j = \mathcal{G}_1 = 1 + \frac{2G_6 r_s}{r^3}$$
$$\mathcal{N}_m = \mathcal{M}_1 = \mathcal{M}_2 = \mu^2$$

- Note: no dependence on $G_{4,X}$ for solutions of vacuum Einstein equations

Applications

Schwarzschild black hole

- Stability for all radii $r \geq r_s$

$$-\frac{1}{2} < \frac{G_6}{r_s^2} < 1$$

- Conclusion: for any non-zero G_6 , there exist sufficiently small black holes subject to instabilities

- Example motivated by **dark energy**

$$G_6 \sim \Lambda^{-2} \quad , \quad \Lambda \sim (M_{\rm Pl} H_0^2)^{1/3} \quad \rightarrow \quad G_6 \sim (10^3\,{\rm km})^2$$

$\rightarrow$ unstable **stellar-mass** BHs, stable **supermassive** BHs

- Potentially interesting for **primordial** BHs with $r_s \sim 10^{-10}\,{\rm m}$

Applications

Reissner-Nordström black hole

- Metric

$$f = g = 1 - \frac{r_s}{r} + \frac{r_Q^2}{4r^2} \qquad\qquad r_s = 2GM\ , \quad r_Q = 2\sqrt{G}\, Q$$

- Coefficient functions

$$\mathcal{H}_1 = \mathcal{H}_2 = 1 - \frac{G_6}{r^2}\left(\frac{r_s}{r} - \frac{r_Q^2}{2r^2}\right)$$

$$\mathcal{N}_j = 1 + \frac{2G_6}{r^2}\left(\frac{r_s}{r} - \frac{3r_Q^2}{4r^2}\right)\ , \qquad \mathcal{G}_1 = 1 + \frac{2G_6}{r^2}\left(\frac{r_s}{r} - \frac{r_Q^2}{4r^2}\right)$$

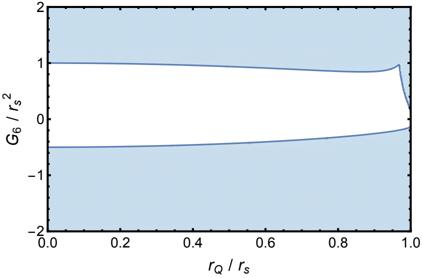
$$\mathcal{M}_1 = \mathcal{M}_2 = \mu^2 + \frac{G_{4,X}\, r_Q^2}{2r^4}\ , \qquad \mathcal{N}_m = \mu^2 - \frac{G_{4,X}\, r_Q^2}{2r^4}$$

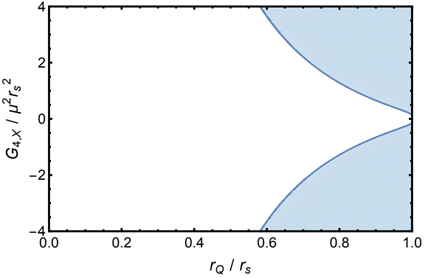
Applications

Reissner-Nordström black hole

- The problem is to derive bounds on G_6 and $G_{4,X}$ such that the coefficient functions are positive for all radii

$$r \geq r_+ = \frac{r_s}{2}\left(1 + \sqrt{1 - \frac{r_Q^2}{r_s^2}}\right) \quad , \qquad r_Q \leq r_s$$





Applications

Reissner-Nordström black hole

- Bounds are most stringent for an **extremal** BH ($r_Q = r_s$)

$$\frac{|G_6|}{r_s^2} < \frac{1}{8} \quad , \qquad \frac{|G_{4,X}|}{\mu^2 r_s^2} < \frac{1}{8}$$

- For small but non-zero charge ($r_Q \ll r_s$)

$$\frac{|G_{4,X}|}{\mu^2 r_s^2} < \frac{2r_s^2}{r_Q^2}$$

- What values of $r_Q/r_s \sim Q/M$ could we expect in realistic situations?

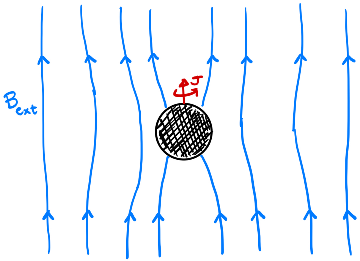
Applications

Digression: **Wald mechanism**

Wald (1974)

- A black hole immersed in an external magnetic field will preferentially accrete charges until acquiring a **net charge**

$$Q = 2B_{\text{ext}}J$$

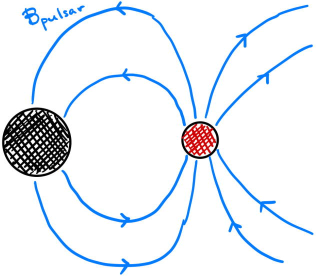


Applications

Digression: **Wald mechanism**

- A sizable BH charge might be achieved in a **NS-BH merger** if the neutron star is a strongly magnetized pulsar

  Levin, D'Orazio, SGS (2018)



Applications

Digression: **Wald mechanism**

- Optimistically, values up to

$$\frac{r_Q}{r_s} \sim 10^{-7}$$

  might be achievable

- Estimate seems robust after more thorough analysis; moreover, BH spin is not necessary

  Chen, Dai (2021); Adari, Berens, Levin (2021)

- Taking $G_{4,X} = \mathcal{O}(1)$ and $r_Q \sim 10^{-7}\, r_s$, $r_s \sim 10\,\mathrm{km}$

$$\mu \gtrsim 10^{-17}\,\mathrm{eV}$$

  Compare with range $10^{-22} - 10^{-20}\,\mathrm{eV}$ for **fuzzy dark matter**

Applications

Perfect fluid stars

- In general, stability conditions must be investigated numerically because metric is not known explicitly
- However, suppose the stability criteria are extremized at the **center of the star**
- Checked for **uniform density** star and **polytropic** star with $p = K\rho^{5/3}$
- Then we can solve the TOV equations analytically in the vicinity of $r = 0$ and obtain bounds on G_6 and $G_{4,X}$
- Plausible that assumption is true for all realistic equations of state, including imperfect fluids

Applications

Perfect fluid stars

- Stability bounds

$$-\frac{3}{2\rho_c} < \frac{G_6}{M_{\rm Pl}^2} < \frac{3}{\rho_c + 3p_c}$$
$$-\frac{1}{2\rho_c} < \frac{G_{4,X}}{\mu^2 M_{\rm Pl}^2} < \frac{1}{2p_c}$$

$\rho_c \rightarrow$ central density

$p_c \rightarrow$ central pressure

- Example: neutron star and typical EFT couplings

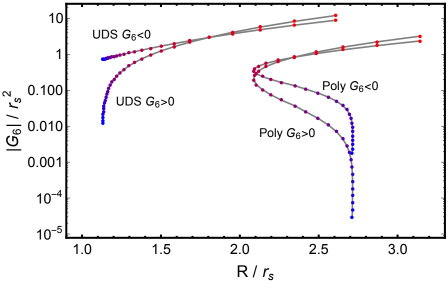
$$\rho_c \sim 10^{18}\,\mathrm{kg\,m^{-3}}\ ,\ |G_6| \sim \frac{|G_{4,X}|}{\mu^2} \sim \Lambda^{-2} \quad \rightarrow \quad \frac{\Lambda}{M_{\rm Pl}} \gtrsim 10^{-38}$$

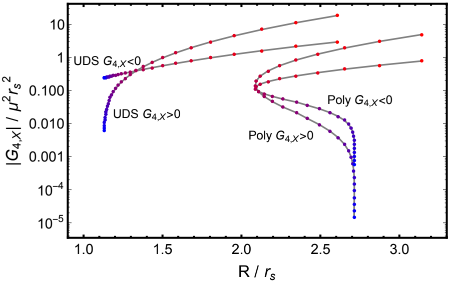
Seemingly mild but relevant to **dark energy**

Applications

Perfect fluid stars

- Interesting dependence on the **equation of state**
- Motivates dedicated analysis of realistic EoS





Global solutions

Global solutions are determined by equations of the form

$$\frac{d^2 u_I}{dr_*^2} + \omega^2 u_I - V_{IJ} u_J = 0$$

$u_I \in \{\text{monopole, axial, polar}_1\text{, polar}_2\}$

$\omega \rightarrow$ complex eigenfrequency

$r_* \rightarrow$ tortoise coordinate

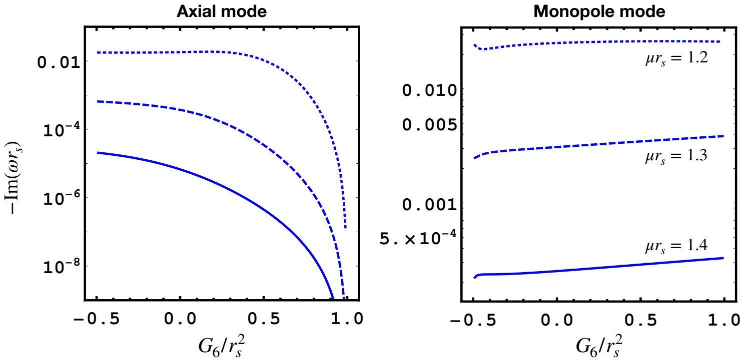
$V_{IJ} \rightarrow$ effective potential

In general $\mathrm{Im}(\omega) \neq 0$ due to the coupling to gravity

$\mathrm{Im}(\omega) < 0 \rightarrow$ decaying mode, stable

$\mathrm{Im}(\omega) > 0 \rightarrow$ growing mode, unstable

Global solutions



- Numerically we find no evidence of unstable global modes
- However our code cannot access values of G_6 arbitrarily close to the bounds $\{-r_s^2/2\,,\ r_s\}$

Global solutions

For values of G_6 close to the bounds we can prove **analytically** the absence of unstable modes

Axial mode:

$$\int_{r_s}^{\infty} dr \left(1 - \frac{r_s}{r}\right) \left[|v'|^2 + V_{\rm axi}(r)|v|^2\right] = -\frac{|\omega|^2 |v(r_s)|^2}{\mathrm{Im}(\omega)}$$

- $v(r)$ is the redefined axial mode function
- Similar to a formula derived originally in asymptotically AdS backgrounds

  Horowitz, Hubeny (1999)

Global solutions

$$\int_{r_s}^{\infty} dr \left(1 - \frac{r_s}{r}\right) \left[|v'|^2 + V_{\rm axi}(r)|v|^2\right] = -\frac{|\omega|^2|v(r_s)|^2}{{\rm Im}(\omega)}$$

- If $V_{\rm axi}$ was positive definite then we could immediately conclude ${\rm Im}(\omega) < 0$
- Unfortunately this is not the case; however for $G_6/r_s^2 = 1 - \epsilon$ we can prove
$$\int_{r_s}^{\infty} dr \left(1 - \frac{r_s}{r}\right) V_{\rm axi}|v|^2 = C \log \frac{1}{\epsilon}$$
to leading order in small ϵ and where $C > 0$
- This proves that ${\rm Im}(\omega) < 0$ and explains the behavior observed numerically
- Proofs for the monopole and polar modes are analogous although more involved

Outlook

What to make of the instabilities?

- If absent, then one has interesting bounds relevant for **dark energy** and ultra-light **dark matter**
- If present, then potentially interesting signatures, but needs understanding of higher-derivative operators, cf. ghost condensate

  Arkani-Hamed, Cheng, Luty, Mukohyama (2003)

Other ways to grow massive vector hair?

- Tachyonic instability (vectorization) via vector-matter coupling

  Minamitsuji (2020)

- Non-linear instabilities
- Quantum phase transitions

Outlook

Extensions of our work (future/ongoing)

- Robustness of no-go result for tachyonic instabilities
- Rotating systems, cosmological constant
- Realistic NS equations of state
- Inclusion of higher-derivative operators

Thank you